Problem 11.2: A zeroth-order natural relation.

This problem studies an $N = 2$ linear sigma model coupled to fermions:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi^i)^2 + \frac{1}{2}\mu^2(\phi^i)^2 - \frac{\lambda}{4}((\phi^i)^2)^2 + \bar{\psi}(i\partial\!\!\!/)\psi - g\bar{\psi}(\phi^1 + i\gamma^5\phi^2)\psi \tag{1}$$

where ϕ^i is a two-component field, $i = 1, 2$.

(a) Show that this theory has the following global symmetry:

$$\begin{aligned}\phi^1 &\to \cos\alpha\phi^1 - \sin\alpha\phi^2,\\ \phi^2 &\to \sin\alpha\phi^1 + \cos\alpha\phi^2,\\ \psi &\to e^{-i\alpha\gamma^5/2}\psi.\end{aligned} \tag{2}$$

Show also that the solution to the classical equations of motion with the minimum energy breaks this symmetry spontaneously.

(b) Denote the vacuum expectation value of the field ϕ^i by v and make the change of variables

$$\phi^i(x) = (v + \sigma(x), \pi(x)) \,. \tag{3}$$

Write out the Lagrangian in these new variables, and show that the fermion acquires a mass given by

$$m_f = gv. \tag{4}$$

(c) Compute the one-loop radiative correction to m_f, choosing renormalization conditions so that v and g (defined as the $\psi\psi\pi$ vertex at zero momentum transfer) receive no radiative corrections. Show that relation (4) receives nonzero corrections but that these corrections are finite. This is in accord with our general discussion in Section 11.6.

Part (a)

To get a better understanding of what (1) describes we expand the sums to get

$$\begin{aligned}\mathcal{L} &= \mathcal{L}_0 + \mathcal{L}_{\text{int}},\\ \mathcal{L}_0 &= \frac{1}{2}\left((\partial_\mu\phi^1)^2 + \mu^2(\phi^1)^2\right) + \frac{1}{2}\left((\partial_\mu\phi^2)^2 + \mu^2(\phi^2)^2\right) + \bar{\psi}(i\partial\!\!\!/)\psi,\\ \mathcal{L}_{\text{int}} &= -\frac{\lambda}{4}(\phi^1)^4 - \frac{\lambda}{4}(\phi^2)^4 - \frac{\lambda}{2}(\phi^1)^2(\phi^2)^2 - g\phi^1\bar{\psi}\psi - ig\phi^2\bar{\psi}\gamma^5\psi.\end{aligned} \tag{5}$$

From the above equation we see that $\mathcal{L}$ describes two real scalar ϕ^4 fields that can interact with each other and a massless fermion field.

To show that equation (2) is indeed a symmetry of the Lagrangian (1), it is easiest to use the unexpanded Lagrangian of (1). The vector dot product $(\phi^i)^2$ is invariant under the transformation (2)

$$\begin{aligned}(\phi^i)^2 &= (\phi^1)^2 + (\phi^2)^2\\ &\to \cos^2\alpha(\phi^1)^2 + \sin^2\alpha(\phi^2)^2 - 2\sin\alpha\cos\alpha\phi^1\phi^2\\ &\quad + \sin^2\alpha(\phi^1)^2 + \cos^2\alpha(\phi^2)^2 + 2\sin\alpha\cos\alpha\phi^1\phi^2\\ &= (\phi^1)^2 + (\phi^2)^2.\end{aligned} \tag{6}$$

This means that the first three terms of (1) are invariant under this symmetry. Next, we show that the fermionic kinetic term is invariant under this transformation

$$\bar{\psi}i\partial\!\!\!/\psi \to \bar{\psi}e^{-i\alpha\gamma^5/2}i\partial\!\!\!/ e^{-i\alpha\gamma^5/2}\psi = \bar{\psi}i\partial\!\!\!/ e^{+i\alpha\gamma^5/2}e^{-i\alpha\gamma^5/2}\psi = \bar{\psi}i\partial\!\!\!/\psi. \tag{7}$$

Finally, we show that the fermion-scalar interactions are invariant under this transformation

$$\begin{aligned}
\phi^1\bar{\psi}\psi + i\phi^2\bar{\psi}\gamma^2\psi \to& \left(\cos\alpha\phi^1 - \sin\alpha\phi^2\right)\bar{\psi}e^{+i\alpha\gamma^5/2}e^{-i\alpha\gamma^5/2}\psi \\
&+ i\left(\sin\alpha\phi^1 + \cos\alpha\phi^2\right)\bar{\psi}e^{-i\alpha\gamma^5/2}\gamma^5 e^{-i\alpha\gamma^5/2}\psi \\
=& \cos\alpha\phi^1\bar{\psi}e^{-i\alpha\gamma^5}\psi + i\sin\alpha\phi^1\bar{\psi}\gamma^5 e^{-i\alpha\gamma^5}\psi \\
&- \sin\alpha\phi^2\bar{\psi}e^{-i\alpha\gamma^5}\psi + i\cos\alpha\phi^2\bar{\psi}\gamma^5 e^{-i\alpha\gamma^5}\psi \\
=& \phi^1\bar{\psi}\left(\cos\alpha + i\sin\alpha\gamma^5\right)e^{-i\alpha\gamma^5}\psi \\
&+ i\phi^2\bar{\psi}\gamma^5\left(\cos\alpha + i\sin\alpha\gamma^5\right)e^{-i\alpha\gamma^5}\psi \\
=& \phi^1\bar{\psi}e^{+i\alpha\gamma^5}e^{-i\alpha\gamma^5}\psi + i\phi^2\bar{\psi}\gamma^5 e^{+i\alpha\gamma^5}e^{-i\alpha\gamma^5}\psi \\
=& \phi^1\bar{\psi}\psi + i\phi^2\bar{\psi}\gamma^5\psi.
\end{aligned} \tag{8}$$

Now we want to show that the classical solutions to the equation of motion that minimize the energy break the symmetry (2). The Hamiltonian density is given by

$$\begin{aligned}
\mathcal{H} =& \sum_{i=1,2}\frac{\mathcal{L}}{\partial(\partial_0\phi^i)}(\partial_0\phi^i) + \frac{\mathcal{L}}{\partial(\partial_0\bar{\psi})}(\partial_0\bar{\psi}) + \frac{\mathcal{L}}{\partial(\partial_0\psi)}(\partial_0\psi) - \mathcal{L} \\
=& \frac{1}{2}(\partial_0\phi_1)^2 + \frac{1}{2}(\nabla\phi_1)^2 - \mu^2(\phi_1)^2 + \frac{\lambda}{4}(\phi_1)^4 \\
&+ \frac{1}{2}(\partial_0\phi_2)^2 + \frac{1}{2}(\nabla\phi_2)^2 - \mu^2(\phi_2)^2 + \frac{\lambda}{4}(\phi_2)^4 \\
&+ \frac{\lambda}{2}(\phi_1)^2(\phi_2)^2 + i\bar{\psi}(\boldsymbol{\gamma}\cdot\boldsymbol{\nabla})\psi + g\phi_1\bar{\psi}\psi + ig\phi_2\bar{\psi}\gamma^5\psi
\end{aligned} \tag{9}$$

Ignoring the fermion part of the Hamiltonian, the Hamiltonian is minimized by the uniform fields, ϕ_0^i that minimize the potential

$$V(\phi) = -\frac{1}{2}\mu^2(\phi^i)^2 + \frac{\lambda}{4}((\phi^i)^2)^2. \tag{10}$$

Taking the first derivative with respect to ϕ^i we obtain the condition that ϕ_0^i must satisfy

$$\begin{aligned}
\left.\frac{\partial V(\phi)}{\partial\phi^i}\right|_{\phi_0^i} &= -\mu^2\phi_0^i + \lambda(\phi_0^i)^2\phi_0^i = 0 \\
\implies (\phi_0^i)^2 &= \frac{\mu^2}{\lambda} \equiv v.
\end{aligned} \tag{11}$$

How can this be done without setting $g = 0$? How does the Fermion fields effect v and spontaneous symmetry breaking?

Part (b)

We denote the vacuum expectation value of ϕ^i by v and make the change of variables $\phi^1 \to v+\sigma$ and $\phi^2 \to \pi$. In terms of the new fields σ and π the Lagrangian is

$$\begin{aligned}
\mathcal{L} &= \frac{1}{2}\left((\partial_\mu(v+\sigma))^2 + \mu^2(v+\sigma)^2\right) + \frac{1}{2}\left((\partial_\mu\pi)^2 + \mu^2(\pi)^2\right) + \bar{\psi}(i\partial\!\!\!/)\psi \\
&\quad - \frac{\lambda}{4}(v+\sigma)^4 - \frac{\lambda}{4}(\pi)^4 - \frac{\lambda}{2}(v+\sigma)^2(\pi)^2 - g(v+\sigma)\bar{\psi}\psi - ig\pi\bar{\psi}\gamma^5\psi \\
&= \frac{1}{2}\left((\partial_\mu\sigma)^2 + \mu^2(\sigma^2+2v\sigma+v^2)\right) + \frac{1}{2}\left((\partial_\mu\pi)^2 + \mu^2\pi^2\right) + \bar{\psi}(i\partial\!\!\!/)\psi \\
&\quad - \frac{\lambda}{4}(\sigma^4 + 4v\sigma^3 + 6v^2\sigma^2 + 4v^3\sigma + v^4) - \frac{\lambda}{4}\pi^4 - \frac{\lambda}{2}(\sigma^2+2v\sigma+v^2)\pi^2 \\
&\qquad - g(v+\sigma)\bar{\psi}\psi - ig\pi\bar{\psi}\gamma^5\psi \\
&= \frac{1}{2}\left((\partial_\mu\sigma)^2 + (\mu^2 - 3\lambda v^2)\sigma^2\right) + \frac{1}{2}\left((\partial_\mu\pi)^2 + (\mu^2-\lambda v^2)\pi^2\right) + \left(\bar{\psi}(i\partial\!\!\!/)\psi - gv\bar{\psi}\psi\right) \\
&\quad + v\left(\mu^2 - \lambda v^2\right)\sigma + \frac{1}{2}\mu^2 v^2 - \lambda v\sigma^3 - \frac{\lambda}{4}\sigma^4 - \frac{\lambda}{4}v^4 - \frac{\lambda}{4}\pi^4 - \frac{\lambda}{2}\sigma^2\pi^2 - \lambda v\sigma\pi^2 \\
&\qquad - g\sigma\bar{\psi}\psi - ig\pi\bar{\psi}\gamma^5\psi.
\end{aligned} \tag{12}$$

The above can be simplified by substituting $v = \mu/\sqrt{\lambda}$ and dropping the constant terms

$$\begin{aligned}
\mathcal{L} &\to \frac{1}{2}\left((\partial_\mu\sigma)^2 + (\mu^2 - 3\lambda\frac{\mu^2}{\lambda})\sigma^2\right) + \frac{1}{2}\left((\partial_\mu\pi)^2 + (\mu^2 - \lambda\frac{\mu^2}{\lambda})\pi^2\right) + \left(\bar{\psi}(i\partial\!\!\!/)\psi - \frac{g\mu}{\sqrt{\lambda}}\bar{\psi}\psi\right) \\
&\quad + \frac{\mu}{\sqrt{\lambda}}\left(\mu^2 - \lambda\frac{\mu^2}{\lambda}\right)\sigma - \lambda\frac{\mu}{\sqrt{\lambda}}\sigma^3 - \lambda\frac{\mu}{\sqrt{\lambda}}\sigma\pi^2 - \frac{\lambda}{4}\sigma^4 - \frac{\lambda}{4}\pi^4 - \frac{\lambda}{2}\sigma^2\pi^2 \\
&\qquad - g\sigma\bar{\psi}\psi - ig\pi\bar{\psi}\gamma^5\psi, \\
&= \frac{1}{2}\left((\partial_\mu\sigma)^2 - 2\mu^2\sigma^2\right) + \frac{1}{2}(\partial_\mu\pi)^2 + \left(\bar{\psi}(i\partial\!\!\!/)\psi - m_f\bar{\psi}\psi\right) \\
&\quad - \sqrt{\lambda}\mu\sigma^3 - \sqrt{\lambda}\mu\sigma\pi^2 - \frac{\lambda}{4}\sigma^4 - \frac{\lambda}{4}\pi^4 - \frac{\lambda}{2}\sigma^2\pi^2 - g\sigma\bar{\psi}\psi - ig\pi\bar{\psi}\gamma^5\psi.
\end{aligned} \tag{13}$$

Notice that the term linear in σ vanishes, as it should (because v is a minimum of the potential). Furthermore, note that the fermion field has gained a mass

$$m_f = gv = \frac{g\mu}{\sqrt{\lambda}}. \tag{14}$$

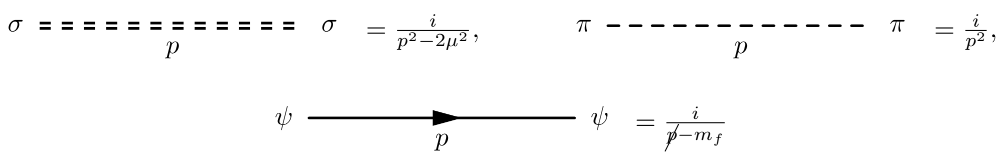


Figure 1: Feynman propagators for the Lagrangian (13).

Part (c)

The Feynman rules for the (un-renormalized) Lagrangian (13) are collected in Figs. 1 and 3.

To get the counter terms for renormalized perturbation theory we rescale the fields in (13) by:

$$\begin{aligned}
\phi^i &\to \sqrt{Z_\phi}\phi^i, \\
\psi &\to \sqrt{Z_f}\psi,
\end{aligned} \tag{15}$$

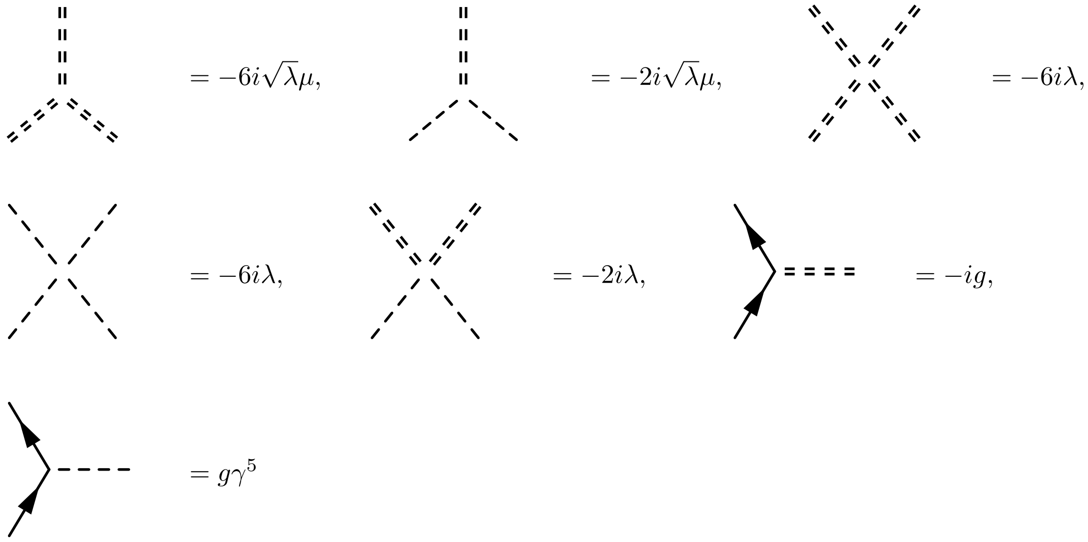


Figure 2: Feynman vertices for the Lagrangian (13).

to obtain

$$\mathcal{L} = Z_\phi \frac{1}{2}(\partial_\mu \phi^i)^2 + \frac{1}{2} Z_\phi \mu^2 (\phi^i)^2 - \frac{\lambda}{4} Z_\phi^2 ((\phi^i)^2)^2 + Z_f \bar{\psi}(i\partial\!\!\!/)\psi - g\sqrt{Z_\phi} Z_f \bar{\psi}(\phi^1 + i\gamma^5\phi^2)\psi. \quad (16)$$

Defining

$$Z_\phi = 1 + \delta_\phi,\ Z_f = 1 + \delta_f,\ Z_\phi \mu^2 = \mu^2 + \delta_\mu,\ \lambda Z_\phi^2 = \lambda + \delta_\lambda \text{ and } g\sqrt{Z_\phi} Z_f = g + \delta_g, \quad (17)$$

we obtain

$$\mathcal{L} = \mathcal{L}_0 + \delta\mathcal{L}, \quad (18)$$

$$\mathcal{L}_0 = \frac{1}{2}(\partial_\mu \phi^i)^2 + \frac{1}{2}\mu^2 (\phi^i)^2 - \frac{\lambda}{4}((\phi^i)^2)^2 + \bar{\psi}(i\partial\!\!\!/)\psi - g\bar{\psi}(\phi^1 + i\gamma^5\phi^2)\psi, \quad (19)$$

$$\delta\mathcal{L} = \frac{\delta_\phi}{2}(\partial_\mu \phi^i)^2 + \frac{\delta_\mu}{2}(\phi^i)^2 - \frac{\delta_\lambda}{4}((\phi^i)^2)^2 + \delta_f \bar{\psi}(i\partial\!\!\!/)\psi - \delta_g \bar{\psi}(\phi^1 + i\gamma^5\phi^2)\psi. \quad (20)$$

Substituting, $\phi^1 = v + \sigma$ and $\phi^2 = \pi$ we obtain the counterterm Lagrangian

$$\begin{aligned}
\delta\mathcal{L} &= \frac{\delta_\phi}{2}(\partial_\mu \phi^i)^2 + \frac{\delta_\mu}{2}(\phi^i)^2 - \frac{\delta_\lambda}{4}((\phi^i)^2)^2 + \delta_f \bar{\psi}(i\partial\!\!\!/)\psi - \delta_g \bar{\psi}(\phi^1 + i\gamma^5\phi^2)\psi \\
&= \frac{\delta_\phi}{2}(\partial_\mu \sigma)^2 + \frac{\delta_\phi}{2}(\partial_\mu \pi)^2 + \frac{\delta_\mu}{2}\left(v^2 + 2v\sigma + \sigma^2 + \pi^2\right) \\
&\quad - \frac{\delta_\lambda}{4}\left(v^4 + 4v^2\sigma^2 + \sigma^4 + \pi^4 + 4v^3\sigma + 2v^2\sigma^2 + 2v^2\pi^2 + 4v\sigma^3 + 4v\sigma\pi^2 + 2\sigma^2\pi^2\right) \\
&\qquad + \delta_f \bar{\psi}(i\partial\!\!\!/)\psi - \delta_g \bar{\psi}(v + \sigma + i\gamma^5\pi)\psi \\
&= \frac{\delta_\phi}{2}(\partial_\mu \sigma)^2 + \frac{\delta_\phi}{2}(\partial_\mu \pi)^2 + \frac{\delta_\mu}{2}\left(v^2 + 2v\sigma + \sigma^2 + \pi^2\right) \\
&\quad - \frac{\delta_\lambda}{4}\left(v^4 + 4v^2\sigma^2 + \sigma^4 + \pi^4 + 4v^3\sigma + 2v^2\sigma^2 + 2v^2\pi^2 + 4v\sigma^3 + 4v\sigma\pi^2 + 2\sigma^2\pi^2\right) \\
&\qquad + \delta_f \bar{\psi}(i\partial\!\!\!/)\psi - \delta_g \bar{\psi}(v + \sigma + i\gamma^5\pi)\psi \\
&= \frac{\delta_\phi}{2}(\partial_\mu \sigma)^2 + \frac{\delta_\phi}{2}(\partial_\mu \pi)^2 + \delta_f \bar{\psi}(i\partial\!\!\!/)\psi + \left(v\delta_\mu - v^3\delta_\lambda\right)\sigma + \left(\frac{\delta_\mu}{2} - \frac{3v^2\delta_\lambda}{2}\right)\sigma^2 \\
&\quad + \left(\frac{\delta_\mu}{2} - \frac{v^2\delta_\lambda}{2}\right)\pi^2 - v\delta_g\bar{\psi}\psi - \delta_g\sigma\bar{\psi}\psi - v\delta_\lambda\sigma^3 - v\delta_\lambda\sigma\pi^2 - \delta_g\pi\bar{\psi}i\gamma^5\psi \\
&\qquad - \frac{\delta_\lambda}{2}\sigma^2\pi^2 - \frac{\delta_\lambda}{4}\sigma^4 - \frac{\delta_\lambda}{4}\pi^4 + \frac{v^2\delta_\mu}{2} - \frac{v^4\delta_\lambda}{4}.
\end{aligned} \quad (21)$$

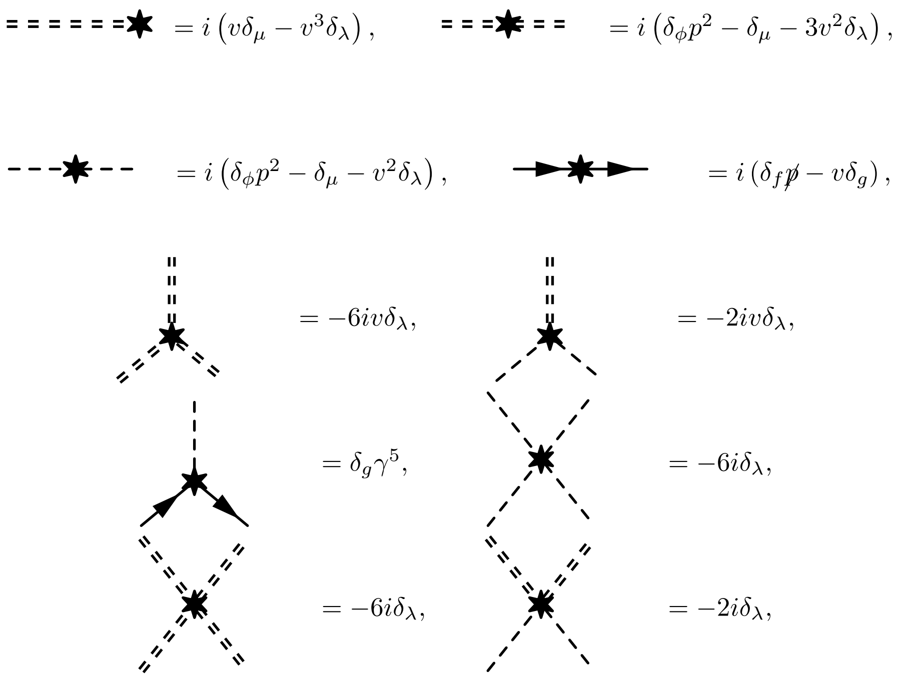
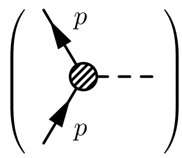


Figure 3: Feynman vertices for the counterterm Lagrangian (21).

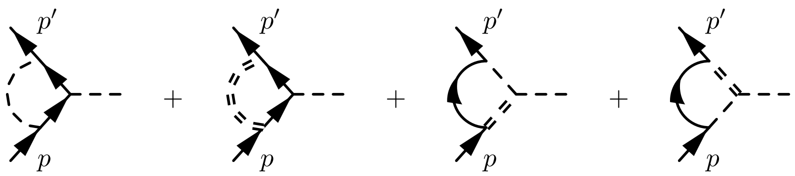
The corrections to the mass are given by $\delta m_f = \Sigma(\not{p})|_{p=m_f}$. Thus, we need to calculate δ_g and any radiative corrections to v. We are to use renormalization conditions such that g and v do not receive any radiative corrections. This fixes δ_g, which must satisfy the equation

$\bar{u}(p)$  $u(p) \quad = \quad g\bar{u}(p)\gamma^5 u(p).$ (22)

More specifically, δ_g is fixed by the condition

$$\bar{u}(p)\delta_g\gamma^5 u(p) = \bar{u}(p))V(p,p)\gamma^5 u(p) \tag{23}$$

where

$V(p,p')\gamma^5 \quad =$  . (24)

Evaluating the sum of the first two diagrams above we obtain

$$\begin{aligned} V_{12}(p,p') = &\int \frac{\mathrm{d}^d k}{(2\pi)^d}\frac{i}{k^2}(g\gamma^5)\frac{i(\not{p}' + \not{k} + m_f)}{(p'+k)^2 - m_f^2}(g\gamma^5)\frac{i(\not{p} + \not{k} + m_f)}{(p+k)^2 - m_f^2}(g\gamma^5)\gamma^5 \\ &+ \int \frac{\mathrm{d}^d k}{(2\pi)^d}\frac{i}{k^2 - 2\mu^2}(-ig)\frac{i(\not{p}' + \not{k} + m_f)}{(p'+k)^2 - m_f^2}(g\gamma^5)\frac{i(\not{p} + \not{k} + m_f)}{(p+k)^2 - m_f^2}(-ig)\gamma^5 \\ = &-ig^3\int \frac{\mathrm{d}^d k}{(2\pi)^d}\left(\frac{1}{k^2}\frac{(-\not{p}' - \not{k} + m_f)}{(p'+k)^2 - m_f^2}\frac{(\not{p} + \not{k} + m_f)}{(p+k)^2 - m_f^2} - \frac{1}{k^2 - 2\mu^2}\frac{(\not{p}' + \not{k} + m_f)}{(p'+k)^2 - m_f^2}\frac{(-\not{p} - \not{k} + m_f)}{(p+k)^2 - m_f^2}\right) \end{aligned} \tag{25}$$

Setting $p' = p$ and $p^2 = m_f^2$, V_{12} simplifies to

$$V(p,p) = -ig^3 \int \frac{\mathrm{d}^d k}{(2\pi)^d} \left(\frac{1}{k^2} - \frac{1}{k^2 - 2\mu^2} \right) \frac{1}{(p+k)^2 - m_f^2}. \tag{26}$$

To finish simplifying $V_{12}(p,p)$ we need to evaluate two integrals of the form

$$\begin{aligned}
I(M) &\equiv \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{1}{(k^2 - M^2)((p+k)^2 - m_f^2)} \\
&= \int \mathrm{d}x \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{1}{(k^2 - (1-x)M^2 + 2xp \cdot k + xp^2 - xm_f^2)^2} \\
&= \int \mathrm{d}x \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{1}{((k+xp)^2 - x^2p^2 - (1-x)M^2 + xp^2 - xm_f^2)^2} \\
&= \int \mathrm{d}x \int \frac{\mathrm{d}^d \ell}{(2\pi)^d} \frac{1}{(\ell^2 - \Delta)^2} \\
&= \int \mathrm{d}x \frac{i}{(4\pi)^{d/2}} \left(\frac{1}{\Delta} \right)^{2-d/2} \Gamma(2 - d/2) \\
&= \frac{i}{16\pi^2} \int \mathrm{d}x \left(\frac{4\pi}{\Delta} \right)^{2-d/2} \Gamma(2 - d/2)
\end{aligned} \tag{27}$$

where $\Delta(M) = x^2p^2 + (1-x)M^2 - xp^2 + xm_f^2 = x^2m_f^2 + (1-x)M^2$. Using I, we obtain

$$\begin{aligned}
V_{12}(p,p) &= -ig^3 \left(I(0) + I(\sqrt{2}\mu) \right) \\
&= \frac{g^3}{16\pi^2} \int \mathrm{d}x \left[1 + \epsilon \log\left(\frac{4\pi}{\Delta(0)} \right) - 1 - \epsilon \log\left(\frac{4\pi}{\Delta(\sqrt{2}\mu)} \right) + \mathcal{O}(\epsilon^2) \right] \left(\frac{1}{\epsilon} + \gamma_E + O(\epsilon) \right) \\
&= \frac{g^3}{16\pi^2} \int \mathrm{d}x \left[\log\left(\frac{4\pi}{\Delta(0)} \right) - \log\left(\frac{4\pi}{\Delta(\sqrt{2}\mu)} \right) + \mathcal{O}(\epsilon) \right] (1 + O(\epsilon)) \\
&= \frac{g^3}{16\pi^2} \int \mathrm{d}x \log\left(\frac{x^2m_f^2 + 2(1-x)\mu^2}{x^2m_f^2} \right).
\end{aligned} \tag{28}$$

Next we evaluate the last two diagrams in V,

$$\begin{aligned}
V_{34}(p,p') = \int \frac{\mathrm{d}^d k}{(2\pi)^d} \Bigg[& (g\gamma^5) \frac{i(\not{k} + m_f)}{k^2 - m_f^2} (-ig) \frac{i}{(k+p)^2 - 2\mu^2} (-2i\mu\sqrt{\lambda}) \frac{i}{(k+p')^2} \\
& (-ig) \frac{i(\not{k} + m_f)}{k^2 - m_f^2} (-ig\gamma^5) \frac{i}{(k+p)^2} (-2i\mu\sqrt{\lambda}) \frac{i}{(k+p')^2 - 2\mu^2} \Bigg] \gamma^5 \\
= 2g^2\mu\sqrt{\lambda} \int \frac{\mathrm{d}^d k}{(2\pi)^d} & \left[\frac{(-\not{k} + m_f)}{(k^2 - m_f^2)((k+p)^2 - 2\mu^2)(k+p')^2} + \frac{(\not{k} + m_f)}{(k^2 - m_f^2)(k+p)^2((k+p')^2 - 2\mu^2)} \right] \\
= 2g^2\mu\sqrt{\lambda}\Gamma(3) & \int \mathrm{d}y \int \mathrm{d}z \int \frac{\mathrm{d}^d \ell}{(2\pi)^d} \left[\frac{(-\not{\ell} + y\not{p} + z\not{p}' + m_f)}{(\ell^2 - \Delta_1)^3} + \frac{(\not{\ell} - y\not{p} - z\not{p}' + m_f)}{(\ell^2 - \Delta_2)^3} \right] \\
= 4g^2\mu\sqrt{\lambda}\Gamma(3) & \int \mathrm{d}y \int \mathrm{d}z \int \frac{\mathrm{d}^d \ell}{(2\pi)^d} \left[\frac{y\not{p} + z\not{p}' + m_f}{(\ell^2 - \Delta_1)^3} - \frac{y\not{p} + z\not{p}' - m_f}{(\ell^2 - \Delta_2)^3} \right]
\end{aligned} \tag{29}$$

where

$$\Delta_1 = -y(1-y)p^2 - z(1-z)p'^2 + 2yzp \cdot p' + (1-y-z)m_f^2 - 2y\mu^2, \tag{30}$$

$$\Delta_2 = -y(1-y)p^2 - z(1-z)p'^2 + 2yzp \cdot p' + (1-y-z)m_f^2 - 2z\mu^2. \tag{31}$$

Setting $p' = p$ and $p^2 = m_f^2$, V_{34} simplifies to

$$\begin{aligned}
V_{34}(p,p) &= 2g^2\mu\sqrt{\lambda}\Gamma(3)\int \mathrm{d}y \int \mathrm{d}z \int \frac{\mathrm{d}^d\ell}{(2\pi)^d}\left[\frac{(y+z)\not{p}+m_f}{(\ell^2-\Delta_1|_{p'=p})^3} - \frac{(y+z)\not{p}-m_f}{(\ell^2-\Delta_2|_{p'=p})^3}\right] \\
&= 2g^2\mu\sqrt{\lambda}\frac{i}{16\pi^2}\int \mathrm{d}y \int \mathrm{d}z \left[\frac{(y+z)\not{p}+m_f}{\Delta_1|_{p'=p}} - \frac{(y+z)\not{p}-m_f}{\Delta_2|_{p'=p}}\right] \\
&= 2g^2\mu\sqrt{\lambda}\frac{i}{16\pi^2}\int \mathrm{d}y \int \mathrm{d}z \left[\frac{(y+z)\not{p}+m_f}{\Delta_1|_{p'=p}} - \frac{(y+z)\not{p}-m_f}{\Delta_1|_{p'=p}}\right] \\
&= 4g^2\mu\sqrt{\lambda}\frac{i}{16\pi^2}\int_0^1 \mathrm{d}x \int_0^{1-x} \mathrm{d}y \frac{m_f}{(1-2x-2y+2xy+x^2+y^2)m_f^2 - 2x\mu^2} \\
&= 4g\lambda\frac{i}{16\pi^2}\int_0^1 \mathrm{d}x \int_0^{1-x} \mathrm{d}y \frac{m_f^2}{(1-2x-2y+2xy+x^2+y^2)m_f^2 - 2x\mu^2} \qquad (32)
\end{aligned}$$

The counterterm, δ_g, is then fixed to be

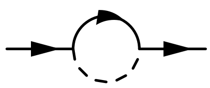
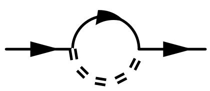
$$\begin{aligned}
\delta_g &= -V(p,p) \\
&= -V_{12}(p,p) - V_{34}(p,p) \\
&= -\frac{g}{16\pi^2}\int_0^1 \mathrm{d}x \Bigg[g^2 \log\left(\frac{x^2m_f^2 + 2(1-x)\mu^2}{x^2m_f^2}\right) \\
&\qquad + 4\lambda i\int_0^{1-x} \mathrm{d}y \frac{m_f^2}{(1-2x-2y+2xy+x^2+y^2)m_f^2 - 2x\mu^2}\Bigg] \qquad (33)
\end{aligned}$$

which is finite.

Next we calculate $\Sigma(\not{p})$. The self-energy is given by

$$-i\Sigma(\not{p}) = -i\Sigma'(\not{p}) \quad + \quad$$  , (34)

where

$$-i\Sigma'(\not{p}) =$$  +  . (35)

The primed self energy is

$$\begin{aligned}
-i\Sigma'(\not{p}) &= \int \frac{\mathrm{d}^dk}{(2\pi)^d}\left[(g\gamma^5)\frac{i(\not{k}+\not{p}+m_f)}{(k+p)^2-m_f^2}(g\gamma^5)\frac{i}{k^2} + (-ig)\frac{i(\not{k}+\not{p}+m_f)}{(k+p)^2-m_f^2}(-ig)\frac{i}{k^2-2\mu^2}\right] \\
&= -g^2\int \frac{\mathrm{d}^dk}{(2\pi)^d}\left[\frac{(-\not{k}-\not{p}+m_f)}{(k+p)^2-m_f^2}\frac{1}{k^2} - \frac{(\not{k}+\not{p}+m_f)}{(k+p)^2-m_f^2}\frac{1}{k^2-2\mu^2}\right] \\
&= -g^2\int \mathrm{d}x \int \frac{\mathrm{d}^dk}{(2\pi)^d}\left[\frac{(-\not{k}-\not{p}+m_f)}{(k^2+2xk\cdot p+xp^2-xm_f^2)^2} - \frac{(\not{k}+\not{p}+m_f)}{(k^2+2xk\cdot p+xp^2-xm_f^2-2(1-x)\mu^2)^2}\right] \\
&= -g^2\int \mathrm{d}x \int \frac{\mathrm{d}^dk}{(2\pi)^d}\left[\frac{(-\not{\ell}+x\not{p}-\not{p}+m_f)}{(\ell^2-x^2p^2+xp^2-xm_f^2)^2} - \frac{(\not{\ell}-x\not{p}+\not{p}+m_f)}{(\ell^2-x^2p^2+xp^2-xm_f^2-2(1-x)\mu^2)^2}\right] \\
&= -g^2\int \mathrm{d}x \int \frac{\mathrm{d}^dk}{(2\pi)^d}\left[\frac{(-(1-x)\not{p}+m_f)}{(\ell^2-\Delta_1)^2} - \frac{((1-x)\not{p}+m_f)}{(\ell^2-\Delta_2)^2}\right] \qquad (36)
\end{aligned}$$

where $\Delta_1 = -x(1-x)p^2 + xm_f^2$ and $\Delta_2 = \Delta_1 + 2(1-x)\mu^2$. Evaluating the momentum integrals we obtain

$$\begin{aligned} -i\Sigma'(\not{p}) &= -g^2 \frac{i}{(4\pi)^2} \int \mathrm{d}x \left[(-(1-x)\not{p} + m_f) \left(\frac{4\pi}{\Delta_1} \right)^{\epsilon} - ((1-x)\not{p} + m_f) \left(\frac{4\pi}{\Delta_2} \right)^{\epsilon} \right] \Gamma(\epsilon) \\ &= -g^2 \frac{i}{(4\pi)^2} \int \mathrm{d}x \left[\frac{-2(1-x)\not{p}}{\epsilon} + m_f \log\left(\frac{\Delta_2}{\Delta_1} \right) - (1-x)\not{p} \log\left(\frac{4\pi e^{-\gamma_E}}{\Delta_2} \frac{4\pi e^{-\gamma_E}}{\Delta_1} \right) \right] \end{aligned} \tag{37}$$

Problem 12.3: Gross-Neveu model.

The Gross-Neveu model is a model in two spacetime dimensions of fermions with a discrete chiral symmetry:

$$\mathcal{L} = \bar{\psi}_i i\partial\!\!\!/ \psi_i + \frac{1}{2}g^2(\bar{\psi}_i\psi_i)^2 \tag{38}$$

with $i = 1, \ldots, N$. The kinetic term of tow-dimensional fermions is built form matrices γ^μ that satisfy the two-dimensional Dirac algebra. These matrices can be 2×2:

$$\gamma^0 = \sigma^2, \quad \gamma^1 = i\sigma^1, \tag{39}$$

where σ^i are the Pauli matrices. Define

$$\gamma^5 = \gamma^0\gamma^1 = \sigma^3; \tag{40}$$

this matrix anti commutes with the γ^μ.

(a)

(b)

(c)

(d)

(e)

(f)