Problem 15.1: Brute-force computations in $SU(3)$.

The standard basis for the fundamental representation of $SU(3)$ is

$$
\begin{aligned}
t^1 &= \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, & t^2 &= \frac{1}{2}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\
t^3 &= \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, & t^4 &= \frac{1}{2}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \\
t^5 &= \frac{1}{2}\begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, & t^6 &= \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \\
t^7 &= \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, & t^8 &= \frac{1}{2\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.
\end{aligned}
\tag{1}
$$

a) Explain why there are exactly eight matrices in the basis.

b) Evaluate all the commutators of these matrices, to determine the structure constants of $SU(3)$. Show that, with the normalization used here, f^{abc} is totally antisymmetric. (This exercise is tedious; you may wish to check only a representative sample of commutators.)

c) Check the orthogonality condition (15.78), and evaluate that constant $C(r)$ for this representation.

d) Compute the quadratic Casimir operator $C_2(r)$ directly form its definition (15.92), and verify the relation (15.94) between $C_2(r)$ and $C(r)$.

Part (a)

The group $SU(N)$ is a subgroup of $U(N)$. Specifically, $SU(N)$ is equal to $U(N)$ with $U(1)$ removed. Since the generators of $SU(N)$ must be orthogonal to the $U(1)$ generator (which is proportional to the identity matrix), the $SU(N)$ generators must satisfy $\mathrm{tr}\,[t^a] = 0$. In general, the number of $N \times N$ traceless matrices is $N^2 - 1$. Thus, for $SU\,(N = 3)$, the number of traceless 3×3 matrices is $9 - 1 = 8$.

Part (b)

Using a simple Mathematica script is is easy to generate the all the commutators of the Gell-Mann matrices

$$
\left[t^a, t^b\right] = i \begin{pmatrix}
0 & t^3 & -t^2 & t^7/2 & -t^6/2 & t^5/2 & t^4/2 & 0 \\
 & 0 & t^1 & t^6/2 & t^7/2 & -t^4/2 & -t^5/2 & 0 \\
 & & 0 & t^5/2 & -t^4/2 & -t^7/2 & t^6/2 & 0 \\
 & & & 0 & \left(\sqrt{3}t^8 + t^3\right)/2 & t^2/2 & t^1/2 & -\sqrt{3}t^5/4 \\
 & & & & 0 & -t^1/2 & t^2/2 & \sqrt{3}t^4/4 \\
 & & & & & 0 & \left(\sqrt{3}t^8 - t^3\right)/2 & -\sqrt{3}t^7/4 \\
 & & & & & & 0 & \sqrt{3}t^6/4 \\
 & & & & & & & 0
\end{pmatrix}
$$

where we only show the upper triangle since the commutator is antisymmetric. The generators t^1, t^2, t^3 form an $SU(2)$ subgroup of $SU(3)$

$$\left[t^a, t^b\right] = i\epsilon^{abc} t^c \text{ for } a, b, c \in \{1, 2, 3\}.$$

The structure constant is

$$\begin{aligned} f^{abc} &= -\frac{i}{C(F)} \text{tr}\left[\left[t^a, t^b\right] t^c\right] \\ &= \frac{\epsilon^{abd}}{C(F)} \text{tr}\left[t^d t^c\right] \\ &= \frac{\epsilon^{abd}}{C(F)} C(F)\, \delta^{dc} \\ &= \epsilon^{abc} \end{aligned}$$

as expected for $SU(2)$. Since the generators are orthogonal, setting $a, b \in \{1, 2, 3\}$ and $c = 4, 5, 6, 7, 8$ results in $f^{abc} = 0$. One can easily prove that the remaining structure constants are

$$f^{147} = -f^{156} = f^{246} = f^{257} = f^{345} = -f^{357} = \frac{1}{2}$$

$$f^{458} = f^{678} = \frac{\sqrt{3}}{2}$$

with all others not related to these by permutations zero. Some examples:

$$\begin{aligned} \left[t^1, t^4\right] &= \frac{1}{2} t^7 &&\implies \text{tr}\left[\left[t^1, t^4\right] t^c\right] = \frac{C(F)}{2} \delta^{c7} &&\implies f^{147} = -f^{417} = \frac{1}{2} \\ \left[t^4, t^7\right] &= \frac{1}{2} t^1 &&\implies \text{tr}\left[\left[t^4, t^7\right] t^c\right] = \frac{C(F)}{2} \delta^{c1} &&\implies f^{471} = -f^{741} = \frac{1}{2} \\ \left[t^1, t^7\right] &= \frac{1}{2} t^4 &&\implies \text{tr}\left[\left[t^1, t^7\right] t^c\right] = \frac{C(F)}{2} \delta^{c4} &&\implies f^{174} = -f^{714} = \frac{1}{2} \end{aligned}$$

Part (c)

One can easily see that the inner product of these matrices satisfy

$$\text{tr}\left[t^a t^b\right] = \frac{1}{2} \delta^{ab},$$

which fixes $C(F) = \frac{1}{2}$.

Part (d)

Using Mathematica,

$$\sum_{a=1}^{8} t^a t^a = \frac{1}{3}$$

which fixes the quadratic Casimir $C_2(F) = \frac{1}{3}$.

Problem 15.2: Adjoint representation of $SU(2)$.

Write down the basis matrices of the adjoint representation of $SU(2)$. Compute $C(G)$ and $C_2(G)$ directly from their definitions (15.78) and (15.92).

The basis matrices for the adjoint representation are defined by the structure constants

$$\left(t_G^b\right)_{ac} = if^{abc} = i\epsilon^{abc}$$

where $a, b, c \in \{1, 2, 3\}$. Explicitly,

$$t_G^1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}$$
$$t_G^2 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}$$
$$t_G^3 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

These satisfy the orthogonality relation

$$\mathrm{tr}\left[t_G^a t_G^b\right] = C(G)\,\delta^{ab}$$

with $\mathrm{tr}\left[\left(t_b^a\right)^2\right] = 2$. The quadratic Casimir is

$$\sum_{i=1}^{3} t_G^a t_G^a = C_2(G)\,\mathbf{1}$$

where $C_2(G) = 2$.

Problem 15.3: Coulomb potential.

a) Using functional integration, compute the expectation value of the Wilson loop in pure quantum electrodynamics without fermions. Show that

$$\langle U_P(z, z)\rangle = \exp\left\{-ie^2 \oint_P dx^\mu \oint_P y^\nu g_{\mu\nu} \frac{1}{2\pi^2 (x-y)^2}\right\}$$

with x and y integrate around the closed curve P.

b) Consider the Wilson loop of a rectangular path of (spacelike) width R and (timelike) length T, $T \gg R$. Compute the expectation value of the Wilson loop in this limit and compare to the general expression for the time evolution

$$\langle U_P\rangle = \exp\{-iE(R)T\},$$

where $E(R)$ is the energy of the electromagnetic sources corresponding to the Wilson loop. Show that the potential energy of these sources is just the Coulomb potential, $V(R) = -e^2/4\pi R$.

c) Assuming that the propagator of the non-Abelian gauge field is given by the Feynman gauge expression

$$\left\langle A_\mu^a(x)\, A_\nu^b(y)\right\rangle = \int \frac{d^4p}{(2\pi)^4}\frac{-ig_{\mu\nu}\delta^{ab}}{p^2}e^{-ip\cdot(x-y)},$$

compute the expectation value of a non-Abelian Wilson loop to order g^2. The result will depend on the representation r of the gauge group in which one chooses the matrices that appear in the exponential. Show that, to this order, the Coulomb potential of the non-Abelian gauge theory is $V(R) = -g^2C_2(r)/4\pi R$.

Part (a)

The Wilson line for an Abelian gauge group is defined to be

$$U_P(x,y) = \exp\left\{-ie\int_P dz^\mu A_\mu(z)\right\}$$

where the path P runs from x to y. To get the Wilson loop, we simply set $y = x$

$$U_P(x,x) = \exp\left\{-ie\oint_P dy^\mu A_\mu(y)\right\}.$$

To get the expectation value of the Wilson loop we integrate over all field configurations A_μ weighted by the pure Maxwell action

$$\langle U_P(x,x)\rangle = \frac{\int \mathcal{D}A\,\exp\left\{iS_{\text{Maxwell}}[A] - ie\oint_P dy^\mu A_\mu(y)\right\}}{\int \mathcal{D}A\,\exp\left\{iS_{\text{Maxwell}}[A]\right\}}.$$

Here, the Maxwell action is

$$\begin{aligned} S_{\text{Maxwell}} &= \int d^4y\,\left(-\frac{1}{4}F^2 - \frac{1}{2\xi}(\partial^\mu A_\mu)^2\right)\\ &= \int d^4y\,\left[-\frac{1}{2}A_\mu\left(-g^{\mu\nu}\partial^2 + \left(1-\frac{1}{\xi}\right)\partial^\mu\partial^\nu\right)A_\nu\right] \end{aligned}$$

where we have included the gauge fixing piece in the action.

$$\begin{aligned}\langle U_P(x,x)\rangle &= \frac{\int \mathcal{D}A\,\exp\left\{iS_{\text{Maxwell}}[A] - ie\oint_P dy^\mu A_\mu(y)\right\}}{\int \mathcal{D}A\,\exp\left\{iS_{\text{Maxwell}}[A]\right\}}\\ &= \frac{\int \mathcal{D}A\,\exp\left\{-i\int d^4y\,\frac{1}{2}A_\mu(y)\,\Box_\xi^{\mu\nu}A_\nu(y) - ie\oint_P dy^\mu A_\mu(y)\right\}}{\int \mathcal{D}A\,\exp\left\{iS_{\text{Maxwell}}[A]\right\}}\end{aligned}$$

where

$$\Box_\xi^{\mu\nu} = -g^{\mu\nu}\partial^2 + \left(1-\frac{1}{\xi}\right)\partial^\mu\partial^\nu.$$

To perform the functional integration, we must put $\langle U_P(x,x)\rangle$ into a Gaussian form. We rewrite the line integral as an integral over all space as

$$\oint_P dy^\mu A_\mu(y) = \int d^4y\; J^\mu(y)\,A_\mu(y)$$

where $J^{\mu}(y)$ is a unit vector that restricts the integral over all space to that over the path P. Then $\langle U_P(x,x)\rangle$ becomes

$$\langle U_P(x,x)\rangle = \frac{\int \mathcal{D}A \, \exp\left\{-i\int d^4y \, \frac{1}{2}A_\mu(y)\,\Box_\xi^{\mu\nu}A_\nu(y) - ie\int d^4y \, J^\mu(y)\,A_\mu(y)\right\}}{\int \mathcal{D}A \, \exp\left\{iS_{\text{Maxwell}}[A]\right\}}$$

which is a Gaussian integral. Notice that J is basically an external delta source. Evaluating the path integral, we obtain

$$\begin{aligned}\langle U_P(x,x)\rangle &= \frac{\int \mathcal{D}A \, \exp\left\{-i\int d^4y \, \frac{1}{2}A_\mu(y)\,\Box_\xi^{\mu\nu}A_\nu(y) - ie\int d^4y \, J^\mu(y)\,A_\mu(y)\right\}}{\int \mathcal{D}A \, \exp\left\{iS_{\text{Maxwell}}[A]\right\}} \\ &= \exp\left\{\frac{1}{2}(-ie)^2\int d^4y \, d^4z \, J_\mu(y)\,G_\xi^{\mu\nu}(y,z)\,J_\nu(z)\right\} \\ &= \exp\left\{-\frac{e^2}{2}\oint_P dy^\mu \oint_P dz^\nu \, G_{\xi\mu\nu}(y,z)\right\}\end{aligned}$$

where $G_\xi^{\mu\nu}(x,y)$ is the Greens function of $\Box_\xi^{\mu\nu}$

$$G_\xi^{\mu\nu}(x,y) = \int \frac{d^4k}{(2\pi)^4}\frac{-i}{k^2+i\epsilon}\left(g^{\mu\nu} - (1-\xi)\frac{k^\mu k^\nu}{k^2}\right)e^{-ik\cdot(x-y)}.$$

To calculate the position space representation of the Green's function, we rewrite the denominator in the integral as

$$\int_0^\infty ds \, e^{is\left(k^2+i\epsilon\right)} = \frac{\lim_{s\to\infty} e^{is\left(k^2+i\epsilon\right)} - 1}{i\left(k^2+i\epsilon\right)} = \frac{i}{k^2+i\epsilon},$$

so that

$$G_\xi^{\mu\nu}(x,y) = -i\int_0^\infty ds \int \frac{d^4k}{(2\pi)^4}\left(g^{\mu\nu} - (1-\xi)\frac{k^\mu k^\nu}{k^2}\right)e^{-ik\cdot(x-y)}e^{is\left(k^2+i\epsilon\right)}.$$

Setting $\xi = 1$ and defining $\zeta = x - y$, the Green's function becomes

$$\begin{aligned}
G^{\mu\nu}_{\xi=1}(x,y) &= -ig^{\mu\nu}\int \frac{d^4k}{(2\pi)^4}\frac{e^{-ik\cdot(x-y)}}{k^2+i\epsilon} \\
&= -ig^{\mu\nu}\int_0^\infty ds \int \frac{d^4k}{(2\pi)^4} e^{is\left(k^2+i\epsilon\right)-ik\cdot(x-y)} \\
&= -ig^{\mu\nu}\int_0^\infty ds \int \frac{d^4k}{(2\pi)^4} e^{is(k-\zeta/2s)^2-i\zeta^2/4s-\epsilon s} \\
&= -ig^{\mu\nu}\int_0^\infty ds\, e^{-i\zeta^2/4s-\epsilon s}\int \frac{d^4\tilde{k}}{(2\pi)^4} e^{is\tilde{k}^2} \\
&= -ig^{\mu\nu}\int_0^\infty ds\, e^{-i\zeta^2/4s-\epsilon s}\left(\int \frac{d\tilde{k}^0}{2\pi}e^{-is\left(\tilde{k}^0\right)^2}\right)\left(\prod_{i=1}^3\int\frac{d\tilde{k}^i}{2\pi}e^{is\left(\tilde{k}^i\right)^2}\right) \\
&= \frac{-ig^{\mu\nu}}{(2\pi)^4}\int_0^\infty ds\, e^{-i\zeta^2/4s-\epsilon s}\left(\frac{\pi}{is}\right)^{1/2}\left(\frac{\pi}{-is}\right)^{3/2} \\
&= \frac{g^{\mu\nu}}{(4\pi)^2}\int_0^\infty ds\,\frac{e^{-i\zeta^2/4s-\epsilon s}}{s^2} \\
&= \frac{g^{\mu\nu}}{(4\pi)^2}\int_0^\infty dt\, e^{-it\zeta^2/4-\epsilon/t} \\
&= \frac{g^{\mu\nu}}{(4\pi)^2}\int_0^\infty dt\, e^{it\left(-\zeta^2+i\epsilon\right)/4} \\
&= \frac{g^{\mu\nu}}{(4\pi)^2}\frac{4i}{\zeta^2-i\epsilon} \\
&= \frac{i}{4\pi^2}\frac{g^{\mu\nu}}{\zeta^2-i\epsilon}
\end{aligned}$$

where we have set $\epsilon \to t\epsilon/4$ in the third last line since the ϵ is only there to tell us how to deform the contour, only its sign matters. Substituting this expression for the Green's function into the expression for $\langle U_P(x,x)\rangle$, we obtain

$$\langle U_P(x,x)\rangle_{\xi=1} = \exp\left\{-\frac{e^2}{8\pi^2}\oint_P dy^\mu\oint_P dz^\nu\frac{g_{\mu\nu}}{(y-z)^2-i\epsilon}\right\}.$$

Part (b)

This part of the problem asks us to perform the path integration in the exponential of $\langle U_P(x,x)\rangle$ for a rectangular path of spacelike width R and timelike length T where $T \gg R$. A convenient parameterization for this path is

$$(0,0,0,0) \to (T,0,0,0) \to (T,R,0,0) \to (0,R,0,0) \to (0,0,0,0)\,.$$

That is we sit at the origin for time T, move in the 1-direction a distance R, move backwards in time for a length T and then move back a distance R to the origin. In the limit $T \gg R$ the largest contributions come from the timelike paths. We also ignore the self-energy contributions – when the y and z paths end and begin at the same points. Thus,

$$\begin{aligned}
\oint_P dy^\mu\oint_P dz^\nu\frac{g_{\mu\nu}}{(y-z)^2} &\sim 2\int_T^0 dz^0\int_0^T dy^0\frac{g_{00}}{\left(y^0-z^0\right)^2-R^2-i\epsilon} \\
&= -2\int_{-T/2}^{T/2}dz\int_{-T/2}^{T/2}dy\frac{g_{00}}{(y-z)^2-R^2-i\epsilon}.
\end{aligned}$$

Taking the limit $T \to \infty$ this is approximately

$$\begin{aligned}\oint_P dy^\mu \oint_P dz^\nu \frac{g_{\mu\nu}}{(y-z)^2} &\sim -2T \int_{-\infty}^{\infty} dy \frac{1}{y^2 - R^2 - i\epsilon} \\ &= -2T \int_{-\infty}^{\infty} dy \frac{1}{(y - R - i\epsilon)(y + R + i\epsilon)} \\ &= -2\pi i \frac{T}{R}.\end{aligned}$$

Thus, the expectation value of the Wilson loop becomes

$$\begin{aligned}\langle U_P(x,x)\rangle &= \exp\left\{-\frac{e^2}{8\pi^2}\oint_P dy^\mu \oint_P dz^\nu \frac{g_{\mu\nu}}{(y-z)^2}\right\} \\ &= \exp\left\{-\frac{ie^2}{4\pi R}T\right\} \\ &= \exp\{-iV(R)T\}\end{aligned}$$

where

$$V(R) = \frac{e^2}{4\pi R}$$

is the Coulomb potential.

Some comments: (it would be nice to work these out)

- The divergences from when $x \to y$ are of the form $e^{-iT\Lambda}$ and renormalize the probe particle Lagrangian by a linear divergence Λ (remember that a Wilson line is generated by a heavy probe quark). This divergence is present because we are treating the probe particle as a classical point particle – the divergence comes from the energy contained in the electric and magnetic fields of the classical point particle.
- At the cusps of the rectangle, there is an IR divergence which depends on the angle that the Wilson lines make at the cusp. This divergence should reproduce the IR divergence calculated in chapter 6 of Peskin.

Part (c)

The non-Abelian Wilson line is

$$U_P(x,y) = P\exp\left\{ig\int_0^1 ds\, \frac{dx^\mu}{ds} A_\mu^a(x(s))\, t^a\right\}$$

while the non-Abelian Wilson loop is

$$U_P(x,x) = \mathrm{tr}\left[P\exp\left\{ig\int_0^1 ds\, \frac{dx^\mu}{ds} A_\mu^a(x(s))\, t^a\right\}\right].$$

Due to the path ordering, the above it is hard to calculate the expectation value of the Wilson loop. However, we only need the result to lowest order in g to extract the potential. To see how this works, note that the potential is the order e^2 term of the Abelian Wilson loop expectation value is

$$\langle U_P(x,x)\rangle \approx 1 + -i\left(\frac{e^2}{4\pi R}\right)T + \mathcal{O}\left(e^4\right) = 1 + -iV(R)T + \mathcal{O}\left(e^4\right).$$

That is the potential is proportional to the expectation value of the $\mathcal{O}\left(e^2\right)$ term of $U_P(x,x)$. That is,

$$V(R) = \frac{i}{T}\left\langle -e^2 \oint_P dy^\mu \oint_P dz^\nu A_\mu(y)\, A_\nu(z)\right\rangle.$$

In the non-Abelian case we have

$$V(R) = \frac{i}{T}\left\langle \mathrm{tr}\left[P\left\{ig\int_0^1 ds\,\frac{dy^\mu}{ds}A_\mu^a(y(s))\,t^a ig\int_0^1 dt\,\frac{dz^\mu}{dt}A_\mu^b(z(t))\,t^b\right\}\right]\right\rangle$$

The non-Abelian case is the same as the Abelian case except with $A_\mu \to A_\mu^a t^a$. Since the propagator at lowest order in g is

$$\left\langle 0\left|TA_\mu^a(x)\,A_\nu^b(y)\right|0\right\rangle = G_{\mu\nu}(x,y)\,\delta^{ab},$$

the exponent is proportional to $\delta^{ab}t^a t^b = C_2$. Therefore, we can simply replace e^2 in part (b) by $g^2 C_2$ to get

$$V(R) = -\frac{g^2 C_2}{4\pi R}.$$