An Introduction to Quantum Field Theory (Peskin and Schroeder) Solutions

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December 12, 2017

Contents

1 The Dirac Equation

1.1 Lorentz group ✓

The Lorentz commutation relations are

$$[J^{\mu\nu}, J^{\rho\sigma}] = i\left(g^{\nu\rho}J^{\mu\sigma} - g^{\mu\rho}J^{\nu\sigma} - g^{\nu\sigma}J^{\mu\rho} + g^{\mu\sigma}J^{\nu\rho}\right).$$

(a) Define the generators of rotations and boots as

$$L^i = \frac{1}{2}\epsilon^{ijk}J^{jk} \text{ and } K^i = J^{0i},$$

where ijk is a permutation of (123). An infinitesimal Lorentz transformation can then be written

$$\Phi \to \left(1 - i\boldsymbol{\theta}\cdot\mathbf{L} - i\boldsymbol{\beta}\cdot\mathbf{K}\right)\Phi.$$

Write the commutation relations of these vector operators explicitly. Show that the combinations

$$\mathbf{J}_+ = \frac{1}{2}\left(\mathbf{L} + i\mathbf{K}\right) \text{ and } \mathbf{J}_- = \frac{1}{2}\left(\mathbf{L} - i\mathbf{K}\right)$$

commute with one another and separately satisfy the combination relation of angular momentum.

Proof: The L operator commutation relations are

$$\begin{aligned}
\left[L^i, L^j\right] &= L^i L^j - L^j L^i \\
&= \frac{1}{4}\left[\epsilon^{ikl} J^{kl}, \epsilon^{jmn} J^{mn}\right] \\
&= \frac{1}{4}\epsilon^{ikl}\epsilon^{jmn}\left[J^{kl}, J^{mn}\right] \\
&= \frac{i}{4}\epsilon^{ikl}\epsilon^{jmn}\left[g^{lm}J^{kn} - g^{km}J^{ln} - g^{ln}J^{km} + g^{kn}J^{lm}\right] \\
&= \frac{i}{4}\left[\epsilon^{ikl}\epsilon^{jmn}g^{lm}J^{kn} - \epsilon^{ikl}\epsilon^{jmn}g^{km}J^{ln} - \epsilon^{ikl}\epsilon^{jmn}g^{ln}J^{km} + \epsilon^{ikl}\epsilon^{jmn}g^{kn}J^{lm}\right] \\
& \text{for the second and last term } k \leftrightarrow l \\
&= \frac{i}{4}\left[\epsilon^{ikl}\epsilon^{jmn}g^{lm}J^{kn} - \epsilon^{ilk}\epsilon^{jmn}g^{lm}J^{kn} - \epsilon^{ikl}\epsilon^{jmn}g^{ln}J^{km} + \epsilon^{ilk}\epsilon^{jmn}g^{ln}J^{km}\right] \\
&= \frac{i}{4}\left[\epsilon^{ikl}\epsilon^{jmn}g^{lm}J^{kn} + \epsilon^{ikl}\epsilon^{jmn}g^{lm}J^{kn} - \epsilon^{ikl}\epsilon^{jmn}g^{ln}J^{km} - \epsilon^{ikl}\epsilon^{jmn}g^{ln}J^{km}\right] \\
& \text{for the third and last term } m \leftrightarrow n \\
&= \frac{i}{4}\left[\epsilon^{ikl}\epsilon^{jmn}g^{lm}J^{kn} + \epsilon^{ikl}\epsilon^{jmn}g^{lm}J^{kn} - \epsilon^{ikl}\epsilon^{jnm}g^{lm}J^{kn} - \epsilon^{ikl}\epsilon^{jnm}g^{lm}J^{kn}\right] \\
&= \frac{i}{4}\left[\epsilon^{ikl}\epsilon^{jmn}g^{lm}J^{kn} + \epsilon^{ikl}\epsilon^{jmn}g^{lm}J^{kn} + \epsilon^{ikl}\epsilon^{jmn}g^{lm}J^{kn} + \epsilon^{ikl}\epsilon^{jmn}g^{lm}J^{kn}\right] \\
&= i\epsilon^{ikl}\epsilon^{jmn}g^{lm}J^{kn} \\
&= i\epsilon^{ikl}\epsilon^{jmn}\left(-\delta^{lm}\right)J^{kn} \\
&= -i\epsilon^{ikl}\epsilon^{jln}J^{kn} \\
&= i\epsilon^{ikl}\epsilon^{jnl}J^{kn} \\
&= i\left(\delta^{ij}\delta^{kn} - \delta^{in}\delta^{kj}\right)J^{kn} \\
&= i\left(\delta^{ij}J^{kk} - J^{ji}\right) \\
&= iJ^{ij} \\
&= i\epsilon^{ijk}L^k
\end{aligned}$$

where

$$\begin{aligned}
J^{kk}_{\alpha\beta} &= i\left(\delta^k_\alpha\delta^k_\beta - \delta^k_\beta\delta^k_\alpha\right) \\
&= 0
\end{aligned}$$

and

$$\begin{aligned}
\epsilon^{ijk}L^k &= \frac{1}{2}\epsilon^{ijk}\epsilon^{klm}J^{lm} \\
&= \frac{1}{2}\epsilon^{kij}\epsilon^{klm}J^{lm} \\
&= \frac{1}{2}\left(\delta^{il}\delta^{jm} - \delta^{im}\delta^{jl}\right)J^{lm} \\
&= \frac{1}{2}\left(J^{ij} - J^{ji}\right) \\
&= J^{ij}.
\end{aligned}$$

The K commutation relations are

$$\begin{aligned}
\left[K^i, K^j\right] &= K^i K^j - K^j K^i \\
&= J^{0i} J^{0j} - J^{0j} J^{0i} \\
&= i\left(g^{i0} J^{0j} - g^{00} J^{ij} - g^{ij} J^{00} + g^{0j} J^{i0}\right).
\end{aligned}$$

This is simplified using properties of the metric $g^{i0} = 0$, $g^{00} = -1$, $g^{ij} = -1$ and the generators $J^{00} = 0$

$$\begin{aligned}
\left[K^i, K^j\right] &= -iJ^{ij} \\
&= -i\epsilon^{ijk} L^k.
\end{aligned}$$

Next, we need

$$\begin{aligned}
\left[L^i, K^j\right] &= \frac{1}{2}\epsilon^{ikl}\left[J^{kl}, J^{0j}\right] \\
&= \frac{i}{2}\epsilon^{ikl}\left(-g^{lj} J^{k0} + g^{kj} J^{l0}\right) \\
&= \frac{i}{2}\epsilon^{ikl}\left(\delta^{lj} J^{k0} - \delta^{kj} J^{l0}\right) \\
&= \frac{i}{2}\left(\epsilon^{ikj} J^{k0} - \epsilon^{ijl} J^{l0}\right) \\
&= \frac{i}{2}\left(\epsilon^{ijk} J^{0k} + \epsilon^{ijk} J^{0k}\right) \\
&= i\epsilon^{ijl} K^l.
\end{aligned}$$

Lastly we compute the angular momentum commutators

$$\begin{aligned}
\left[\mathbf{J}_+, \mathbf{J}_-\right] &= \left[\frac{1}{2}\left(\mathbf{L} + i\mathbf{K}\right), \frac{1}{2}\left(\mathbf{L} - i\mathbf{K}\right)\right] \\
&= \frac{1}{4}\left\{\left[\mathbf{L}, \mathbf{L}\right] - i\left[\mathbf{L}, \mathbf{K}\right] + i\left[\mathbf{K}, \mathbf{L}\right] + \left[\mathbf{K}, \mathbf{K}\right]\right\} \\
&= \frac{i}{2}\left[\mathbf{K}, \mathbf{L}\right] \\
&= \frac{i}{2}\left[K^i, L^j\right] \hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_j \\
&= \frac{i}{2}\left[K^i, L^i\right] \\
&= 0
\end{aligned}$$

$$\begin{aligned}
\left[J^i_\pm, J^j_\pm\right] &= \left[\frac{1}{2}\left(L^i \pm iK^i\right), \frac{1}{2}\left(L^j \pm iK^j\right)\right] \\
&= \frac{1}{4}\left\{\left[L^i, L^j\right] \pm i\left[L^i, K^j\right] \pm i\left[K^i, L^j\right] - \left[K^i, K^j\right]\right\} \\
&= \frac{1}{4}\left\{i\epsilon^{ijk} L^k \pm i\left(i\epsilon^{ijl} K^l\right) \mp i\left(i\epsilon^{jil} K^l\right) + i\epsilon^{ijk} L^k\right\} \\
&= \left\{i\epsilon^{ijk} L^k \mp \epsilon^{ijl} K^l \pm \epsilon^{ijl} K^l + i\epsilon^{ijk} L^k\right\} \\
&= \frac{1}{2}\left\{i\epsilon^{ijk} L^k \mp \epsilon^{ijl} K^l\right\} \\
&= \frac{i}{2}\epsilon^{ijk}\left\{L^k \pm iK^k\right\} \\
&= \frac{i}{2}\epsilon^{ijk} J^k_\pm
\end{aligned}$$

(b) The finite-dimensional representations of the rotation group correspond precisely the to the allowed values for angular momentum: integers or half-integers. The result of part (a) implies that all finite-dimensional representations of the Lorentz group correspond to pairs of integers or half integers, (j_+, j_-), corresponding to pairs of representations of the rotation group. Using the fact that $\mathbf{J} = \boldsymbol{\sigma}/2$ in the spin-1/2 representation of angular momentum, write explicitly the transformation laws of the 2-component objects transforming according to the $\left(\frac{1}{2}, 0\right)$ and $\left(0, \frac{1}{2}\right)$ representations of the Lorentz group. Show that these correspond precisely to the transformations of ψ_L and ψ_R giving in (3.37).

Proof: The representations of the Lorentz group are denoted by $(m, n) \equiv \pi_{m,n}$ where m, n are either half-integers or integers. The irreducible representations are given by

$$\begin{aligned}\pi_{m,n}\left(L^i\right) &= \mathbb{I}_{(2m+1)} \otimes J_i^{(n)} + J_i^{(m)} \otimes \mathbb{I}_{(2n+1)} \\ \pi_{m,n}\left(K^i\right) &= i\left(\mathbb{I}_{(2m+1)} \otimes J_i^{(n)} - J_i^{(m)} \otimes \mathbb{I}_{(2n+1)}\right).\end{aligned}$$

With $\mathbf{J} = \boldsymbol{\sigma}/2$ the $\left(\frac{1}{2}, 0\right)$ representation is found to be

$$\begin{aligned}\pi_{m,n}\left(L^i\right) &= \mathbb{I}_2 \otimes \mathbb{I}_1 + \frac{\sigma^i}{2} \otimes \mathbb{I}_1 = \left(\mathbb{I}_2 + \frac{\sigma^i}{2}\right) \otimes \mathbb{I}_1 = \frac{\sigma^i}{2} \otimes \mathbb{I}_1 = \frac{\sigma^i}{2} \\ \pi_{m,n}\left(K^i\right) &= i\left(\mathbb{I}_2 \otimes \mathbb{I}_1 - \frac{\sigma^i}{2} \otimes \mathbb{I}_1\right) = i\left(\mathbb{I}_2 - \frac{\sigma^i}{2}\right) \otimes \mathbb{I}_1 = -i\frac{\sigma^i}{2} \otimes \mathbb{I}_1 = -i\frac{\sigma^i}{2}.\end{aligned}$$

Thus, with $L^i = \sigma^i/2$ and $K^i = -i\sigma^i/2$ the transformation law becomes

$$\begin{aligned}\mathbf{\Phi}_{\left(\frac{1}{2},0\right)} &\rightarrow e^{-\frac{i}{2}\left(\omega_{\mu\nu} J^{\mu\nu}\right)} \mathbf{\Phi}_{\left(\frac{1}{2},0\right)} \\ &= \left(1 - \frac{i}{2}\omega_{\mu\nu} J^{\mu\nu}\right) \mathbf{\Phi}_{\left(\frac{1}{2},0\right)} \\ &= \left(1 - \frac{i}{2}\omega_{0\nu} J^{0\nu} - \frac{i}{2}\omega_{i\nu} J^{i\nu}\right) \mathbf{\Phi}_{\left(\frac{1}{2},0\right)} \\ &= \left(1 - \frac{i}{2}\omega_{00} J^{00} - \frac{i}{2}\omega_{0i} J^{0i} - \frac{i}{2}\omega_{i0} J^{i0} - \frac{i}{2}\omega_{ij} J^{ij}\right) \mathbf{\Phi}_{\left(\frac{1}{2},0\right)} \\ &= \left(1 - \frac{i}{2}\omega_{0i} J^{0i} - \frac{i}{2}\omega_{0i} J^{0i} - \frac{i}{2}\omega_{ij} J^{ij}\right) \mathbf{\Phi}_{\left(\frac{1}{2},0\right)} \\ &= \left(1 - i\omega_{0i} K^i - \frac{i}{2}\omega_{ij}\epsilon^{ijk} L^k\right) \mathbf{\Phi}_{\left(\frac{1}{2},0\right)} \\ &= \left(1 - i\boldsymbol{\beta} \cdot \mathbf{K} - i\boldsymbol{\theta} \cdot \mathbf{L}\right) \mathbf{\Phi}_{\left(\frac{1}{2},0\right)} \\ &= \left(1 - \frac{1}{2}\boldsymbol{\beta} \cdot \boldsymbol{\sigma} - \frac{i}{2}\boldsymbol{\theta} \cdot \boldsymbol{\sigma}\right) \mathbf{\Phi}_{\left(\frac{1}{2},0\right)}\end{aligned}$$

where we have defined $\beta^i = \omega_{0i} = -\omega_{i0}$ and $\theta^k = \omega_{ij}\epsilon^{ijk}$.

Now for the $\left(0, \frac{1}{2}\right)$ representation we have

$$\begin{aligned}\pi_{m,n}\left(L^i\right) &= \mathbb{I}_1 \otimes \frac{\sigma^i}{2} + \mathbb{I}_1 \otimes \mathbb{I}_2 = \mathbb{I}_1 \otimes \left(\frac{\sigma^i}{2} + \mathbb{I}_2\right) = \frac{\sigma^i}{2} \\ \pi_{m,n}\left(K^i\right) &= i\left(\mathbb{I}_1 \otimes \frac{\sigma^i}{2} - \mathbb{I}_1 \otimes \mathbb{I}_2\right) = i\left(\frac{\sigma^i}{2} - \mathbb{I}_2\right) \otimes \mathbb{I}_1 = i\frac{\sigma^i}{2} \otimes \mathbb{I}_1.\end{aligned}$$

Thus, with $L^i = \sigma^i/2$ and $K^i = i\sigma^i/2$ the transformation law becomes

$$\begin{aligned}
\mathbf{\Phi}_{\left(0,\frac{1}{2}\right)} &\rightarrow e^{-\frac{i}{2}(\omega_{\mu\nu}J^{\mu\nu})}\mathbf{\Phi}_{\left(0,\frac{1}{2}\right)} \\
&= \left(1 - i\boldsymbol{\beta}\cdot\mathbf{K} - i\boldsymbol{\theta}\cdot\mathbf{L}\right)\mathbf{\Phi}_{\left(0,\frac{1}{2}\right)} \\
&= \left(1 + \frac{1}{2}\boldsymbol{\beta}\cdot\boldsymbol{\sigma} - \frac{i}{2}\boldsymbol{\theta}\cdot\boldsymbol{\sigma}\right)\mathbf{\Phi}_{\left(0,\frac{1}{2}\right)}.
\end{aligned}$$

Upon comparison with equation (3.37) we identify

$$\mathbf{\Phi}_{\left(\frac{1}{2},0\right)} = \psi_L \quad \text{and} \quad \mathbf{\Phi}_{\left(0,\frac{1}{2}\right)} = \psi_R.$$

Thus, the left- and right-handed spinor transform according to separate representations of the Lorentz group.

■

(c) The identity $\boldsymbol{\sigma}^T = -\sigma^2\boldsymbol{\sigma}\sigma^2$ allows us to rewrite the ψ_L transformation in the unitarily equivalent form

$$\psi' \rightarrow \psi'\left(1 + i\boldsymbol{\theta}\cdot\boldsymbol{\sigma}/2 + \boldsymbol{\beta}\cdot\boldsymbol{\sigma}/2\right),$$

where $\psi' = \psi_L^T\sigma^2$. Using this law, we can represent the object that transforms as $\left(\frac{1}{2},\frac{1}{2}\right)$ as a 2×2 matrix that has the ψ_R transformation law on the left and simultaneously, the transposed ψ_L transforms on the right. Parametrize this matrix as

$$\begin{pmatrix} V^0 + V^3 & V^1 - iV^2 \\ V^1 + iV^2 & V^0 - V^3 \end{pmatrix}.$$

Show that the object V^μ transforms as a 4-vector.

Proof: Left-handed spinors transform according to

$$\psi_L \rightarrow \left(1 - \frac{1}{2}\boldsymbol{\beta}\cdot\boldsymbol{\sigma} - \frac{i}{2}\boldsymbol{\theta}\cdot\boldsymbol{\sigma}\right)\psi_L.$$

With $\psi' = \psi_L^T\sigma^2$ we verify the transformation law

$$\begin{aligned}
\psi' &\rightarrow \left(\left(1 - \frac{1}{2}\boldsymbol{\beta}\cdot\boldsymbol{\sigma} - \frac{i}{2}\boldsymbol{\theta}\cdot\boldsymbol{\sigma}\right)\psi_L\right)^T\sigma^2 \\
&= \psi_L^T\sigma^2\sigma^2\left(1 - \frac{1}{2}\boldsymbol{\beta}\cdot\boldsymbol{\sigma} - \frac{i}{2}\boldsymbol{\theta}\cdot\boldsymbol{\sigma}\right)^T\sigma^2 \\
&= \psi'\sigma^2\left(1 - \frac{1}{2}\boldsymbol{\beta}\cdot\boldsymbol{\sigma}^T - \frac{i}{2}\boldsymbol{\theta}\cdot\boldsymbol{\sigma}^T\right)\sigma^2 \\
&= \psi'\left(1 + \frac{1}{2}\boldsymbol{\beta}\cdot\boldsymbol{\sigma} + \frac{i}{2}\boldsymbol{\theta}\cdot\boldsymbol{\sigma}\right).
\end{aligned}$$

Now we are interested in the transformation properties of a $\left(\frac{1}{2},\frac{1}{2}\right)$ object. We parameterize the $\left(\frac{1}{2},\frac{1}{2}\right)$ object as the matrix

$$\mathbf{\Phi}_{\left(\frac{1}{2},\frac{1}{2}\right)} = \begin{pmatrix} V^0 + V^3 & V^1 - iV^2 \\ V^1 + iV^2 & V^0 - V^3 \end{pmatrix} = V^\mu\sigma_\mu$$

where it will be shown that V^μ is a 4-vector. Applying the transformation to $\mathbf{\Phi}_{\left(\frac{1}{2},\frac{1}{2}\right)}$ we get

$$\begin{aligned}
\mathbf{\Phi}_{\left(\frac{1}{2},\frac{1}{2}\right)} &\rightarrow \left(1+\frac{1}{2}\boldsymbol{\beta}\cdot\boldsymbol{\sigma}-\frac{i}{2}\boldsymbol{\theta}\cdot\boldsymbol{\sigma}\right)\mathbf{\Phi}_{\left(\frac{1}{2},\frac{1}{2}\right)}\left(1+\frac{1}{2}\boldsymbol{\beta}\cdot\boldsymbol{\sigma}+\frac{i}{2}\boldsymbol{\theta}\cdot\boldsymbol{\sigma}\right)\\
&= \left(1+\frac{1}{2}\left(\boldsymbol{\beta}-i\boldsymbol{\theta}\right)\cdot\boldsymbol{\sigma}\right)V^\mu\sigma_\mu\left(1+\frac{1}{2}\left(\boldsymbol{\beta}\cdot\boldsymbol{\sigma}+i\boldsymbol{\theta}\right)\cdot\boldsymbol{\sigma}\right)\\
&= V^\mu\sigma_\mu+\frac{1}{2}\left(\boldsymbol{\beta}-i\boldsymbol{\theta}\right)\cdot\boldsymbol{\sigma}V^\mu\sigma_\mu+\frac{1}{2}V^\mu\sigma_\mu\left(\boldsymbol{\beta}+i\boldsymbol{\theta}\right)\cdot\boldsymbol{\sigma}+\mathcal{O}\left(\theta^2,\beta^2\right)\\
&= V^\mu\sigma_\mu+\frac{V^\mu}{2}\boldsymbol{\beta}\cdot\left(\boldsymbol{\sigma}\sigma_\mu+\sigma_\mu\boldsymbol{\sigma}\right)-\frac{V^\mu}{2}i\boldsymbol{\theta}\cdot\left(\boldsymbol{\sigma}\sigma_\mu-\sigma_\mu\boldsymbol{\sigma}\right)\\
&= V^\mu\sigma_\mu+\frac{V^\mu}{2}\beta^i\cdot\{\sigma_i,\sigma_\mu\}-\frac{V^\mu}{2}i\theta^i\cdot[\sigma_i,\sigma_\mu]\\
&= V^\mu\sigma_\mu+\frac{V^0}{2}\beta^i\cdot\{\sigma_i,\sigma_0\}+\frac{V^j}{2}\beta^i\cdot\{\sigma_i,\sigma_j\}-\frac{V^0}{2}i\theta^i\cdot[\sigma_i,\sigma_0]-\frac{V^j}{2}i\theta^i\cdot[\sigma_i,\sigma_j]\\
&= V^\mu\sigma_\mu+\frac{V^0}{2}\beta^i\cdot\{\sigma_i,\mathbb{I}\}+\frac{V^j}{2}\beta^i\{\sigma_i,\sigma_j\}-\frac{V^0}{2}i\theta^i[\sigma_i,\mathbb{I}]-\frac{V^j}{2}i\theta^i[\sigma_i,\sigma_j]\\
&= V^\mu\sigma_\mu+\frac{V^0}{2}\beta^i\left(2\sigma_i\right)+\frac{V^j}{2}\beta^i\left(2\delta_{ij}\right)-\frac{V^j}{2}i\theta^i\left(2i\epsilon_{ijk}\sigma^k\right)\\
&= V^\mu\sigma_\mu+V^0\beta^i\sigma_i-V^i\beta_i+V^j\theta^i\epsilon_{ijk}\sigma^k
\end{aligned}$$

where $\{\sigma_i,\mathbb{I}\}=2\sigma^i$, $[\sigma_i,\mathbb{I}]=0$, $\{\sigma_i,\sigma_j\}=2\delta_{ij}$, $[\sigma_i,\mathbb{I}]=2i\epsilon_{ijk}\sigma^k$. Also note that $\beta^iV^j\delta_{ij}=-\beta^iV_i$ because $g_{ij}=-\delta_{ij}$. Recall that we have defined the anti-symmetric tensor $\omega_{0i}=\beta_i$ and $\omega_{ij}=\epsilon_{ijk}\theta^k$. Inserting these expressions into the above, we have

$$\begin{aligned}
V^\mu\sigma_\mu+V^0\beta^i\sigma_i-V^i\beta_i+V^j\theta^i\epsilon_{ijk}\sigma^k &= V^\mu\sigma_\mu+V^0\omega_{0i}\sigma^i-V^i\omega_{0i}+V^j\omega_{jk}\sigma^k\\
&= V^\mu\sigma_\mu+V^0\omega_{0i}\sigma^i+V^i\omega_{i0}\sigma^0+\omega_{ij}V^i\sigma^j\\
&= V^\mu\sigma_\mu+\omega_{\mu\nu}\sigma^\nu V^\mu\\
&= \left(\delta_\mu^\nu+\omega^\nu{}_\mu\right)V_\nu\sigma^\mu.
\end{aligned}$$

We would like to show that this is identical to equation (3.19) in P&S. P&S assert that a 4-vector V^μ transforms as follows

$$V^\alpha\rightarrow\left(\delta_\beta^\alpha-\frac{i}{2}\omega_{\mu\nu}\left(\mathcal{J}^{\mu\nu}\right)_\beta^\alpha\right)V^\beta$$

for $\left(\mathcal{J}^{\mu\nu}\right)_{\alpha\beta}=\delta_\alpha^\mu\delta_\beta^\nu-\delta_\alpha^\nu\delta_\beta^\mu$. With this definition of $\left(\mathcal{J}^{\mu\nu}\right)_\beta^\alpha$ the transformation condition becomes

$$\begin{aligned}
\delta_\beta^\alpha-\frac{i}{2}\omega_{\mu\nu}\left(\mathcal{J}^{\mu\nu}\right)_\beta^\alpha &= \delta_\beta^\alpha-\frac{i}{2}\omega_{\mu\nu}\left(\mathcal{J}^{\mu\nu}\right)_{\gamma\beta}g^{\gamma\alpha}\\
&= \delta_\beta^\alpha-\frac{i}{2}\omega_{\mu\nu}\left(\delta_\gamma^\mu\delta_\beta^\nu-\delta_\gamma^\nu\delta_\beta^\mu\right)g^{\gamma\alpha}\\
&= \delta_\beta^\alpha-\frac{i}{2}\left(\omega_{\gamma\beta}-\omega_{\beta\gamma}\right)g^{\gamma\alpha}\\
&= \delta_\beta^\alpha-i\omega_{\gamma\beta}g^{\gamma\alpha}\\
&= \delta_\beta^\alpha-i\omega_\beta^\alpha.
\end{aligned}$$

This is the identical result obtained from transforming $V^\mu\sigma_\mu$. Thus, we see that V^μ is indeed a 4-vector.

■

1.2 Gordon Identity ✓

Derive the Gordon identity,

$$\bar{u}(p')\gamma^{\mu}u(p)=\bar{u}(p')\left[\frac{p'^{\mu}+p^{\mu}}{2m}+\frac{i\sigma^{\mu\nu}q_{\nu}}{2m}\right]u(p),$$

where $q=(p'-p)$.

Proof:

The computation is straightforward:

$$\begin{aligned}
\sigma^{\mu\nu}q_{\nu} &= \frac{i}{2}\left[\gamma^{\mu},\gamma^{\nu}\right](p'_{\nu}-p_{\nu})\\
&= \frac{i}{2}\left(\gamma^{\mu}\not{p}'-\not{p}'\gamma^{\mu}\right)-\frac{i}{2}\left(\gamma^{\mu}\not{p}-\not{p}\gamma^{\mu}\right)\\
&= \frac{i}{2}\left(2g^{\mu\nu}p'_{\nu}-\not{p}'\gamma^{\mu}-\not{p}'\gamma^{\mu}\right)-\frac{i}{2}\left(\gamma^{\mu}\not{p}-2g^{\mu\nu}p_{\nu}+\gamma^{\mu}\not{p}\right)\\
&= i\left(p'^{\mu}-\not{p}'\gamma^{\mu}\right)-i\left(\gamma^{\mu}\not{p}-p^{\mu}\right)
\end{aligned}$$

$$\begin{aligned}
\bar{u}(p')\left[\frac{p'^{\mu}+p^{\mu}}{2m}+\frac{i\sigma^{\mu\nu}q_{\nu}}{2m}\right]u(p) &= \bar{u}(p')\left[\frac{p'^{\mu}+p^{\mu}-(p'^{\mu}-\not{p}'\gamma^{\mu})+(\gamma^{\mu}\not{p}-p^{\mu})}{2m}\right]u(p)\\
&= \bar{u}(p')\left[\frac{\not{p}'\gamma^{\mu}+\gamma^{\mu}\not{p}}{2m}\right]u(p)\\
&= \bar{u}(p')\left[\frac{m\gamma^{\mu}+\gamma^{\mu}m}{2m}\right]u(p)\\
&= \bar{u}(p')\gamma^{\mu}u(p)
\end{aligned}$$

where we have used the Fourier transformed Dirac equation and it's adjoint equation

$$(\not{p}-m)u(p)=0 \text{ and } \bar{u}(p)(\not{p}-m)=0.$$

■

1.3 Spinor products ✓

Together is Problems 5.3 and 5.6 we develop an efficient computational method for processes involving massless particles. Let k_0^{μ}, k_1^{μ} be fixed 4-vectors satisfying $k_0^2=0$, $k_1^2=-1$, $k_0\cdot k_1=0$. Define basic spinors in the following way: Let u_{L0} be the left-handed spinor for for a fermion with momentum k_0. Let $u_{R0}=\not{k}_1u_{L0}$. Then, for any p such that p is lightlike $\left(p^2=0\right)$ define

$$u_L(p)=\tfrac{1}{\sqrt{2p\cdot k_0}}\not{p}u_{R0} \quad \text{and} \quad u_R(p)=\tfrac{1}{\sqrt{2p\cdot k_0}}\not{p}u_{L0}\ .$$

This set of conventions defines the phases of spinors unambiguously (except when p is parallel to k_0).

(a) Show that $\not{k}_0u_{R0}=0$. Show that, for any lightlike p, $\not{p}u_R(p)=0$.

Proof:

$$\begin{aligned}
\not{k}_0u_{R0} &= \not{k}_0\not{k}_1u_{L0}\\
&= k_0^{\mu}k_1^{\nu}\gamma_{\mu}\gamma_{\nu}u_{L0}\\
&= k_0^{\mu}k_1^{\nu}\left(2g_{\mu\nu}-\gamma_{\nu}\gamma_{\mu}\right)u_{L0}\\
&= 2k_0\cdot k_1u_{L0}-\not{k}_1\not{k}_0u_{L0}\\
&= 0
\end{aligned}$$

where we have used the Dirac equation for a massless particle $\not{k}_0 u_{L0} = 0$ and the dot product $k_0 \cdot k_1$. Now for any lightlike 4-momentum p the definition of the spinors above satisfy the massless Dirac equation

$$\begin{aligned}
\not{p} u_R(p) &= \frac{1}{\sqrt{2p \cdot k_0}} \not{p}\not{p} u_{L0} \\
&= \frac{1}{\sqrt{2p \cdot k_0}} p^2 u_{L0} \\
&= 0.
\end{aligned}$$

■

(b) For the choices $k_0 = (E, 0, 0, -E)$, $k_1 = (0, 1, 0, 0)$, construct u_{L0}, u_{R0}, $u_L(p)$, and $u_R(p)$ explicitly.

Proof: For this problem we will need the operator $\not{k}$ which in the Chiral or Weyl basis is given by

$$\begin{aligned}
\not{k} &= \begin{pmatrix} 0 & k^0 - \mathbf{k} \cdot \boldsymbol{\sigma} \\ k^0 + \mathbf{k} \cdot \boldsymbol{\sigma} & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 & k^0 - k^3 & -\left(k^1 - ik^2\right) \\ 0 & 0 & -\left(k^1 + ik^2\right) & k^0 + k^3 \\ k^0 + k^3 & k^1 - ik^2 & 0 & 0 \\ k^1 + ik^2 & k^0 - k^3 & 0 & 0 \end{pmatrix}.
\end{aligned}$$

From equation (3.50) in P&S the Dirac spinors for are given by

$$\begin{aligned}
u(k_0) &= \begin{pmatrix} \sqrt{k_0 \cdot \sigma} \xi \\ \sqrt{k_0 \cdot \bar{\sigma}} \xi \end{pmatrix} \\
&= \begin{pmatrix} \sqrt{E\left(\sigma^0 + \sigma^3\right)} \xi \\ \sqrt{E\left(\sigma^0 - \sigma^3\right)} \xi \end{pmatrix} \\
&= \sqrt{E} \begin{pmatrix} \sqrt{\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}} \xi \\ \sqrt{\begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}} \xi \end{pmatrix} \\
&= \sqrt{2E} \begin{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \xi \\ \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \xi \end{pmatrix}
\end{aligned}$$

where $\xi \in \left\{ \begin{pmatrix} 1 & 0 \end{pmatrix}^T, \begin{pmatrix} 0 & 1 \end{pmatrix}^T \right\}$ and $\bar{\sigma} = \left(\sigma^0 - \boldsymbol{\sigma}\right)$. Taking $\xi = \begin{pmatrix} 1 & 0 \end{pmatrix}^T$ the left-handed spinor is

$$u_{L0} = \sqrt{2E} \left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right) = \sqrt{2E} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

while the right-handed spinor is given by taking $\xi = \begin{pmatrix} 0 & 1 \end{pmatrix}^T$

$$u_{R0} = \sqrt{2E} \left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} \right) = \sqrt{2E} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

The chiral spinors for any momentum p are then given by

$$\begin{aligned}
u_L(p) &= \frac{1}{\sqrt{2p\cdot k_0}}\not{p}\,u_{R0} \\
&= \frac{\sqrt{2E}}{\sqrt{2E\left(p^0+p^3\right)}}\begin{pmatrix} 0 & 0 & p^0-p^3 & -\left(p^1-ip^2\right) \\ 0 & 0 & -\left(p^1+ip^2\right) & p^0+p^3 \\ p^0+p^3 & p^1-ip^2 & 0 & 0 \\ p^1+ip^2 & p^0-p^3 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \\
&= \sqrt{\frac{1}{p^0+p^3}}\begin{pmatrix} -p^1+ip^2 \\ p^0+p^3 \\ 0 \\ 0 \end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
u_R(p) &= \frac{1}{\sqrt{2p\cdot k_0}}\not{p}\,u_{L0} \\
&= \frac{\sqrt{2E}}{\sqrt{2E\left(p^0+p^3\right)}}\begin{pmatrix} 0 & 0 & p^0-p^3 & -\left(p^1-ip^2\right) \\ 0 & 0 & -\left(p^1+ip^2\right) & p^0+p^3 \\ p^0+p^3 & p^1-ip^2 & 0 & 0 \\ p^1+ip^2 & p^0-p^3 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \\
&= \sqrt{\frac{1}{p^0+p^3}}\begin{pmatrix} 0 \\ 0 \\ p^0+p^3 \\ p^1+ip^2 \end{pmatrix}
\end{aligned}$$

■

(c) Define the spinor products $s(p_1,p_2)$ and $t(p_1p_2)$, for p_1,p_2 lightlike, by

$$s(p_1,p_2)=\bar{u}_R(p_1)\,u_L(p_2) \quad \text{and} \quad t(p_1,p_2)=\bar{u}_L(p_1)\,u_R(p_2)\,.$$

Using the explicit forms for the u_λ given in part (b), compute the spinor products explicitly and show that $t(p_1,p_2)=s(p_1,p_2)^*$ and $s(p_1,p_2)=-s(p_2,p_1)$. In addition, show that

$$|s(p_1,p_2)|^2=2p_1\cdot p_2.$$

Thus the spinor products are the square roots of 4-vector dot products.

Proof:

$$
\begin{aligned}
s(p_1,p_2) &= \bar{u}_R(p_1)\,u_L(p_2) \\
&= \sqrt{\frac{1}{(p_1^0+p_1^3)(p_2^0+p_2^3)}}\begin{pmatrix} 0 & 0 & p_1^0+p_1^3 & p_1^1-ip_1^2 \end{pmatrix}\begin{pmatrix} 0&0&1&0\\0&0&0&1\\1&0&0&0\\0&1&0&0 \end{pmatrix}\begin{pmatrix} -p_2^1+ip_2^2 \\ p_2^0+p_2^3 \\ 0 \\ 0 \end{pmatrix} \\
&= \sqrt{\frac{1}{(p_1^0+p_1^3)(p_2^0+p_2^3)}}\begin{pmatrix} 0 & 0 & p_1^0+p_1^3 & p_1^1-ip_1^2 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ -p_2^1+ip_2^2 \\ p_2^0+p_2^3 \end{pmatrix} \\
&= \frac{\left(p_1^0+p_1^3\right)\left(-p_2^1+ip_2^2\right)+\left(p_1^1-ip_1^2\right)\left(p_2^0+p_2^3\right)}{\sqrt{(p_1^0+p_1^3)(p_2^0+p_2^3)}} \\
&= \frac{-p_1^0p_2^1-p_1^3p_2^1+ip_1^0p_2^2+ip_1^3p_2^2+p_1^1p_2^0-ip_1^2p_2^0+p_1^1p_2^3-ip_1^2p_2^3}{\sqrt{(p_1^0+p_1^3)(p_2^0+p_2^3)}} \\
&= \frac{p_1^1\left(p_2^0+p_2^3\right)-\left(p_1^0-p_1^3\right)p_2^1+i\left(\left(p_1^0+p_1^3\right)p_2^2-p_1^2\left(p_2^0+p_2^3\right)\right)}{\sqrt{(p_1^0+p_1^3)(p_2^0+p_2^3)}} \\
&= -s(p_2,p_1)
\end{aligned}
$$

$$
\begin{aligned}
|s(p_1,p_2)|^2 &= \frac{\left|p_1^1\left(p_2^0+p_2^3\right)-\left(p_1^0-p_1^3\right)p_2^1+i\left(\left(p_1^0+p_1^3\right)p_2^2-p_1^2\left(p_2^0+p_2^3\right)\right)\right|^2}{(p_1^0+p_1^3)(p_2^0+p_2^3)} \\
&= \frac{\left|p_1^1\left(p_2^0+p_2^3\right)-\left(p_1^0-p_1^3\right)p_2^1+i\left(\left(p_1^0+p_1^3\right)p_2^2-p_1^2\left(p_2^0+p_2^3\right)\right)\right|^2}{(p_1^0+p_1^3)(p_2^0+p_2^3)} \\
&= \frac{\left(p_1^1\left(p_2^0+p_2^3\right)-\left(p_1^0-p_1^3\right)p_2^1\right)^2+\left(\left(p_1^0+p_1^3\right)p_2^2-p_1^2\left(p_2^0+p_2^3\right)\right)^2}{(p_1^0+p_1^3)(p_2^0+p_2^3)} \\
&= \frac{\left(p_1^1\right)^2\left(p_2^0+p_2^3\right)^2-2p_1^1\left(p_2^0+p_2^3\right)\left(p_1^0-p_1^3\right)p_2^1-+\left(p_1^0-p_1^3\right)\left(p_2^1\right)^2}{(p_1^0+p_1^3)(p_2^0+p_2^3)} \\
&\quad\times\frac{\left(p_1^0+p_1^3\right)^2\left(p_2^2\right)^2-2\left(p_1^0+p_1^3\right)p_2^2p_1^2\left(p_2^0+p_2^3\right)+\left(p_1^2\right)^2\left(p_2^0+p_2^3\right)^2}{(p_1^0+p_1^3)(p_2^0+p_2^3)} \\
&= \frac{(p_2^0+p_2^3)}{(p_1^0+p_1^3)}\left(\left(p_1^1\right)^2+\left(p_1^2\right)^2\right)+\frac{(p_1^0+p_1^3)}{(p_2^0+p_2^3)}\left(\left(p_2^1\right)^2+\left(p_2^2\right)^2\right)-2\left(p_1^1p_2^1+p_2^2p_1^2\right) \\
&= \frac{(p_2^0+p_2^3)}{(p_1^0+p_1^3)}\left(\left(p_1^1\right)^2+\left(p_1^2\right)^2\right)+\frac{(p_1^0+p_1^3)}{(p_2^0+p_2^3)}\left(\left(p_2^1\right)^2+\left(p_2^2\right)^2\right)-2\left(p_1^1p_2^1+p_2^2p_1^2\right)
\end{aligned}
$$

Since p_1 and p_2 are lightlike

$$
\left(p_i^1\right)^2+\left(p_i^2\right)^2=\left(p_i^0\right)^2-\left(p_i^3\right)^2=\left(p_i^0+p_i^3\right)\left(p_i^0-p_i^3\right)
$$

we have

$$
\begin{aligned}
|s(p_1,p_2)|^2 &= \left(p_2^0+p_2^3\right)\left(p_1^0-p_1^3\right)+\left(p_1^0+p_1^3\right)\left(p_2^0-p_2^3\right)-2\left(p_1^1p_2^1+p_2^2p_1^2\right) \\
&= p_1^0p_2^0+p_1^0p_2^3-p_1^3p_2^0-p_1^3p_2^3+p_1^0p_2^0+p_1^3p_2^0-p_1^0p_2^3-p_1^3p_2^3-2\left(p_1^1p_2^1+p_2^2p_1^2\right) \\
&= 2p_1^0p_2^0-2\left(p_1^1p_2^1+p_2^2p_1^2+p_1^3p_2^3\right) \\
&= 2p_1\cdot p_2.
\end{aligned}
$$

■

1.4 Majorana fermions ✓

Recall form Eq. (3.40) that one can write a relativistic equation for massless 2-component fermion field that transforms as the upper two components of a Dirac spinor (ψ_L). Call such a 2-component field $\chi_a(x)$, $a = 1, 2$.

(a) Show that it is possible to write an equation for $\chi(x)$ as a massive field in the following way:

$$i\bar{\sigma} \cdot \partial\chi - im\sigma^2\chi^* = 0.$$

That is, show, that this equation is relativistically invariant and, second, that it implies the Klein-Gordon equation, $\left(\partial^2 + m^2\right)\chi = 0$. This form of the fermion mass is called Majorana mass term.

Proof:

The unitary matrix, $\Lambda_{\frac{1}{2}}$, which Lorentz transforms a fermion field (spinor) is given by equation (3.30)

$$\Lambda_{\frac{1}{2}}(\boldsymbol{\theta}, \boldsymbol{\beta}) = \exp\left(-\frac{i}{2}\omega_{\mu\nu}S^{\mu\nu}\right) = \begin{pmatrix} \Lambda_{\left(\frac{1}{2},0\right)} & 0 \\ 0 & \Lambda_{\left(0,\frac{1}{2}\right)} \end{pmatrix}$$

where

$$S^{\mu\nu} = \frac{i}{4}\left[\gamma^\mu, \gamma^\nu\right]$$

is the Lorentz transformation generator for spinor, $\beta^i = \omega_{0i} = -\omega_{i0}$ and $\theta^k = \omega_{ij}\epsilon^{ijk}$. This transformation matrix is block diagonal. This is seen from the block diagonal form of the generators of boost (3.26) and (3.27)

$$\begin{aligned} S^{0i} &= -\frac{i}{2}\begin{pmatrix} \sigma^i & 0 \\ 0 & -\sigma^i \end{pmatrix} \\ S^{ij} &= \frac{1}{2}\epsilon^{ijk}\Sigma^k \end{aligned}$$

where the spin operator of Dirac theory is

$$\Sigma^k = \begin{pmatrix} \sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix}.$$

The blocks $\Lambda_{\left(\frac{1}{2},0\right)}$ and $\Lambda_{\left(0,\frac{1}{2}\right)}$ are the left and right handed representations of the Lorentz group. A spinor which transforms according to $\Lambda_{\frac{1}{2}}$ is called a Dirac spinor.
Because of the block diagonal form left and right spinors transform in different representations of the Lorentz group

$$\Lambda_{\frac{1}{2}}\psi = \begin{pmatrix} \Lambda_{\left(\frac{1}{2},0\right)} & 0 \\ 0 & \Lambda_{\left(0,\frac{1}{2}\right)} \end{pmatrix}\begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} = \begin{pmatrix} \Lambda_{\left(\frac{1}{2},0\right)}\psi_L \\ \Lambda_{\left(0,\frac{1}{2}\right)}\psi_R \end{pmatrix}.$$

Under infinitesimal rotations by $\boldsymbol{\theta}$ and boosts $\boldsymbol{\beta}$ the transformation laws for the left and right handed spinors are

$$\begin{aligned} \Lambda_{\left(\frac{1}{2},0\right)} &= \left(1 - \frac{1}{2}\boldsymbol{\beta}\cdot\boldsymbol{\sigma} - \frac{i}{2}\boldsymbol{\theta}\cdot\boldsymbol{\sigma}\right) \\ \Lambda_{\left(0,\frac{1}{2}\right)} &= \left(1 - \frac{1}{2}\boldsymbol{\beta}\cdot\boldsymbol{\sigma} + \frac{i}{2}\boldsymbol{\theta}\cdot\boldsymbol{\sigma}\right). \end{aligned}$$

Now equation (3.40) tells us that we can write the relativistic field equation for a massless fermion which transforms under $\Lambda_{\left(\frac{1}{2},0\right)}$ as $i\bar{\sigma}\cdot\partial\chi = 0$. Furthermore, we know how γ^μ transforms. This is given by equation (3.29)

$$\Lambda_{\frac{1}{2}}^{-1}\gamma^\mu\Lambda_{\frac{1}{2}} = \Lambda^\mu_{\ \nu}\gamma^\nu.$$

On the left the Lorentz transformations are acting on the spinor indices of γ^μ while on the right the Lorentz transformation acts on the space time index μ. We can work out the transformation for $\bar{\sigma}$ because $\bar{\sigma}^\mu$ is just the off-diagonal blocks of γ^μ

$$\begin{aligned}\Lambda^\mu{}_\nu\gamma^\nu &= \Lambda_{\frac{1}{2}}^{-1}\gamma^\mu\Lambda_{\frac{1}{2}}\\ \begin{pmatrix} 0 & \Lambda^\mu{}_\nu\sigma^\nu \\ \Lambda^\mu{}_\nu\bar{\sigma}^\nu & 0\end{pmatrix} &= \begin{pmatrix} 0 & \Lambda^{-1}_{\left(\frac{1}{2},0\right)}\sigma^\mu\Lambda_{\left(0,\frac{1}{2}\right)} \\ \Lambda^{-1}_{\left(0,\frac{1}{2}\right)}\bar{\sigma}^\mu\Lambda_{\left(\frac{1}{2},0\right)} & 0\end{pmatrix}.\end{aligned}$$

From the above we conclude that $\bar{\sigma}^\mu\left(\Lambda^{-1}\right)^\nu{}_\mu = \Lambda_{\left(0,\frac{1}{2}\right)}\bar{\sigma}^\nu\Lambda^{-1}_{\left(\frac{1}{2},0\right)}$. We will also need the fact that if χtransforms according to $\Lambda_{\left(\frac{1}{2},0\right)}$ then $\sigma^2\chi^*$ transforms under $\Lambda_{\left(0,\frac{1}{2}\right)}$ (i.e., χ is left-handed and $\sigma^2\chi^*$ is right-handed). We prove this here. Applying $\Lambda_{\left(\frac{1}{2},0\right)}$ to $\sigma^2\chi^*$ yields

$$\begin{aligned}\sigma^2\chi^* &\rightarrow \sigma^2\left(\left(1-\frac{1}{2}\boldsymbol{\beta}\cdot\boldsymbol{\sigma}-\frac{i}{2}\boldsymbol{\theta}\cdot\boldsymbol{\sigma}\right)\chi\right)^*\\ &= \sigma^2\left(1-\frac{1}{2}\boldsymbol{\beta}\cdot\boldsymbol{\sigma}^*+\frac{i}{2}\boldsymbol{\theta}\cdot\boldsymbol{\sigma}^*\right)\chi^*\\ &= \left(1+\frac{1}{2}\boldsymbol{\beta}\cdot\boldsymbol{\sigma}-\frac{i}{2}\boldsymbol{\theta}\cdot\boldsymbol{\sigma}\right)\chi^*\\ &= \Lambda_{\left(0,\frac{1}{2}\right)}\chi^*.\end{aligned}$$

Lorentz transforms; review of a scalar field χ (no spatial orientation).

1. χ: Under Lorentz transformations the scalar χ transforms as $\chi(x)\rightarrow\chi'(x') = \chi\left(\Lambda^{-1}x'\right)$ where $x\rightarrow x' = \Lambda x$.
2. $\partial_\mu\chi$: Under Lorentz transformations $\partial_\mu\chi$ transforms as

$$\begin{aligned}\partial_\mu\chi(x)\rightarrow\partial'_\mu\chi'(x') &= \frac{\partial x^\nu}{\partial x'^\mu}\frac{\partial}{\partial x^\nu}\chi\left(\left(\Lambda^{-1}\right)^\beta{}_\alpha x'^\alpha\right)\\ &= \frac{\partial\left(\Lambda^{-1}\right)^\nu{}_\rho x'^\rho}{\partial x'^\mu}\frac{\partial}{\partial x^\nu}\chi\left(\left(\Lambda^{-1}\right)^\beta{}_\alpha x'^\alpha\right)\\ &= \left(\Lambda^{-1}\right)^\nu{}_\rho g^\rho{}_\mu\frac{\partial}{\partial x^\nu}\chi\left(\left(\Lambda^{-1}\right)^\beta{}_\alpha x'^\alpha\right)\\ &= \left(\Lambda^{-1}\right)^\nu{}_\mu\left(\partial_\nu\chi\right)\left(\Lambda^{-1}x'\right)\end{aligned}$$

where $x\rightarrow x' = \Lambda x$. Notice that this is opposite to how the vector x^μ transforms $x^\mu\rightarrow x'^\mu = \Lambda^\mu{}_\nu x^\nu$. This shows that while x^μ is a contravariant vector, ∂_μ is a covariant vector.

Applying the Lorentz transformation to the field equation we have

$$i\bar{\sigma}\cdot\partial\chi(x) - im\sigma^2\chi^*(x) = 0 \quad\rightarrow\quad i\bar{\sigma}^\mu\left(\Lambda^{-1}\right)^\nu{}_\mu\partial_\nu\Lambda_{\left(\frac{1}{2},0\right)}\chi\left(\Lambda^{-1}x\right) - im\Lambda_{\left(0,\frac{1}{2}\right)}\sigma^2\chi^*\left(\Lambda^{-1}x\right) = 0$$

where we know how $\partial_\mu\chi$ transforms. Using $\bar{\sigma}^\mu\left(\Lambda^{-1}\right)^\nu{}_\mu = \Lambda_{\left(0,\frac{1}{2}\right)}\bar{\sigma}^\nu\Lambda^{-1}_{\left(\frac{1}{2},0\right)}$ we conclude

$$\begin{aligned}i\bar{\sigma}\cdot\partial\chi(x) - im\sigma^2\chi^*(x) = 0 \quad&\rightarrow\quad i\Lambda_{\left(0,\frac{1}{2}\right)}\bar{\sigma}^\nu\Lambda^{-1}_{\left(\frac{1}{2},0\right)}\partial_\nu\Lambda_{\left(\frac{1}{2},0\right)}\chi\left(\Lambda^{-1}x\right) - im\Lambda_{\left(0,\frac{1}{2}\right)}\sigma^2\chi^*\left(\Lambda^{-1}x\right) = 0\\ &= \quad\Lambda_{\left(0,\frac{1}{2}\right)}\left\{i\bar{\sigma}^\nu\partial_\nu\chi\left(\Lambda^{-1}x\right) - im\sigma^2\chi^*\left(\Lambda^{-1}x\right)\right\} = 0.\end{aligned}$$

Thus, we see that the field equation $i\bar{\sigma}\cdot\partial\chi(x) - im\sigma^2\chi^*(x) = 0$ is relativistically invariant.

Next, we must show that $i\bar{\sigma}\cdot\partial\chi(x) - im\sigma^2\chi^*(x) = 0$ implies the Klein-Gordon equation. The equation $-i\bar{\sigma}^*\cdot\partial\chi^*(x) + im\sigma^{2*}\chi(x) = 0$ implies $\chi^*(x) = \frac{1}{m}\sigma^2\bar{\sigma}\cdot\partial\chi(x)$. Substitution into the complex conjugate of $i\bar{\sigma}\cdot\partial\chi(x) - im\sigma^2\chi^*(x) = 0$

we have

$$\begin{aligned}
i\bar{\sigma}^{\mu *}\partial_\mu \chi^*(x) - im\sigma^{2*}\chi(x) &= 0 \\
i\bar{\sigma}^{\mu *}\partial_\mu \frac{1}{m}\sigma^2\bar{\sigma}^\nu\partial_\nu\chi(x) - im\sigma^{2*}\chi(x) &= 0 \\
\sigma^2\sigma^2\bar{\sigma}^{\mu *}\sigma^2\bar{\sigma}^\nu\partial_\mu\partial_\nu\chi(x) + m^2\sigma^{2*}\chi(x) &= 0 \\
\sigma^2\sigma^\mu\bar{\sigma}^\nu\partial_\mu\partial_\nu\chi(x) + m^2\sigma^{2*}\chi(x) &= 0 \\
\sigma^\mu\bar{\sigma}^\nu\partial_\mu\partial_\nu\chi(x) - m^2\chi(x) &= 0 \\
-g^{\mu\nu}\partial_\mu\partial_\nu\chi(x) - m^2\chi(x) &= 0 \\
\partial^2\chi(x) + m^2\chi(x) &= 0
\end{aligned}$$

as required.

■

(b) Does the Majorana equation follow from a Lagrangian? The mass term would seem to be the variation of $(\sigma^2)_{ab}\chi_a^*\chi_b^*$; however, since σ^2 is antisymmetric, this expression would vanish if $\chi(x)$ were an ordinary c-number field. When we go to quantum field theory, we know that $\chi(x)$ will become an anti-commuting quantum field. Therefore, it makes sense to develop its classical theory by considering $\chi(x)$ as a classical anti-commuting field, that is, as a field that takes as values Grassmann numbers which satisfy

$$\alpha\beta = -\beta\alpha \text{ for any } \alpha, \beta.$$

Note that this relation implies that $\alpha^2 = 0$. A Grassmann field $\xi(x)$ can be expanded in a basis of functions as

$$\xi(x) = \sum_n \alpha_n\phi_n(x),$$

where the $\phi_n(x)$ are orthogonal c-number functions and the α_n are a set of independent Grassmann numbers. Define the complex conjugate of a product of Grassmann numbers to reverse the order:

$$(\alpha\beta)^* \equiv \beta^*\alpha^* = -\alpha^*\beta^*.$$

This rule imitates the Hermitian conjugation of quantum fields. Show that the classical action,

$$S = \int d^4x \left[\chi^\dagger i\bar{\sigma}\cdot\partial\chi + \frac{im}{2}\left(\chi^T\sigma^2\chi - \chi^\dagger\sigma^2\chi^*\right)\right],$$

(where $\chi^\dagger = (\chi^*)^T$) is real ($S^* = S$), and that varying this S with respect to χ and χ^* yields the Majorana equation.

Proof:

The trick to this problem is realizing that $S \in \mathbb{C}$ and so $S^* = S^\dagger$. The complex conjugate of the action is therefore

$$\begin{aligned}
S^* &= \int d^4x \left[\left(\chi^\dagger i\bar{\sigma}^\mu\partial_\mu\chi\right)^\dagger + \left(\frac{im}{2}\left(\chi^T\sigma^2\chi - \chi^\dagger\sigma^2\chi^*\right)\right)^\dagger\right] \\
&= \int d^4x \left[-\left(\partial_\mu\chi\right)^\dagger i\bar{\sigma}^{\mu\dagger}\chi^{\dagger\dagger} - \frac{im}{2}\left(\chi^\dagger\sigma^{2\dagger}\chi^{T\dagger} - \chi^{*\dagger}\sigma^{2\dagger}\chi^{\dagger\dagger}\right)\right] \\
&= \int d^4x \left[-\left(\partial_\mu\chi\right)^\dagger i\bar{\sigma}^\mu\chi - \frac{im}{2}\left(\chi^\dagger\sigma^2\chi^* - \chi^T\sigma^2\chi\right)\right] \\
&= \int d^4x \left[-\left(\partial_\mu\chi\right)^\dagger i\bar{\sigma}^\mu\chi\right] + \int d^4x \left[\frac{im}{2}\left(\chi^T\sigma^2\chi - \chi^\dagger\sigma^2\chi^*\right)\right] \\
&=_{IBP} \oint dS_\mu \left[-i\chi^\dagger\bar{\sigma}^\mu\chi\right] - \int d^4x \left[-\left(\chi^\dagger i\bar{\sigma}^\mu\partial_\mu\chi\right)\right] + \int d^4x \left[\frac{im}{2}\left(\chi^T\sigma^2\chi - \chi^\dagger\sigma^2\chi^*\right)\right] \\
&= \int d^4x \left[\left(\chi^\dagger i\bar{\sigma}^\mu\partial_\mu\chi\right)\right] + \int d^4x \left[\frac{im}{2}\left(\chi^T\sigma^2\chi - \chi^\dagger\sigma^2\chi^*\right)\right] \\
&= S.
\end{aligned}$$

Notice that we have assumed that the field χ vanishes at the boundary of the integration region.

Recall that the action is the integral of the Lagrange density $\mathcal{L}$. In our case we have

$$\begin{aligned}\mathcal{L} &= \chi^{\dagger} i\bar{\sigma}\cdot\partial\chi + \frac{im}{2}\left(\chi^T\sigma^2\chi - \chi^{\dagger}\sigma^2\chi^*\right) \\ &= \chi_a^* i\bar{\sigma}^{\mu}_{ab}\partial_{\mu}\chi_b + \frac{im}{2}\left(\sigma^2_{ab}\chi_a\chi_b - \sigma^2_{ab}\chi_a^*\chi_b^*\right)\end{aligned}$$

. The Euler-Lagrange equations are derived from variation of the action, requiring that to first order $\delta\chi$ that $\delta S = 0$. However, we cannot just use the Euler-Lagrange equations because χ is a Grassmann field. Varying the action with respect to χ^* yields

$$\begin{aligned}\delta S &\equiv S\left[\chi^* + \delta\chi^*\right] - S\left[\chi^*\right] \\ &= \int d^4x\mathcal{L}\left(\chi^* + \delta\chi^*, \partial_{\mu}\chi^* + \partial_{\mu}\delta\chi^*\right) - \int d^4x\mathcal{L}\left(\chi^*, \partial_{\mu}\chi^*\right) \\ &= \int d^4x\left[\left(\chi_a^* + \delta\chi_a^*\right) i\bar{\sigma}^{\mu}_{ab}\partial_{\mu}\chi_b + \frac{im}{2}\sigma^2_{ab}\left(\chi_a\chi_b - \left(\chi_a^* + \delta\chi_a^*\right)\left(\chi_b^* + \delta\chi_b^*\right)\right)\right] \\ &\quad - \int d^4x\left[\chi_a^* i\bar{\sigma}^{\mu}_{ab}\partial_{\mu}\chi_b + \frac{im}{2}\sigma^2_{ab}\left(\chi_a\chi_b - \chi_a^*\chi_b^*\right)\right] \\ &= \int d^4x\left[\delta\chi_a^* i\bar{\sigma}^{\mu}_{ab}\partial_{\mu}\chi_b + \frac{im}{2}\sigma^2_{ab}\left(-\chi_a^*\delta\chi_b^* - \delta\chi_a^*\chi_b^*\right)\right] \\ &= \int d^4x\left[\delta\chi_a^* i\bar{\sigma}^{\mu}_{ab}\partial_{\mu}\chi_b + \frac{im}{2}\left(-\sigma^2_{ab}\delta\chi_a^*\chi_b^* - \sigma^2_{ab}\delta\chi_a^*\chi_b^*\right)\right] \\ &= \int d^4x\, \delta\chi_a^*\left[i\bar{\sigma}^{\mu}_{ab}\partial_{\mu}\chi_b - im\sigma^2_{ab}\chi_b^*\right] \\ &= 0\end{aligned}$$

Since this holds for all integration volumes

$$i\bar{\sigma}^{\mu}\partial_{\mu}\chi - im\sigma^2\chi^* = 0$$

which is the Majorana equation. Similarly, with the action

$$S = S^{\dagger} = \int d^4x\left[-\left(\partial_{\mu}\chi\right)^{\dagger} i\bar{\sigma}^{\mu}\chi - \frac{im}{2}\left(\chi^{\dagger}\sigma^2\chi^* - \chi^T\sigma^2\chi\right)\right]$$

and varying with respect to χ yields the complex conjugate equation

$$i\bar{\sigma}^{\mu}\partial_{\mu}\chi^* - im\sigma^2\chi = 0.$$

■

(c) Let us write a 4-component Dirac field as

$$\psi\left(x\right) = \begin{pmatrix}\psi_L \\ \psi_R\end{pmatrix},$$

and recall that the lower components of ψ transform in a way equivalent by a unitary transformation to the complex conjugate of the representation ψ_L. In this way we can rewrite the 4-component Dirac field in terms of two 2-component spinors:

$$\psi_L\left(x\right) = \chi_1\left(x\right) \text{ and } \psi_R\left(x\right) = i\sigma^2\chi_2^*\left(x\right).$$

Rewrite the Dirac Lagrangian in terms of χ_1 and χ_2 and note the form of the mass term.

Proof: The Dirac Lagrange density is $\mathcal{L}_D = \bar{\psi}\left(i\gamma^\mu\partial_\mu - m\right)\psi$. The Dirac spinor is $\psi = \begin{pmatrix} \psi_L & \psi_R \end{pmatrix}^T = \begin{pmatrix} \chi_1 & i\sigma^2\chi_2^* \end{pmatrix}^T$.

$$\begin{aligned}
\mathcal{L}_D &= \bar{\psi}\left(i\gamma^\mu\partial_\mu - m\right)\psi \\
&= \begin{pmatrix} \chi_1^\dagger & -i\chi_2^T\sigma^2 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} -m & i\sigma^\mu\partial_\mu \\ i\bar{\sigma}^\mu\partial_\mu & -m \end{pmatrix}\begin{pmatrix} \chi_1 \\ i\sigma^2\chi_2^* \end{pmatrix} \\
&= \begin{pmatrix} -i\chi_2^T\sigma^2 & \chi_1^\dagger \end{pmatrix}\begin{pmatrix} -m\chi_1 + i\sigma^\mu\partial_\mu i\sigma^2\chi_2^* \\ i\bar{\sigma}^\mu\partial_\mu\chi_1 - mi\sigma^2\chi_2^* \end{pmatrix} \\
&= -i\chi_2^T\sigma^2\left(-m\chi_1 + i\sigma^\mu\partial_\mu i\sigma^2\chi_2^*\right) + \chi_1^\dagger\left(i\bar{\sigma}^\mu\partial_\mu\chi_1 - mi\sigma^2\chi_2^*\right) \\
&= i\chi_2^T\sigma^2\sigma\cdot\partial\left(\sigma^2\chi_2^*\right) + i\chi_1^\dagger\bar{\sigma}\cdot\partial\chi_1 + im\chi_2^T\sigma^2\chi_1 - mi\chi_1^\dagger\sigma^2\chi_2^* \\
&= i\chi_1^\dagger\bar{\sigma}\cdot\partial\chi_1 + i\chi_2^T\sigma^2\sigma\sigma^2\cdot\partial\chi_2^* + im\left(\chi_2^T\sigma^2\chi_1 - \chi_1^\dagger\sigma^2\chi_2^*\right) \\
&= i\chi_1^\dagger\bar{\sigma}\cdot\partial\chi_1 + i\chi_2^T\bar{\sigma}^*\cdot\partial\chi_2^* + im\left(\chi_2^T\sigma^2\chi_1 - \chi_1^\dagger\sigma^2\chi_2^*\right) \\
&= i\chi_1^\dagger\bar{\sigma}\cdot\partial\chi_1 + i\chi_2^T\bar{\sigma}^*\cdot\partial\chi_2^* + im\left(\chi_2^T\sigma^2\chi_1 - \chi_1^\dagger\sigma^2\chi_2^*\right)
\end{aligned}$$

where we have used $\sigma^2\sigma^\mu\sigma^2 = \bar{\sigma}^{\mu*}$.

Since the second term, $i\chi_2^T\bar{\sigma}^*\cdot\partial\chi_2^*$, is just a number we can simply it further using the fact that $\left(\bar{\sigma}^\mu\right)^T = -\sigma^\mu$

$$\begin{aligned}
i\chi_2^T\bar{\sigma}^{\mu*}\partial_\mu\chi_2^* &= \left(i\chi_2^T\bar{\sigma}^{\mu*}\partial_\mu\chi_2^*\right)^T \\
&= i\partial_\mu\chi_2^\dagger\bar{\sigma}^{\mu\dagger}\chi_2 \\
&= i\partial_\mu\chi_2^\dagger\bar{\sigma}^\mu\chi_2.
\end{aligned}$$

Thus, the Lagrange density is

$$\mathcal{L}_D = i\chi_1^\dagger\bar{\sigma}\cdot\partial\chi_1 + i\chi_2^\dagger\bar{\sigma}\cdot\partial\chi_2 + im\left(\chi_2^T\sigma^2\chi_1 - \chi_1^\dagger\sigma^2\chi_2^*\right).$$

Notice that if $\chi_1 = \chi_2$ we have twice the Majorana Lagrange density from part (b):

$$\begin{aligned}
\mathcal{L}_M &= i\chi_1^\dagger\bar{\sigma}\cdot\partial\chi_1 + i\chi_1^\dagger\bar{\sigma}\cdot\partial\chi_1 + im\left(\chi_1^T\sigma^2\chi_1 - \chi_1^\dagger\sigma^2\chi_1^*\right) \\
&= 2\left\{i\chi_1^\dagger\bar{\sigma}\cdot\partial\chi_1 + \frac{im}{2}\left(\chi_1^T\sigma^2\chi_1 - \chi_1^\dagger\sigma^2\chi_1^*\right)\right\}.
\end{aligned}$$

■

(d) Show that the action of part (c) has a global symmetry. Compute the divergences of the currents

$$J^\mu = \chi^\dagger\bar{\sigma}^\mu\chi \text{ and } J^\mu = \chi_1^\dagger\bar{\sigma}^\mu\chi_1 - \chi_2^\dagger\bar{\sigma}^\mu\chi_2,$$

for the theories of parts (b) and (c), respectively, and relate your results to the symmetries of these theories. Construct a theory of N free massive 2-component fermion fields with $O(N)$ symmetry (that is, the symmetry of rotations in an N-dimensional space).

Proof:

The action

$$S = \int d^4x\left[i\chi_1^\dagger\bar{\sigma}\cdot\partial\chi_1 + i\chi_2^\dagger\bar{\sigma}\cdot\partial\chi_2 + im\left(\chi_2^T\sigma^2\chi_1 - \chi_1^\dagger\sigma^2\chi_2^*\right)\right]$$

is invariant under the $U(1)$ symmetry $\psi \to e^{i\theta}\psi$.

To verify this we write

$$\begin{aligned}\psi &\rightarrow e^{i\theta}\psi \\ \begin{pmatrix}\chi_1 \\ i\sigma^2\chi_2^*\end{pmatrix} &\rightarrow \begin{pmatrix} e^{i\theta}\chi_1 \\ i\sigma^2\left(e^{-i\theta}\chi_2\right)^*\end{pmatrix}.\end{aligned}$$

Thus, in terms of the $\chi' s$ the transformation $\psi \rightarrow e^{i\theta}\psi$ is $\chi_1 \rightarrow e^{i\theta}\chi_1$ and $\chi_2 \rightarrow e^{-i\theta}\chi_2$. Because in each term of the action χ_1 is partnered with either its complex conjugate or χ_2 and vice versa for χ_2 the action is invariant under the $U(1)$ symmetry $\chi_1 \rightarrow e^{i\theta}\chi_1$ and $\chi_2 \rightarrow e^{-i\theta}\chi_2$.

We now compute the divergences of the currents,

$$\begin{aligned}J^\mu &= \chi^\dagger\bar{\sigma}^\mu\chi \\ J^\mu &= \chi_1^\dagger\bar{\sigma}^\mu\chi_1 - \chi_2^\dagger\bar{\sigma}^\mu\chi_2,\end{aligned}$$

for the theories of part (b) and (c). To derive these currents we apply equation (2.12).

Part (c): Under the $U(1)$ transformation the Lagrangian of part (c) is invariant. To compute the currents we need to know the equations of motion. The equation of motion is obtained from the Euler-Lagrange equations

$$\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\chi_i\right)}\right) - \frac{\partial\mathcal{L}}{\partial\chi_i} = 0.$$

With $\chi_i = \chi_1$ the EL equation yields

$$\begin{aligned}\partial_\mu\left(i\chi_1^\dagger\bar{\sigma}^\mu\right) - im\chi_2^T\sigma^2 &= 0 \\ \left(\partial_\mu\chi_1^\dagger\right)\bar{\sigma}^\mu - m\chi_2^T\sigma^2 &= 0 \\ \bar{\sigma}^{\mu\dagger}\left(\partial_\mu\chi_1^\dagger\right)^\dagger - m\sigma^{2\dagger}\chi_2^{T\dagger} &= 0 \\ \bar{\sigma}^\mu\partial_\mu\chi_1 - m\sigma^2\chi_2^* &= 0.\end{aligned}$$

Similarly, for $\chi_i = \chi_2$

$$\begin{aligned}\partial_\mu\left(i\chi_2^\dagger\bar{\sigma}^\mu\right) + im\frac{\partial}{\partial\chi_2}\left(\chi_2^T\sigma^2\chi_1\right) &= 0 \\ \left(\partial_\mu\chi_2^\dagger\right)\bar{\sigma}^\mu - m\frac{\partial}{\partial\chi_2}\left(\chi_1^T\sigma^2\chi_2\right) &= 0 \\ \left(\partial_\mu\chi_2^\dagger\right)\bar{\sigma}^\mu - m\chi_1^T\sigma^2 &= 0 \\ \bar{\sigma}^\mu\left(\partial_\mu\chi_2\right) - m\sigma^2\chi_1^* &= 0.\end{aligned}$$

Now the divergence of the current can be computed

$$\begin{aligned}\partial_\mu J^\mu &= \partial_\mu\left(\chi_1^\dagger\bar{\sigma}^\mu\chi_1 - \chi_2^\dagger\bar{\sigma}^\mu\chi_2\right) \\ &= \partial_\mu\chi_1^\dagger\bar{\sigma}^\mu\chi_1 - \partial_\mu\chi_2^\dagger\bar{\sigma}^\mu\chi_2 + \chi_1^\dagger\bar{\sigma}^\mu\partial_\mu\chi_1 - \chi_2^\dagger\bar{\sigma}^\mu\partial_\mu\chi_2 \\ &= \left(\bar{\sigma}^\mu\partial_\mu\chi_1\right)^\dagger\chi_1 - \left(\bar{\sigma}^\mu\partial_\mu\chi_2\right)^\dagger\chi_2 + \chi_1^\dagger\left(\bar{\sigma}^\mu\partial_\mu\chi_1\right) - \chi_2^\dagger\left(\bar{\sigma}^\mu\partial_\mu\chi_2\right) \\ &= \left(m\sigma^2\chi_2^*\right)^\dagger\chi_1 - \left(m\sigma^2\chi_1^*\right)^\dagger\chi_2 + \chi_1^\dagger\left(m\sigma^2\chi_2^*\right) - \chi_2^\dagger\left(m\sigma^2\chi_1^*\right) \\ &= m\chi_2^T\sigma^2\chi_1 - m\chi_1^T\sigma^2\chi_2 + m\chi_1^\dagger\sigma^2\chi_2^* - m\chi_2^\dagger\sigma^2\chi_1^* \\ &= 0.\end{aligned}$$

Since the divergence of the current vanishes, the current is conserved under $U(1)$ transformations.

Part (c): Under the $U(1)$ transformation the Lagrangian of part (b) is NOT invariant

$$\begin{aligned}\chi^\dagger i\bar{\sigma}\cdot\partial\chi + \frac{im}{2}\left(\chi^T\sigma^2\chi - \chi^\dagger\sigma^2\chi^*\right) &\rightarrow \left(\chi e^{i\phi}\right)^\dagger i\bar{\sigma}\cdot\partial\left(\chi e^{i\phi}\right) + \frac{im}{2}\left(\left(\chi e^{i\phi}\right)^T\sigma^2\chi e^{i\phi} - \left(\chi e^{i\phi}\right)^\dagger\sigma^2\left(\chi e^{i\phi}\right)^*\right) \\ &= \chi^\dagger i\bar{\sigma}\cdot\partial\chi + \frac{im}{2}\left(\chi^T\sigma^2\chi e^{2i\phi} - \chi^\dagger\sigma^2\chi^* e^{-2i\phi}\right) \\ &\neq \mathcal{L}.\end{aligned}$$

However, if $m = 0$ then the Lagrangian is symmetric under $U(1)$ phase rotations.

The equations of motion are

$$\begin{aligned}\bar{\sigma}^\mu\partial_\mu\chi - m\sigma^2\chi^* &= 0 \\ \left(\partial_\mu\chi^\dagger\right)\bar{\sigma}^\mu - m\chi^T\sigma^2 &= 0.\end{aligned}$$

Thus the divergence of the current is

$$\begin{aligned}\partial_\mu\left(\chi^\dagger\bar{\sigma}^\mu\chi\right) &= \left(\partial_\mu\chi^\dagger\bar{\sigma}^\mu\right)\chi + \chi^\dagger\left(\bar{\sigma}^\mu\partial_\mu\chi\right) \\ &= \left(m\chi^T\sigma^2\right)\chi + \chi^\dagger\left(m\sigma^2\chi^*\right) \\ &= m\chi^T\sigma^2\chi + m\chi^\dagger\sigma^2\chi^*.\end{aligned}$$

Like the Lagrangian the current is only conserved if $m = 0$.

For the last part of this question we construct a theory of N free massive 2-component fermion fields with $O(N)$ symmetry (the symmetry of rotations in an N-dimensional space). Each free massive particle is described by the Lagrangian of part (b)

$$\mathcal{L}_a = \chi_a^\dagger i\bar{\sigma}\cdot\partial\chi_a + \frac{im}{2}\left(\chi_a^T\sigma^2\chi_a - \chi_a^\dagger\sigma^2\chi_a^*\right)$$

where $a \in \{1, 2, ..., N\}$. The total Lagrangian is the sum of the individual Lagrangians

$$\mathcal{L} = \sum_a \chi_a^\dagger i\bar{\sigma}\cdot\partial\chi_a + \frac{im}{2}\left(\chi_a^T\sigma^2\chi_a - \chi_a^\dagger\sigma^2\chi_a^*\right).$$

Each fermion field satisfies the equation of motion

$$\begin{aligned}\bar{\sigma}^\mu\partial_\mu\chi_a - m\sigma^2\chi_a^* &= 0 \\ \left(\partial_\mu\chi_a^\dagger\right)\bar{\sigma}^\mu - m\chi_a^T\sigma^2 &= 0.\end{aligned}$$

To have $O(N)$ symmetry the Lagrangian must be invariant under rotations in N-dimensional space. Let the rotation operator be denoted by R_{ab} where a, b are the components of the operator. Applied to the b^{th} fermion field the rotation operator takes the b^{th} fermion field to the a^{th} fermion field

$$R_{ab}\chi_b = \chi_a.$$

Since R is a rotation, it is an orthogonal matrix (i.e., $R^{-1} = R^T$ and $R_{ab} \in \mathbb{R}$). Application of R to the Lagrangian yields

$$\begin{aligned}\mathcal{L} &\rightarrow \sum_{ab}\left(R_{ab}\chi_b\right)^\dagger i\bar{\sigma}\cdot\partial\left(R_{ab}\chi_b\right) + \frac{im}{2}\left(\left(R_{ab}\chi_b\right)^T\sigma^2\left(R_{ab}\chi_b\right) - \left(R_{ab}\chi_b\right)^\dagger\sigma^2\left(R_{ab}\chi_b\right)^*\right) \\ &= \sum_{ab} R_{ab}^\dagger R_{ab}\chi_b^\dagger i\bar{\sigma}\cdot\partial\chi_b + \frac{im}{2}\left(R_{ab}^T R_{ab}\chi_b^T\sigma^2\chi_b - R_{ab}^\dagger R_{ab}^*\chi_b^\dagger\sigma^2\chi_b^*\right) \\ &= \sum_{ab} R_{ba}R_{ab}\chi_b^\dagger i\bar{\sigma}\cdot\partial\chi_b + \frac{im}{2}\left(R_{ba}R_{ab}\chi_b^T\sigma^2\chi_b - R_{ba}R_{ab}\chi_b^\dagger\sigma^2\chi_b^*\right) \\ &= \sum_b \chi_b^\dagger i\bar{\sigma}\cdot\partial\chi_b + \frac{im}{2}\left(\chi_b^T\sigma^2\chi_b - \chi_b^\dagger\sigma^2\chi_b^*\right) \\ &= \mathcal{L}.\end{aligned}$$

Thus, we see that the Lagrangian for massive two-component fermions is invariant under $O(N)$ transformations.

■

(e) Quantize the Majorana theory of parts (a) and (b). That is, promote $\chi(x)$ to a quantum fields satisfying the canonical anti-commutation relation

$$\left\{\chi_a(\mathbf{x}), \chi_b^\dagger(\mathbf{y})\right\} = \delta_{ab}\delta^{(3)}(\mathbf{x}-\mathbf{y}),$$

construct a Hermitian Hamiltonian, and find a representation of the canonical commutation relations that diagonalizes the Hamiltonian in terms of a set of creation and annihilation operators. (Hint: Compare $\chi(x)$ to the top two components of the quantized Dirac field.)

Proof: The book suggests comparing χ to the upper two component of the quantized Dirac field

$$\begin{aligned}
\psi &= \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\sum_s \left(a^s_{\mathbf{p}} u^s(p)e^{-ip\cdot x} + b^{s\dagger}_{\mathbf{p}} v^s(p)e^{ip\cdot x}\right) \\
&= \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\sum_s \left(a^s_{\mathbf{p}}\begin{pmatrix}\sqrt{p\cdot\sigma}\xi^s \\ \sqrt{p\cdot\bar{\sigma}}\xi^s\end{pmatrix} e^{-ip\cdot x} + b^{s\dagger}_{\mathbf{p}}\begin{pmatrix}\sqrt{p\cdot\sigma}\eta^s \\ -\sqrt{p\cdot\bar{\sigma}}\eta^s\end{pmatrix} e^{ip\cdot x}\right).
\end{aligned}$$

Setting $\psi_R = i\sigma^2\psi_L$ in the Dirac Lagrangian yields the Majorana Lagrangian. Thus, we expect the Dirac field to be a solution to the Majorana equation under certain restrictions (i.e., $\eta^s = -i\sigma^2\xi$). Setting χ equal to ψ_L, the quantized Majorana field becomes

$$\chi = \int \frac{d^3p}{(2\pi)^3}\sqrt{\frac{p\cdot\sigma}{2E_{\mathbf{p}}}}\sum_s \left(a^s_{\mathbf{p}}\xi^s e^{-ip\cdot x} + b^{s\dagger}_{\mathbf{p}}\eta^s e^{ip\cdot x}\right).$$

The condition of charge conjugation

$$u^s(p) = -i\gamma^2\left(v^s(p)\right)^* \text{ and } v^s(p) = -i\gamma^2\left(u^s(p)\right)^*$$

places a restriction on the spinors, namely $\eta^s = -i\sigma^2\xi$. Thus, the Majorana field becomes

$$\chi = \int \frac{d^3p}{(2\pi)^3}\sqrt{\frac{p\cdot\sigma}{2E_{\mathbf{p}}}}\sum_s \left(a^s_{\mathbf{p}} i\sigma^2\eta^{s*} e^{-ip\cdot x} + b^{s\dagger}_{\mathbf{p}}\eta^s e^{ip\cdot x}\right).$$

We will now test our solution in the Majorana equation of motion $i\bar{\sigma}^\mu\partial_\mu\chi = im\sigma^2\chi^*$. The RHS becomes

$$\begin{aligned}
i\bar{\sigma}^\mu\partial_\mu\chi &= i\bar{\sigma}^\mu\partial_\mu\int \frac{d^3p}{(2\pi)^3}\sqrt{\frac{p\cdot\sigma}{2E_{\mathbf{p}}}}\sum_s \left(a^s_{\mathbf{p}} i\sigma^2\eta^{s*} e^{-ip\cdot x} + b^{s\dagger}_{\mathbf{p}}\eta^s e^{ip\cdot x}\right) \\
&= i\bar{\sigma}^\mu\int \frac{d^3p}{(2\pi)^3}\sqrt{\frac{p\cdot\sigma}{2E_{\mathbf{p}}}}\sum_s \left(a^s_{\mathbf{p}} i\sigma^2\eta^{s*}\left(-ip_\mu\right) e^{-ip\cdot x} + \left(ip_\mu\right) b^{s\dagger}_{\mathbf{p}}\eta^s e^{ip\cdot x}\right) \\
&= i\int \frac{d^3p}{(2\pi)^3}\sqrt{\bar{\sigma}\cdot p}\sqrt{\frac{\left(\bar{\sigma}\cdot p\right)\left(p\cdot\sigma\right)}{2E_{\mathbf{p}}}}\sum_s \left(a^s_{\mathbf{p}}\sigma^2\eta^{s*} e^{-ip\cdot x} + ib^{s\dagger}_{\mathbf{p}}\eta^s e^{ip\cdot x}\right) \\
&= im\int \frac{d^3p}{(2\pi)^3}\sqrt{\frac{\bar{\sigma}\cdot p}{2E_{\mathbf{p}}}}\sum_s \left(a^s_{\mathbf{p}}\sigma^2\eta^{s*} e^{-ip\cdot x} + ib^{s\dagger}_{\mathbf{p}}\eta^s e^{ip\cdot x}\right)
\end{aligned}$$

where we have used the fact that $\left(\bar{\sigma}\cdot p\right)\left(\sigma\cdot p\right) = m^2$. The LHS is

$$\begin{aligned}
im\sigma^2\chi^* &= im\sigma^2\left(\int \frac{d^3p}{(2\pi)^3}\sqrt{\frac{p\cdot\sigma}{2E_{\mathbf{p}}}}\sum_s \left(a^s_{\mathbf{p}} i\sigma^2\eta^{s*} e^{-ip\cdot x} + b^{s\dagger}_{\mathbf{p}}\eta^s e^{ip\cdot x}\right)\right)^* \\
&= im\sigma^2\int \frac{d^3p}{(2\pi)^3}\sqrt{\frac{p\cdot\sigma^*}{2E_{\mathbf{p}}}}\sum_s \left(-ia^{s\dagger}_{\mathbf{p}}\sigma^{2*}\eta^s e^{ip\cdot x} + b^s_{\mathbf{p}}\eta^{s*} e^{-ip\cdot x}\right).
\end{aligned}$$

To deal with the square root of σ^* we note that the square root is defined as a taylor series. Inserting unity in the form $\sigma^2\sigma^2$ between each consecutive term of σ^*, $\sigma^{*n} = \sigma^2\bar{\sigma}\sigma^2$ and the taylor series becomes one in $\bar{\sigma}$ instead of σ^*. With this in mind the LHS becomes

$$\begin{aligned} im\sigma^2\chi^* &= im\sigma^2 \int \frac{d^3p}{(2\pi)^3}\sigma^2\sqrt{\frac{p\cdot\bar{\sigma}}{2E_{\mathbf{p}}}}\sigma^2 \sum_s \left(-ia_{\mathbf{p}}^{s\dagger}\sigma^{2*}\eta^s e^{ip\cdot x} + b_{\mathbf{p}}^s \eta^{s*} e^{-ip\cdot x}\right). \\ &= im \int \frac{d^3p}{(2\pi)^3}\sqrt{\frac{p\cdot\bar{\sigma}}{2E_{\mathbf{p}}}} \sum_s \left(a_{\mathbf{p}}^{s\dagger} i\eta^s e^{ip\cdot x} + b_{\mathbf{p}}^s \sigma^2 \eta^{s*} e^{-ip\cdot x}\right). \end{aligned}$$

Comparing the LHS and RHS we see that they are equal only if

$$b_{\mathbf{p}}^s = a_{\mathbf{p}}^s.$$

Thus, the quantized Majorana field is

$$\chi = \int \frac{d^3p}{(2\pi)^3}\sqrt{\frac{p\cdot\sigma}{2E_{\mathbf{p}}}} \sum_s \left(a_{\mathbf{p}}^s i\sigma^2\eta^{s*} e^{-ip\cdot x} + a_{\mathbf{p}}^{s\dagger}\eta^s e^{ip\cdot x}\right).$$

From this mode expansion of the Majorana field we can see that the Majorana particle is its own anti-particle.

To determine the commutation relations of the particle operator $a_{\mathbf{p}}^s$ we simplify the commutator

$$\left\{\chi_a(\mathbf{x}), \chi_b^\dagger(\mathbf{y})\right\} = \delta_{ab}\delta^{(3)}(\mathbf{x}-\mathbf{y})$$

in terms of $a_{\mathbf{p}}^s$ commutators. Simplifying, we have

$$\begin{aligned} \delta_{ad}\delta^{(3)}(\mathbf{x}-\mathbf{y}) &= \left\{\chi_a(\mathbf{x}), \chi_d^\dagger(\mathbf{y})\right\} \\ &= \int \frac{d^3p}{(2\pi)^3}\int \frac{d^3p'}{(2\pi)^3} \sum_{sr} \left[\frac{p\cdot\sigma}{2E_{\mathbf{p}}}\right]_{ab}^{1/2} \left[\frac{p'\cdot\sigma}{2E_{\mathbf{p}'}}\right]_{ed}^{1/2} \\ &\quad \left\{\left(ia_{\mathbf{p}}^s\sigma_{bc}^2\eta_c^{s*}e^{-ip\cdot x} + a_{\mathbf{p}}^{s\dagger}\eta_b^s e^{ip\cdot x}\right), \left(-ia_{\mathbf{p}'}^{r\dagger}\sigma_{fe}^2\eta_f^r e^{ip'\cdot y} + a_{\mathbf{p}'}^r\eta_e^{r*}e^{-ip'\cdot y}\right)\right\} \\ &= \int \frac{d^3p}{(2\pi)^3}\int \frac{d^3p'}{(2\pi)^3} \sum_{sr} \left[\frac{p\cdot\sigma}{2E_{\mathbf{p}}}\right]_{ab}^{1/2} \left[\frac{p\cdot\sigma}{2E_{\mathbf{p}}}\right]_{ed}^{1/2} \\ &\quad \Big[\sigma_{bc}^2\eta_c^{s*}\sigma_{fe}^2\eta_f^r \left\{a_{\mathbf{p}}^s, a_{\mathbf{p}'}^{r\dagger}\right\} e^{-ip\cdot x+ip'\cdot y} + i\sigma_{bc}^2\eta_c^{s*}\eta_e^{r*}\left\{a_{\mathbf{p}}^s, a_{\mathbf{p}'}^r\right\} e^{-ip\cdot x-ip'\cdot y} \\ &\quad -i\eta_b^s\sigma_{fe}^2\eta_f^r\left\{a_{\mathbf{p}}^{s\dagger}, a_{\mathbf{p}'}^{r\dagger}\right\} e^{ip\cdot x+ip'\cdot y} + \eta_b^s\eta_e^{r*}\left\{a_{\mathbf{p}}^{s\dagger}, a_{\mathbf{p}'}^r\right\} e^{ip\cdot x-ip'\cdot y}\Big]. \end{aligned}$$

This is very complicated and it is not obvious what commutation relations for $a_{\mathbf{p}}^s$ yields the RHS. However, since we are using a modified Dirac field we would expect that $a_{\mathbf{p}}^s$ satisfies the usual relations

$$\begin{aligned} \left\{a_{\mathbf{p}}^s, a_{\mathbf{p}'}^r\right\} &= 0 = \left\{a_{\mathbf{p}}^{s\dagger}, a_{\mathbf{p}'}^{r\dagger}\right\} \\ \left\{a_{\mathbf{p}}^s, a_{\mathbf{p}'}^{r\dagger}\right\} &= (2\pi)^3\delta_{rs}\delta(\mathbf{p}-\mathbf{p}'). \end{aligned}$$

Lets test if these commutation relations are indeed the ones we require

$$\begin{aligned}
\left\{\chi_a(\mathbf{x}),\chi_d^\dagger(\mathbf{y})\right\} &= \int\frac{d^3p}{(2\pi)^3}\int\frac{d^3p'}{(2\pi)^3}\sum_{sr}\left[\frac{p\cdot\sigma}{2E_{\mathbf{p}}}\right]_{ab}^{1/2}\left[\frac{p\cdot\sigma}{2E_{\mathbf{p}}}\right]_{ed}^{1/2}\\
&\quad\times\left[\sigma_{bc}^2\eta_c^{s*}\sigma_{fe}^2\eta_f^r\left\{a_{\mathbf{p}}^s,a_{\mathbf{p}'}^{r\dagger}\right\}e^{-ip\cdot x+ip'\cdot y}+\eta_b^s\eta_e^{r*}\left\{a_{\mathbf{p}}^{s\dagger},a_{\mathbf{p}'}^r\right\}e^{ip\cdot x-ip'\cdot y}\right]\\
&= \int\frac{d^3p}{(2\pi)^3}\int d^3p'\sum_{sr}\left[\frac{p\cdot\sigma}{2E_{\mathbf{p}}}\right]_{ab}^{1/2}\left[\frac{p\cdot\sigma}{2E_{\mathbf{p}}}\right]_{ed}^{1/2}\\
&\quad\times\delta_{rs}\delta(\mathbf{p}-\mathbf{p}')\left[\sigma_{bc}^2\eta_c^{s*}\sigma_{fe}^2\eta_f^r e^{-ip\cdot x+ip'\cdot y}+\eta_b^s\eta_e^{r*}e^{ip\cdot x-ip'\cdot y}\right]\\
&= \int\frac{d^3p}{(2\pi)^3}\sum_{s}\left[\frac{p\cdot\sigma}{2E_{\mathbf{p}}}\right]_{ab}^{1/2}\left[\frac{p\cdot\sigma}{2E_{\mathbf{p}}}\right]_{ed}^{1/2}\\
&\quad\times\left[\sigma_{bc}^2\eta_c^{s*}\sigma_{fe}^2\eta_f^s e^{-i\left(E_{\mathbf{p}}x^0-\mathbf{p}\cdot\mathbf{x}\right)+i\left(E_{\mathbf{p}}y^0-\mathbf{p}\cdot\mathbf{y}\right)}+\eta_b^s\eta_e^{s*}e^{i\left(E_{\mathbf{p}}x^0-\mathbf{p}\cdot\mathbf{x}\right)-i\left(E_{\mathbf{p}}y^0-\mathbf{p}\cdot\mathbf{y}\right)}\right]
\end{aligned}$$

When taken at equal times the commutator becomes

$$\begin{aligned}
\left\{\chi_a(\mathbf{x}),\chi_d^\dagger(\mathbf{y})\right\} &= \int\frac{d^3p}{(2\pi)^3}\sum_{s}\left[\frac{p\cdot\sigma}{2E_{\mathbf{p}}}\right]_{ab}^{1/2}\left[\frac{p\cdot\sigma}{2E_{\mathbf{p}}}\right]_{ed}^{1/2}\left[\sigma_{bc}^2\eta_c^{s*}\sigma_{fe}^2\eta_f^r e^{i\mathbf{p}\cdot\mathbf{x}-i\mathbf{p}\cdot\mathbf{y}}+\eta_b^s\eta_e^{s*}e^{-i\mathbf{p}\cdot\mathbf{x}+i\mathbf{p}\cdot\mathbf{y}}\right]\\
&= \int\frac{d^3p}{(2\pi)^3}\sum_{s}\left[\left[\frac{p\cdot\sigma}{2E_{\mathbf{p}}}\right]_{ab}^{1/2}\sigma_{bc}^2\eta_c^{s*}\eta_f^s\sigma_{fe}^2\left[\frac{p\cdot\sigma}{2E_{\mathbf{p}}}\right]_{ed}^{1/2}e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}+\left[\frac{p\cdot\sigma}{2E_{\mathbf{p}}}\right]_{ab}^{1/2}\eta_b^s\eta_e^{s*}\left[\frac{p\cdot\sigma}{2E_{\mathbf{p}}}\right]_{ed}^{1/2}e^{-i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}\right]
\end{aligned}$$

To simplify this further we sum over $s$and use the identity,

$$\sum_s\eta_a^{s*}\eta_b^s=\delta_{ab},$$

to get

$$\begin{aligned}
\left\{\chi_a(\mathbf{x}),\chi_d^\dagger(\mathbf{y})\right\} &= \int\frac{d^3p}{(2\pi)^3}\left[\left[\frac{p\cdot\sigma}{2E_{\mathbf{p}}}\right]_{ab}^{1/2}\sigma_{bc}^2\delta_{cf}\sigma_{fe}^2\left[\frac{p\cdot\sigma}{2E_{\mathbf{p}}}\right]_{ed}^{1/2}e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}+\left[\frac{p\cdot\sigma}{2E_{\mathbf{p}}}\right]_{ab}^{1/2}\delta_{be}\left[\frac{p\cdot\sigma}{2E_{\mathbf{p}}}\right]_{ed}^{1/2}e^{-i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}\right]\\
&= \int\frac{d^3p}{(2\pi)^3}\left[\left[\frac{p\cdot\sigma}{2E_{\mathbf{p}}}\right]_{ad}e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}+\left[\frac{p\cdot\sigma}{2E_{\mathbf{p}}}\right]_{ad}e^{-i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}\right]\\
&= \int\frac{d^3p}{(2\pi)^3}\left[\left[\frac{p\cdot\sigma}{2E_{\mathbf{p}}}\right]_{ad}e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}+\left[\frac{p\cdot\sigma}{2E_{\mathbf{p}}}\right]_{ad}e^{-i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}\right].
\end{aligned}$$

To simplify further we need to know $(p\cdot\sigma)_{ad}$. Here, we work out the matrix elements of $p\cdot\sigma$,

$$\begin{aligned}
(p\cdot\sigma)_{ad} &= (E_{\mathbf{p}}-\mathbf{p}\cdot\boldsymbol{\sigma})_{ad}\\
&= E_{\mathbf{p}}\delta_{ad}-p^i\cdot\sigma_{ad}^i.
\end{aligned}$$

Substitution into the commutation relation yields

$$\begin{aligned}
\left\{\chi_a(\mathbf{x}),\chi_d^\dagger(\mathbf{y})\right\} &= \int\frac{d^3p}{(2\pi)^3}\left[\frac{E_{\mathbf{p}}\delta_{ad}-p^i\cdot\sigma_{ad}^i}{2E_{\mathbf{p}}}e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}+\frac{E_{\mathbf{p}}\delta_{ad}-p^i\cdot\sigma_{ad}^i}{2E_{\mathbf{p}}}e^{-i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}\right]\\
&= \int\frac{d^3p}{(2\pi)^3}\left[\frac{\delta_{ad}}{2}e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}+\frac{\delta_{ad}}{2}e^{-i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}\right]-\int\frac{d^3p}{(2\pi)^3}\left[\frac{p^i\cdot\sigma_{ad}^i}{2E_{\mathbf{p}}}e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}+\frac{p^i\cdot\sigma_{ad}^i}{2E_{\mathbf{p}}}e^{-i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}\right]\\
&= \delta_{ad}\delta(\mathbf{x}-\mathbf{y})
\end{aligned}$$

where the second integral is odd in $\mathbf{p}$.

We now turn to calculating the Hamiltonian for Majorana fields. The Hamiltonian is the Legendre transform of the Lagrangian,

$$\begin{aligned} H &= \int d^3x \left(\pi\dot{\chi} - \mathcal{L}\right) \\ &= \int d^3x \left(i\chi^\dagger\dot{\chi} - i\chi^\dagger\bar{\sigma}\cdot\partial\chi + \frac{im}{2}\left(\chi^T\sigma^2\chi - \chi^\dagger\sigma^2\chi^*\right)\right) \end{aligned}$$

where

$$\begin{aligned} \mathcal{L} &= i\chi^\dagger\bar{\sigma}\cdot\partial\chi + \frac{im}{2}\left(\chi^T\sigma^2\chi - \chi^\dagger\sigma^2\chi^*\right), \\ \pi &= \frac{\partial\mathcal{L}}{\partial\left(\dot{\chi}\right)} = i\chi^\dagger. \end{aligned}$$

The simplification is straight forward and messy.

We just note the result here

$$H = \int \frac{d^3p}{(2\pi)^3} E_{\mathbf{p}} \sum_s a_{\mathbf{p}}^{s\dagger} a_{\mathbf{p}}^s.$$

Which is exactly 1/2 the Dirac Hamiltonian,

$$H_D = \int \frac{d^3p}{(2\pi)^3} E_{\mathbf{p}} \sum_s \left(a_{\mathbf{p}}^{s\dagger} a_{\mathbf{p}}^s + b_{\mathbf{p}}^{s\dagger} b_{\mathbf{p}}^s\right),$$

with the restriction $b_{\mathbf{p}}^s = a_{\mathbf{p}}^s$.

We simplify the Hamiltonian term by term. The first term is

$$\begin{aligned} i\chi^\dagger\dot{\chi} &= i\int \frac{d^3p}{(2\pi)^3} \int \frac{d^3p'}{(2\pi)^3} \sum_{rs} \left(-ia_{\mathbf{p}}^s \eta^{sT}\sigma^2 e^{ip\cdot x} + a_{\mathbf{p}}^{s\dagger}\eta^{s\dagger} e^{-ip\cdot x}\right) \sqrt{\frac{p\cdot\sigma}{2E_{\mathbf{p}}}}\partial_0\sqrt{\frac{p'\cdot\sigma}{2E_{\mathbf{p}'}}} \left(a_{\mathbf{p}'}^r i\sigma^2\eta^{r*} e^{-ip'\cdot x} + a_{\mathbf{p}'}^{r\dagger}\eta^r e^{ip'\cdot x}\right) \\ &= i\int \frac{d^3p}{(2\pi)^3} \int \frac{d^3p'}{(2\pi)^3} E_{\mathbf{p}'} \sum_{rs} \left(-ia_{\mathbf{p}}^s \eta^{sT}\sigma^2 e^{ip\cdot x} + a_{\mathbf{p}}^{s\dagger}\eta^{s\dagger} e^{-ip\cdot x}\right) \sqrt{\frac{p\cdot\sigma}{2E_{\mathbf{p}}}}\sqrt{\frac{p'\cdot\sigma}{2E_{\mathbf{p}'}}} \left(a_{\mathbf{p}'}^r \sigma^2\eta^{r*} e^{-ip'\cdot x} + ia_{\mathbf{p}'}^{r\dagger}\eta^r e^{ip'\cdot x}\right) \end{aligned}$$

■

1.5 Supersymmetry

It is possible to write field theories with continuous symmetries linking fermions and bosons; such transformations are called super-symmetries.

(a) The simplest example of a supersymmetric field theory is the theory of a free complex boson and a free Weyl fermion, written in the form

$$\mathcal{L} = \partial_\mu\phi^*\partial^\mu\phi + \chi^\dagger i\bar{\sigma}\cdot\partial\chi + F^*F.$$

Here F is an auxiliary complex scalar field whose field equation is $F = 0$. Show that this Lagrangian is invariant (up to a total divergence) under the infinitesimal transformation

$$\begin{aligned} \delta\phi &= -i\epsilon^T\sigma^2\chi, \\ \delta\chi &= \epsilon F - \sigma\cdot\partial\phi\sigma^2\epsilon^*, \\ \delta F &= -i\epsilon^\dagger\bar{\sigma}\cdot\partial\chi, \end{aligned}$$

where the parameter ϵ_a is a 2-component spinor of Grassmann numbers.

(b) Show that the term

$$\Delta\mathcal{L} = \left[m\phi F + \frac{1}{2}im\chi^T\sigma^2\chi\right] + \text{(complex conjugate)}$$

is also left invariant by the transformation given in part (a). Eliminate F from the complete Lagrangian $\mathcal{L}+\Delta\mathcal{L}$ by solving its field equation, and show that the fermion and boson fields ϕ and $\chi$are given the same mass.

(c) It is possible to write supersymmetric nonlinear field equations by adding cubic and higher-order terms to the Lagrangian. Show that the following rather general field theory, containing the field (ϕ_i, χ_i), $i = 1, ..., n$, is supersymmetric:

$$\begin{aligned}\mathcal{L} &= \partial_\mu\phi_i^*\partial^\mu\phi_i + \chi_i^\dagger i\bar{\sigma}\cdot\partial\chi_i + F_i^*F_i \\ &\quad + \left(F_i\frac{\partial W[\phi]}{\partial\phi_i} + \frac{i}{2}\frac{\partial^2 W[\phi]}{\partial\phi_i\partial\phi_j}\chi_i^T\sigma^2\chi_j + c.c\right),\end{aligned}$$

where $W[\phi]$ is an arbitrary function of the ϕ_i, called the super-potential. For the simple case $n = 1$ and $W = g\phi^3/3$, write out the field equations for ϕ and χ (after elimination of F).

1.6 Fierz transformations ✓

Let u_i, $i = 1, ...4$, be four 4-component Dirac spinors. In the text, we proved the Fierz rearrangement formulae (3.78) and (3.79). The first of these formulae can be written in 4-component notation as

$$\bar{u}_1\gamma^\mu\left(\frac{1+\gamma^5}{2}\right)u_2\bar{u}_3\gamma_\mu\left(\frac{1+\gamma^5}{2}\right)u_4 = -\bar{u}_1\gamma^\mu\left(\frac{1+\gamma^5}{2}\right)u_4\bar{u}_3\gamma_\mu\left(\frac{1+\gamma^5}{2}\right)u_2.$$

The above identity plays an important part in the weak interaction. Thus, we derive this equation from (3.78) and (3.79)

$$\begin{aligned}(\bar{u}_{1R}\sigma^\mu u_{2R})(\bar{u}_{3R}\sigma_\mu u_{4R}) &= -(\bar{u}_{1R}\sigma^\mu u_{4R})(\bar{u}_{3R}\sigma_\mu u_{2R}) \\ (\bar{u}_{1L}\bar{\sigma}^\mu u_{2L})(\bar{u}_{3L}\bar{\sigma}_\mu u_{4L}) &= -(\bar{u}_{3L}\bar{\sigma}^\mu u_{4L})(\bar{u}_{1L}\bar{\sigma}_\mu u_{2L}).\end{aligned}$$

Denoting the handedness operators by $P_R = \frac{1+\gamma^5}{2}$ and $P_L = \frac{1-\gamma^5}{2}$ we have the following:

$$\begin{aligned}(\bar{u}_1\gamma^\mu P_R u_2)(\bar{u}_3\gamma_\mu P_R u_4) &= (\bar{u}_{1R}\gamma^\mu u_{2R})(\bar{u}_{3R}\gamma_\mu u_{4R}) \\ &= (\bar{u}_{1R}\sigma^\mu u_{2R})(\bar{u}_{3R}\sigma_\mu u_{4R}) \\ &= -(\bar{u}_{1R}\sigma^\mu u_{4R})(\bar{u}_{3R}\sigma_\mu u_{2R}) \\ &= -(\bar{u}_1\gamma_\mu P_R u_4)(\bar{u}_3\gamma_\mu P_R u_2)\end{aligned}$$

$$(\bar{u}_1\gamma^\mu P_L u_2)(\bar{u}_3\gamma_\mu P_L u_4) = -(\bar{u}_{3L}\gamma_\mu P_L u_{4L})(\bar{u}_{1L}\gamma_\mu P_L u_{2L}).$$

In fact, there are similar rearrangement formulae for any product

$$\left(\bar{u}_1\Gamma^A u_2\right)\left(\bar{u}_3\Gamma^B u_4\right),$$

where Γ^A, Γ^B are any of the 16 combinations of Dirac matrices listed in Section 3.4.

(a) To begin, normalize the 16 matrices Γ^A to the convention

$$\text{Tr}\left[\Gamma^A\Gamma^B\right] = 4\delta^{AB}.$$

This gives $\Gamma^A = \left\{1, \gamma^0, i\gamma^j, ...\right\}$; write all 16 elements of this set.

Proof: We already know that the 16 Dirac matrices are orthogonal subject to the inner product $\text{Tr}[AB]$. Thus, we only have to find the normalization constant.

1. Scalars ($\Gamma^A = 1$): $\mathrm{Tr}\,[1] = 4$ thus, the identity is properly normalized.
2. Vectors ($\Gamma^A = \gamma^\mu$):

$$\mathrm{Tr}\left[\left(\gamma^0\right)^2\right] = \mathrm{Tr}\,[1] = 4$$

$$\mathrm{Tr}\left[\left(\gamma^i\right)^2\right] = \mathrm{Tr}\left[\begin{pmatrix} -\left(\sigma^i\right)^2 & 0 \\ 0 & -\left(\sigma^i\right)^2 \end{pmatrix}\right] = -\mathrm{Tr}\,[1] - 4$$

Thus, the normalized vector matrices are: γ^0 and $i\gamma^i$.

3. Tensor ($\Gamma^A = \sigma^{\mu\nu}$):

$$\begin{aligned}
\mathrm{Tr}\left[(\sigma^{\mu\nu})^2\right] &= \left(\frac{i}{2}\right)^2 \mathrm{Tr}\left[([\gamma^\mu, \gamma^\nu])^2\right] \\
&= -\frac{1}{4}\mathrm{Tr}\left[\gamma^\mu\gamma^\nu\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu\gamma^\nu\gamma^\mu\right] \\
&= -\frac{1}{4}\left\{2\mathrm{Tr}\left[\gamma^\mu\gamma^\nu\gamma^\mu\gamma^\nu\right] - 2\mathrm{Tr}\left[\gamma^\nu\gamma^\mu\gamma^\mu\gamma^\nu\right]\right\} \\
&= -\frac{1}{4}\left\{2\mathrm{Tr}\left[\gamma^\mu\gamma^\nu\gamma^\mu\gamma^\nu\right] + 2\mathrm{Tr}\left[\gamma^\mu\gamma^\nu\gamma^\mu\gamma^\nu\right] - 4g^{\mu\nu}\mathrm{Tr}\left[\gamma^\mu\gamma^\nu\right]\right\} \\
&= -\frac{1}{4}\left\{4\mathrm{Tr}\left[\gamma^\mu\gamma^\nu\gamma^\mu\gamma^\nu\right] - 16g^{\mu\nu}g^{\mu\nu}\right\} \\
&= -\frac{1}{4}\left\{16\left[g^{\mu\nu}g^{\mu\nu} - g^{\mu\mu}g^{\nu\nu} + g^{\mu\nu}g^{\mu\nu}\right] - 16g^{\mu\nu}g^{\mu\nu}\right\} \\
&= -\frac{1}{4}16\left[g^{\mu\nu}g^{\mu\nu} - g^{\mu\mu}g^{\nu\nu}\right] \\
&= -4\left\{g^{\mu\nu}g^{\mu\nu} - g^{\mu\mu}g^{\nu\nu}\right\} \\
&= \begin{cases} 0 & \mu = \nu \\ 4g^{\mu\mu}g^{\nu\nu} & \mu \neq \nu \end{cases} \\
&= \begin{cases} 0 & \mu = \nu \\ 4 & \mu \neq \nu \text{ and both } \mu, \nu \text{ are spacelike or timelike} \\ -4 & \mu \neq \nu \text{ and } \mu \text{ is spacelike (timelike) and } \nu \text{ is timelike (spacelike)} \end{cases}
\end{aligned}$$

Thus, the normalized tensor matrices are: $-\sigma^{0i}, -\sigma^{i0}, \sigma^{00}, \sigma^{ii}$.

4. Pseudo-vector ($\Gamma^A = \gamma^\mu\gamma^5$):

$$\begin{aligned}
\mathrm{Tr}\left[\left(\gamma^\mu\gamma^5\right)\right] &= \mathrm{Tr}\left[\gamma^\mu\gamma^5\gamma^\mu\gamma^5\right] \\
&= \mathrm{Tr}\left[\gamma^\mu\gamma^5\gamma^5\gamma^\mu\right] \\
&= \mathrm{Tr}\left[\gamma^\mu\gamma^\mu\right] \\
&= 4g^{\mu\mu} \\
&= \begin{cases} -4 & \mu = 0 \\ 4 & \mu = i \end{cases}
\end{aligned}$$

Thus, the normalized pseudo-vector matrices are: $-\gamma^0\gamma^5, \gamma^i\gamma^5$.

5. Pseudo-scalar ($\Gamma^A = \gamma^5$):

$$\mathrm{Tr}\left[\gamma^5\gamma^5\right] = 1$$

Thus, γ^5 is already properly normalized.

■

(b) Write the general Fierz identity as an equation

$$\left(\bar{u}_1\Gamma^A u_2\right)\left(\bar{u}_3\Gamma^B u_4\right) = \sum_{C,D} C^{AB}_{\ \ CD}\left(\bar{u}_1\Gamma^C u_4\right)\left(\bar{u}_3\Gamma^D u_2\right)$$

with unknown coefficients $C^{AB}_{\ \ CD}$. Using the completeness of the 16 Γ^A matrices, show that

$$C^{AB}_{\ \ CD} = \frac{1}{16}\mathrm{Tr}\left[\Gamma^C\Gamma^A\Gamma^D\Gamma^B\right].$$

Proof:

Using the completeness of the Gamma matrices we write

$$\Gamma^A_{ab}\Gamma^B_{cd} = \sum_C C^{AB}_{\ \ C}\Gamma^C_{ad}\sum_D C^{AB}_{\ \ D}\Gamma^D_{bc} = \sum_{CD} C^{AB}_{\ \ CD}\Gamma^C_{ad}\Gamma^D_{bc}$$

from which it follows

$$\left(\bar{u}_1\Gamma^A u_2\right)\left(\bar{u}_3\Gamma^B u_4\right) = \sum_{C,D} C^{AB}_{\ \ CD}\left(\bar{u}_1\Gamma^C u_4\right)\left(\bar{u}_3\Gamma^D u_2\right).$$

We now calculate $C^{AB}_{\ \ C'D'}$. Multiplying by $\Gamma^{C'}_{da}\Gamma^{D'}_{bc}$ we have

$$\Gamma^{C'}_{da}\Gamma^{D'}_{bc}\Gamma^A_{ab}\Gamma^B_{cd} = \sum_{C,D} C^{AB}_{\ \ CD}\Gamma^C_{da}\Gamma^D_{bc}\Gamma^C_{ad}\Gamma^D_{bc}$$

which implies

$$\begin{aligned}\mathrm{Tr}\left[\Gamma^{C'}\Gamma^A\Gamma^{D'}\Gamma^B\right] &= \sum_{C,D} C^{AB}_{\ \ CD}\mathrm{Tr}\left[\Gamma^{C'}\Gamma^C\right]\mathrm{Tr}\left[\Gamma^{D'}\Gamma^D\right] \\ &= 16\sum_{C,D} C^{AB}_{\ \ CD}\delta^{CC'}\delta^{DD'} \\ &= 16C^{AB}_{\ \ C'D'}.\end{aligned}$$

Thus, we see that $C^{AB}_{\ \ CD} = \frac{1}{16}\mathrm{Tr}\left[\Gamma^C\Gamma^A\Gamma^D\Gamma^B\right]$.

■

(c) Work out explicitly the Fierz transformation laws for the products $(\bar{u}_1u_2)(\bar{u}_3u_4)$ and $(\bar{u}_1\gamma^\mu u_2)(\bar{u}_3\gamma_\mu u_4)$.

Proof:

First, the product $(\bar{u}_1u_2)(\bar{u}_3u_4) = \left(\bar{u}_1\Gamma^A u_2\right)\left(\bar{u}_3\Gamma^B u_4\right)$ where $\Gamma^{A,B} = 1$. Thus, we need all nonzero traces of the form

$$\begin{aligned}C^{11}_{\ \ CD} &= \frac{1}{16}\mathrm{Tr}\left[\Gamma^C 1\Gamma^D 1\right] \\ &= \frac{1}{16}\mathrm{Tr}\left[\Gamma^C\Gamma^D\right] \\ &= \frac{1}{4}\delta_{CD}.\end{aligned}$$

Thus, the product rule for $\Gamma^{A,B} = 1$ becomes

$$\begin{aligned}(\bar{u}_1u_2)(\bar{u}_3u_4) &= \sum_{C,D} C^{AB}_{\ \ CD}\left(\bar{u}_1\Gamma^C u_4\right)\left(\bar{u}_3\Gamma^D u_2\right) \\ &= \sum_{C,D}\frac{1}{4}\delta_{CD}\left(\bar{u}_1\Gamma^C u_4\right)\left(\bar{u}_3\Gamma^D u_2\right) \\ &= \frac{1}{4}\sum_C\left(\bar{u}_1\Gamma^C u_4\right)\left(\bar{u}_3\Gamma^C u_2\right)\end{aligned}$$

where the index C cycles through all 16 Γ^C.

Next, we examine the product $(\bar{u}_1\gamma^\mu u_2)(\bar{u}_3\gamma_\mu u_4)$. The coefficients C^{AB}_{CD} are determined as usual

$$C^{\mu\mu}_{CD} = \frac{1}{16}\mathrm{Tr}\left[\Gamma^C\gamma^\mu\Gamma^D\gamma_\mu\right].$$

Using the matrices from our normalized basis of part (a) we define $\Gamma^\mu = \left(\gamma^0, i\boldsymbol{\gamma}\right)$. In terms of Γ^μ we have

$$\begin{aligned}
C^{\mu\mu}_{CD} &= \frac{1}{16}\left\{\mathrm{Tr}\left[\Gamma^C\Gamma^0\Gamma^D\Gamma^0\right] - \sum_{i=1}^{3}\mathrm{Tr}\left[\Gamma^C\left(-i\Gamma^i\right)\Gamma^D\left(-i\Gamma^i\right)\right]\right\} \\
&= \frac{1}{16}\left\{\mathrm{Tr}\left[\Gamma^C\Gamma^0\Gamma^D\Gamma^0\right] + \sum_{i=1}^{3}\mathrm{Tr}\left[\Gamma^C\Gamma^i\Gamma^D\Gamma^i\right]\right\}.
\end{aligned}$$

Suppose that $\Gamma^C = \gamma^0$ then

$$\begin{aligned}
C^{\mu\mu}_{0D} &= \frac{1}{16}\left\{\mathrm{Tr}\left[\Gamma^0\Gamma^0\Gamma^D\Gamma^0\right] + \sum_{i=1}^{3}\mathrm{Tr}\left[\Gamma^0\Gamma^i\Gamma^D\Gamma^i\right]\right\} \\
&= \frac{1}{16}\left\{\mathrm{Tr}\left[\Gamma^D\Gamma^0\right] - \sum_{i=1}^{3}\mathrm{Tr}\left[\Gamma^0\Gamma^D\right]\right\} \\
&= -\delta^{D0}.
\end{aligned}$$

Similarly, if $\Gamma^C = \gamma^j$ then

$$\begin{aligned}
C^{\mu\mu}_{jD} &= \frac{1}{16}\left\{\mathrm{Tr}\left[\Gamma^j\Gamma^0\Gamma^D\Gamma^0\right] + \sum_{i=1}^{3}\mathrm{Tr}\left[\Gamma^j\Gamma^i\Gamma^D\Gamma^i\right]\right\} \\
&= \frac{1}{16}\left\{-\mathrm{Tr}\left[\Gamma^j\Gamma^D\right] + \sum_{i=1}^{3}\mathrm{Tr}\left[\left(-2g^{ij} - \Gamma^i\Gamma^j\right)\Gamma^D\Gamma^i\right]\right\} \\
&= \frac{1}{16}\left\{-4\delta^{Dj} - 2\sum_{i=1}^{3}g^{ij}\mathrm{Tr}\left[\Gamma^D\Gamma^i\right] - \sum_{i=1}^{3}\mathrm{Tr}\left[\Gamma^i\Gamma^j\Gamma^D\Gamma^i\right]\right\} \\
&= \frac{1}{16}\left\{-4\delta^{Dj} - 2\mathrm{Tr}\left[\Gamma^D\Gamma^j\right] - \sum_{i=1}^{3}\mathrm{Tr}\left[\Gamma^j\Gamma^D\right]\right\} \\
&= \frac{1}{16}\left\{-4\delta^{Dj} - 8\delta^{Dj} - 12\delta^{Dj}\right\} \\
&= -\frac{24}{16}\delta^{Dj} \\
&= -\frac{3}{2}\delta^{Dj}
\end{aligned}$$

Thus, we see that for $\Gamma^C = \gamma^\nu$ we must have $\Gamma^D = \gamma^\nu$ and vice versa.

For $\Gamma^{C,D} = 1$ we have $C^{\mu\mu}_{CD} = 0$.

The remaining matrices are the anti-symmetric matrices form the set $\Gamma = \{\sigma^{\mu\nu}, \gamma^\mu\gamma^5, \gamma^5\}$we have for: $\Gamma^C = \gamma^5$

$$\begin{aligned}
C^{\mu\mu}_{\ \ 5D} &= \frac{1}{16}\left\{\mathrm{Tr}\left[\Gamma^5\Gamma^0\Gamma^D\Gamma^0\right] + \sum_{i=1}^{3}\mathrm{Tr}\left[\Gamma^5\Gamma^i\Gamma^D\Gamma^i\right]\right\} \\
&= \frac{1}{16}\left\{-\mathrm{Tr}\left[\Gamma^5\Gamma^D\right] - \sum_{i=1}^{3}\mathrm{Tr}\left[\Gamma^5\Gamma^D\right]\right\} \\
&= -\delta^{D5}
\end{aligned}$$

for: $\Gamma^C = \gamma^0\gamma^5$

$$\begin{aligned}
C^{\mu\mu}_{\ \ \nu5,D} &= \frac{1}{16}\left\{\mathrm{Tr}\left[\Gamma^{05}\Gamma^0\Gamma^D\Gamma^0\right] + \sum_{i=1}^{3}\mathrm{Tr}\left[\Gamma^{05}\Gamma^i\Gamma^D\Gamma^i\right]\right\} \\
&= \frac{1}{16}\left\{\mathrm{Tr}\left[-\gamma^0\gamma^5\gamma^0\Gamma^D\gamma^0\right] + \sum_{i=1}^{3}\mathrm{Tr}\left[\left(-\gamma^0\gamma^5\right)\left(i\gamma^i\right)\Gamma^D\Gamma^i\right]\right\} \\
&= \frac{1}{16}\left\{-\mathrm{Tr}\left[\gamma^5\gamma^0\Gamma^D\right] + \sum_{i=1}^{3}\mathrm{Tr}\left[\left(i\gamma^i\right)\left(-\gamma^0\gamma^5\right)\Gamma^D\Gamma^i\right]\right\} \\
&= \frac{1}{16}\left\{\mathrm{Tr}\left[\Gamma^{05}\Gamma^D\right] + \sum_{i=1}^{3}\mathrm{Tr}\left[\Gamma^{05}\Gamma^D\right]\right\} \\
&= \frac{1}{16}16\delta^{05,D} \\
&= \delta^{05,D}
\end{aligned}$$

for: $\Gamma^C = \gamma^i\gamma^5$

$$\begin{aligned}
C^{\mu\mu}_{\ \ j5,D} &= \frac{1}{16}\left\{\mathrm{Tr}\left[\Gamma^{j5}\Gamma^0\Gamma^D\Gamma^0\right] + \sum_{i=1}^{3}\mathrm{Tr}\left[\Gamma^{j5}\Gamma^i\Gamma^D\Gamma^i\right]\right\} \\
&= \frac{1}{16}\left\{\mathrm{Tr}\left[\Gamma^0\Gamma^{j5}\Gamma^D\Gamma^0\right] + \sum_{i=1}^{3}\mathrm{Tr}\left[\left(i\gamma^j\gamma^5\right)\left(i\gamma^i\right)\Gamma^D\Gamma^i\right]\right\} \\
&= \frac{1}{16}\left\{\mathrm{Tr}\left[\Gamma^{j5}\Gamma^D\right] - \sum_{i=1}^{3}\mathrm{Tr}\left[\left(i\gamma^j\left(i\gamma^i\right)\gamma^5\right)\Gamma^D\Gamma^i\right]\right\} \\
&= \frac{1}{16}\left\{4\delta^{j5,D} - \sum_{i=1}^{3}\mathrm{Tr}\left[\left(i^2 2g^{ij} - \left(i\gamma^i\right)i\gamma^j\right)\gamma^5\Gamma^D\Gamma^i\right]\right\} \\
&= \frac{1}{16}\left\{4\delta^{j5,D} + 2\sum_{i=1}^{3}\mathrm{Tr}\left[g^{ij}\gamma^5\Gamma^D\Gamma^i\right] + \sum_{i=1}^{3}\mathrm{Tr}\left[\left(i\gamma^i\right)i\gamma^j\gamma^5\Gamma^D\Gamma^i\right]\right\} \\
&= \frac{1}{16}\left\{4\delta^{j5,D} + 2\mathrm{Tr}\left[\gamma^5\Gamma^D\Gamma^j\right] + \sum_{i=1}^{3}\mathrm{Tr}\left[\Gamma^{j5}\Gamma^D\right]\right\} \\
&= \frac{1}{16}\left\{4\delta^{j5,D} + 2\mathrm{Tr}\left[\Gamma^D\Gamma^{j5}\right] + \sum_{i=1}^{3}\mathrm{Tr}\left[\Gamma^{j5}\Gamma^D\right]\right\} \\
&= \frac{24}{16}\delta^{j5,D} \\
&= \frac{3}{2}\delta^{j5,D}
\end{aligned}$$

for: $\Gamma^C = \Gamma^{\mu\nu} = -\sigma^{0i}, -\sigma^{i0}, \sigma^{00}, \sigma^{ii}$

$$\begin{aligned}
\{\sigma^{\mu\nu}, \gamma^{\alpha}\} &= \frac{i}{2}\{\gamma^{\mu}\gamma^{\nu} - \gamma^{\nu}\gamma^{\mu}, \gamma^{\alpha}\} \\
&= \frac{i}{2}\{\gamma^{\mu}\gamma^{\nu}, \gamma^{\alpha}\} - \frac{i}{2}\{\gamma^{\nu}\gamma^{\mu}, \gamma^{\alpha}\} \\
&= \frac{i}{2}\{\gamma^{\mu}, \gamma^{\alpha}\}\gamma^{\nu} + \frac{i}{2}\gamma^{\mu}[\gamma^{\nu}, \gamma^{\alpha}] - \frac{i}{2}\{\gamma^{\nu}, \gamma^{\alpha}\}\gamma^{\mu} - \frac{i}{2}\gamma^{\nu}[\gamma^{\mu}, \gamma^{\alpha}] \\
&= ig^{\mu\alpha}\gamma^{\nu} + i\gamma^{\mu}(\gamma^{\nu}\gamma^{\alpha} - g^{\nu\alpha}) - ig^{\nu\alpha}\gamma^{\mu} - i\gamma^{\nu}(\gamma^{\mu}\gamma^{\alpha} - g^{\mu\alpha}) \\
&= ig^{\mu\alpha}\gamma^{\nu} + i\gamma^{\mu}\gamma^{\nu}\gamma^{\alpha} - i\gamma^{\mu}g^{\nu\alpha} - ig^{\nu\alpha}\gamma^{\mu} - i\gamma^{\nu}\gamma^{\mu}\gamma^{\alpha} + i\gamma^{\nu}g^{\mu\alpha} \\
&= 2ig^{\mu\alpha}\gamma^{\nu} - 2ig^{\nu\alpha}\gamma^{\mu} + i\gamma^{\mu}\gamma^{\nu}\gamma^{\alpha} - i\gamma^{\nu}\gamma^{\mu}\gamma^{\alpha} \\
&= 2ig^{\mu\alpha}\gamma^{\nu} - 2ig^{\nu\alpha}\gamma^{\mu} + i\gamma^{\mu}\gamma^{\nu}\gamma^{\alpha} - i\gamma^{\nu}\gamma^{\mu}\gamma^{\alpha}
\end{aligned}$$

1.7 Discrete symmetries of the Dirac field ✓

This problem concerns the discrete symmetries P, C, and T.

(a) Compute the transformation properties under P, C, and T of the antisymmetric tensor fermion bilinears, $\bar{\psi}\sigma^{\mu\nu}\psi$, with $\sigma^{\mu\nu} = \frac{i}{2}[\gamma^{\mu}, \gamma^{\nu}]$. This completes the table of the transformation properties of bilinears at the end of this chapter.

Proof: Starting with the parity transformation we have

$$\begin{aligned}
P\bar{\psi}\sigma^{\mu\nu}\psi P &= P\bar{\psi}PP\frac{i}{2}[\gamma^{\mu}, \gamma^{\nu}]PP\psi P \\
&= \eta_a^*\eta_a\bar{\psi}(t, -\mathbf{x})\gamma^0\frac{i}{2}[\gamma^{\mu}, \gamma^{\nu}]\gamma^0\psi(t, -\mathbf{x}) \\
&= \bar{\psi}(t, -\mathbf{x})\gamma^0\frac{i}{2}(\gamma^{\mu}\gamma^{\nu} - \gamma^{\nu}\gamma^{\mu})\gamma^0\psi(t, -\mathbf{x}).
\end{aligned}$$

Notice that $\mu \neq \nu$ otherwise the commutator vanishes. There are two cases:

1. $\mu \neq \nu$ where $\mu, \nu = 1, 2, 3$.

$$\begin{aligned}
P\bar{\psi}\sigma^{ij}\psi P &= (-1)^2\bar{\psi}(t, -\mathbf{x})\frac{i}{2}\left(\gamma^i\gamma^j - \gamma^j\gamma^i\right)\gamma^0\gamma^0\psi(t, -\mathbf{x}) \\
&= \bar{\psi}(t, -\mathbf{x})\sigma^{ij}\psi(t, -\mathbf{x}).
\end{aligned}$$

2. $\mu \neq \nu$ and either $\mu = 0$ or $\nu = 0$. For $\nu = 0$ the parity transformation is

$$\begin{aligned}
P\bar{\psi}\sigma^{i0}\psi P &= \bar{\psi}(t, -\mathbf{x})\gamma^0\frac{i}{2}\left(\gamma^i\gamma^0 - \gamma^0\gamma^i\right)\gamma^0\psi(t, -\mathbf{x}) \\
&= (-1)\bar{\psi}(t, -\mathbf{x})\frac{i}{2}\left(\gamma^i\gamma^0 - \gamma^0\gamma^i\right)\gamma^0\gamma^0\psi(t, -\mathbf{x}) \\
&= -\bar{\psi}(t, -\mathbf{x})\sigma^{i0}\psi(t, -\mathbf{x}).
\end{aligned}$$

Similarly, for $\mu = 0$ the parity transformation is

$$P\bar{\psi}\sigma^{0j}\psi P = -\bar{\psi}(t, -\mathbf{x})\sigma^{0j}\psi(t, -\mathbf{x}).$$

The parity transformation of the bilinear tensor can be summarized by the following formula

$$P\bar{\psi}\sigma^{\mu\nu}\psi P = (-1)^{\mu}(-1)^{\nu}\bar{\psi}(t,-\mathbf{x})\sigma^{\mu\nu}\psi(t,-\mathbf{x})$$

where

$$(-1)^{\mu} = \begin{cases} 1 & \mu = 0 \\ -1 & \mu = 1,2,3. \end{cases}$$

Next we look at the time reversal transformation

$$\begin{aligned} T\bar{\psi}\sigma^{\mu\nu}\psi T &= T\bar{\psi}TT\frac{i}{2}[\gamma^{\mu},\gamma^{\nu}]TT\psi T \\ &= \bar{\psi}(-t,\mathbf{x})\left(-\gamma^1\gamma^3\right)\left(\frac{i}{2}[\gamma^{\mu},\gamma^{\nu}]\right)^{*}\left(\gamma^1\gamma^3\right)\psi(-t,\mathbf{x}) \\ &= -\bar{\psi}(-t,\mathbf{x})\left(-\gamma^1\gamma^3\right)\frac{i}{2}\left(\gamma^{\mu*}\gamma^{\nu*}-\gamma^{\nu*}\gamma^{\mu*}\right)\left(\gamma^1\gamma^3\right)\psi(-t,\mathbf{x}) \end{aligned}$$

Since $\gamma^{2*} = -\gamma^2$,

$$\gamma^{\mu*}\left(\gamma^1\gamma^3\right) = \begin{cases} \left(\gamma^1\gamma^3\right)\gamma^{\mu} & \mu = 0 \\ -\left(\gamma^1\gamma^3\right)\gamma^{\mu} & \mu = 1,2,3 \end{cases}$$

and we can simplify the transformation law. Like the parity transformation there are two cases:

1. $\mu \neq \nu$ and $\mu,\nu = 1,2,3$.

$$\begin{aligned} T\bar{\psi}\sigma^{ij}\psi T &= -(-1)^2\bar{\psi}(-t,\mathbf{x})\left(-\gamma^1\gamma^3\right)\left(\gamma^1\gamma^3\right)\frac{i}{2}\left(\gamma^i\gamma^j-\gamma^j\gamma^i\right)\psi(-t,\mathbf{x}) \\ &= -\bar{\psi}(-t,\mathbf{x})\sigma^{ij}\psi(-t,\mathbf{x}). \end{aligned}$$

2. $\mu \neq \nu$ and either $\mu = 0$ or $\nu = 0$. For $\nu = 0$ the time reversal transformation is

$$\begin{aligned} T\bar{\psi}\sigma^{i0}\psi T &= -\bar{\psi}(-t,\mathbf{x})\left(-\gamma^1\gamma^3\right)\frac{i}{2}\left(\gamma^{i*}\gamma^{0*}-\gamma^{0*}\gamma^{i*}\right)\left(\gamma^1\gamma^3\right)\psi(-t,\mathbf{x}) \\ &= -(-1)\bar{\psi}(-t,\mathbf{x})\left(-\gamma^1\gamma^3\right)\left(\gamma^1\gamma^3\right)\frac{i}{2}\left(\gamma^i\gamma^0-\gamma^0\gamma^i\right)\psi(-t,\mathbf{x}) \\ &= \bar{\psi}(-t,\mathbf{x})\sigma^{\mu 0}\psi(-t,\mathbf{x}). \end{aligned}$$

Similarly, for $\mu = 0$ the parity transformation is

$$T\bar{\psi}\sigma^{0j}\psi T = \bar{\psi}(t,-\mathbf{x})\sigma^{0j}\psi(t,-\mathbf{x}).$$

The time reversal transformation of the bilinear tensor can be summarized by the following formula

$$T\bar{\psi}\sigma^{\mu\nu}\psi T = -(-1)^{\mu}(-1)^{\nu}\bar{\psi}(t,-\mathbf{x})\sigma^{\mu\nu}\psi(t,-\mathbf{x}).$$

Lastly, we work out how the bilinear tensor transforms under charge conjugation

$$\begin{aligned} C\bar{\psi}\sigma^{\mu\nu}\psi C &= C\bar{\psi}C\sigma^{\mu\nu}C\psi C \\ &= \left(-i\gamma^0\gamma^2\psi\right)^T\sigma^{\mu\nu}\left(-i\bar{\psi}\gamma^0\gamma^2\right)^T \\ &= -\left[\bar{\psi}\gamma^0\gamma^2\left(\sigma^{\mu\nu}\right)^T\gamma^0\gamma^2\psi\right]^T \\ &= -\bar{\psi}\gamma^0\gamma^2\left(\sigma^{\mu\nu}\right)^T\gamma^0\gamma^2\psi \end{aligned}$$

where in the last line we have used the fact that $\left(-i\gamma^0\gamma^2\psi\right)^T \sigma^{\mu\nu} \left(-i\bar{\psi}\gamma^0\gamma^2\right)^T$ is just a number so the transpose does not change anything. The transpose of the gamma matrices are

$$(\gamma^\mu)^T = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\nu & 0 \end{pmatrix}^T = \begin{pmatrix} 0 & (\bar{\sigma}^\mu)^T \\ (\sigma^\nu)^T & 0 \end{pmatrix}.$$

Since,

$$\begin{aligned} \left(\sigma^0\right)^T &= \sigma^0 \\ \left(\sigma^1\right)^T &= \sigma^1 \\ \left(\sigma^2\right)^T &= -\sigma^2 \\ \left(\sigma^3\right)^T &= \sigma^3 \end{aligned}$$

we see that

$$(\gamma^\mu)^T = \begin{cases} \gamma^\mu & \mu = 0, 2 \\ -\gamma^\mu & \mu = 1, 3. \end{cases}$$

Therefore,

$$\left(\gamma^0\gamma^2\right)(\gamma^\mu)^T = -\gamma^\mu\left(\gamma^0\gamma^2\right)$$

and we can show that $\gamma^0\gamma^2 \left(\sigma^{\mu\nu}\right)^T = -\sigma^{\mu\nu}\gamma^0\gamma^2$. Thus, the bilinear tensor under charge conjugation becomes

$$\begin{aligned} C\bar{\psi}\sigma^{\mu\nu}\psi C &= \bar{\psi}\sigma^{\mu\nu}\gamma^0\gamma^2\gamma^0\gamma^2\psi \\ &= \bar{\psi}\sigma^{\mu\nu}\psi. \end{aligned}$$

The charge conjugation transformation of the bilinear tensor can be summarized by the following formula

$$C\bar{\psi}\sigma^{\mu\nu}\psi C = -\bar{\psi}(t, -\mathbf{x})\,\sigma^{\mu\nu}\psi(t, -\mathbf{x}).$$

■

(b) Let $\phi(x)$ be a complex-valued Klein-Gordon field, such as we considered in Problem 2.2. Find unitary operators P, C and an anti-unitary operator T (all defined in terms of their action on the annihilation operators a_p and b_p for the Klein-Gordon particles and antiparticles) that give the following transformations of the Klein-Gordon fields:

$$\begin{aligned} P\phi(t,\mathbf{x})P &= \phi(t,-\mathbf{x}); \\ T\phi(t,\mathbf{x})T &= \phi(-t,\mathbf{x}); \\ C\phi(t,\mathbf{x})C &= \phi^*(t,\mathbf{x}). \end{aligned}$$

Find the transformation properties of the components of the current

$$J^\mu = i\left(\phi^*\partial^\mu\phi - \partial^\mu\phi^*\phi\right)$$

under P, C and T.

Proof: The quantized Klein-Gordon field is

$$\begin{aligned} \phi(t,\mathbf{x}) &= \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_\mathbf{p}}}\left(a_\mathbf{p}e^{-ip\cdot x} + b_\mathbf{p}^\dagger e^{ip\cdot x}\right) \\ \phi^*(t,\mathbf{x}) &= \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_\mathbf{p}}}\left(a_\mathbf{p}^\dagger e^{ip\cdot x} + b_\mathbf{p}e^{-ip\cdot x}\right) \end{aligned}$$

The natural definition for the parity operator is

$$
\begin{aligned}
P a_{\mathbf{p}} P &= a_{-\mathbf{p}} \\
P b_{\mathbf{p}} P &= b_{-\mathbf{p}}.
\end{aligned}
$$

Let us test this definition on the KG field. Application of P yields

$$
\begin{aligned}
P\phi P &= \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\left(P a_{\mathbf{p}} P e^{-ip\cdot x} + \left(P b_{\mathbf{p}} P\right)^{\dagger} e^{ip\cdot x}\right) \\
&= \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\left(a_{-\mathbf{p}} e^{-iE_{\mathbf{p}}t+i\mathbf{p}\cdot\mathbf{x}} + b^{\dagger}_{-\mathbf{p}} e^{iE_{\mathbf{p}}t-i\mathbf{p}\cdot\mathbf{x}}\right).
\end{aligned}
$$

Defining $\tilde{p} = (E_{\tilde{\mathbf{p}}}, \tilde{\mathbf{p}}) \equiv (E_{\mathbf{p}}, -\mathbf{p})$, the above equation may be written as

$$
\begin{aligned}
P\phi P &=_{\mathbf{p}\leftrightarrow -\mathbf{p}} \int \frac{d^3(-p)}{(2\pi)^3}\frac{1}{\sqrt{2E_{-\mathbf{p}}}}\left(a_{-\mathbf{p}} e^{-iE_{-\mathbf{p}}t-i\mathbf{p}\cdot\mathbf{x}} + b^{\dagger}_{-\mathbf{p}} e^{iE_{-\mathbf{p}}t+i\mathbf{p}\cdot\mathbf{x}}\right) \\
&= \int \frac{d^3\tilde{p}}{(2\pi)^3}\frac{1}{\sqrt{2E_{\tilde{\mathbf{p}}}}}\left(a_{\tilde{\mathbf{p}}} e^{-i\tilde{p}\cdot(t,-\mathbf{x})} + b^{\dagger}_{\tilde{\mathbf{p}}} e^{i\tilde{p}\cdot(t,-\mathbf{x})}\right) \\
&= \phi(t,-\mathbf{x}).
\end{aligned}
$$

Thus, P takes $\phi(t,\mathbf{x})$ to $\phi(t,-\mathbf{x})$ as required. We note here that

$$
P\phi^* P = \phi^*(t,-\mathbf{x}).
$$

This will be used later to find the transformation properties of the current.

Define the time reversal operator to be anti-linear so that when commuted past a c-number it conjugates the c-number. The time reversal operator acting on the creation operators is defined to be:

$$
\begin{aligned}
T a_{\mathbf{p}} T &= a_{-\mathbf{p}} \\
T b_{\mathbf{p}} T &= b_{-\mathbf{p}}.
\end{aligned}
$$

Let us test this definition on the KG field. Application of T yields

$$
\begin{aligned}
T\phi T &= \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\left(T a_{\mathbf{p}} T e^{ip\cdot x} + \left(T b_{\mathbf{p}} T\right)^{\dagger} e^{-ip\cdot x}\right) \\
&= \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\left(a_{-\mathbf{p}} e^{ip\cdot x} + b^{\dagger}_{-\mathbf{p}} e^{-ip\cdot x}\right) \\
&= \int \frac{d^3\tilde{p}}{(2\pi)^3}\frac{1}{\sqrt{2E_{\tilde{\mathbf{p}}}}}\left(a_{\tilde{\mathbf{p}}} e^{i\tilde{p}\cdot(t,-\mathbf{x})} + b^{\dagger}_{\tilde{\mathbf{p}}} e^{-i\tilde{p}\cdot(t,-\mathbf{x})}\right) \\
&= \int \frac{d^3\tilde{p}}{(2\pi)^3}\frac{1}{\sqrt{2E_{\tilde{\mathbf{p}}}}}\left(a_{\tilde{\mathbf{p}}} e^{-i\tilde{p}\cdot(-t,\mathbf{x})} + b^{\dagger}_{\tilde{\mathbf{p}}} e^{-i\tilde{p}\cdot(-t,\mathbf{x})}\right) \\
&= \phi(-t,\mathbf{x})
\end{aligned}
$$

as required. We note here that

$$
T\phi^* T = \phi^*(-t,\mathbf{x}).
$$

This will be used later to find the transformation properties of the current.

Define the time reversal operator to be anti-linear so that when commuted past a c-number it conjugates the c-number. The time reversal operator acting on the creation operators is defined to be:

$$
\begin{aligned}
C a_{\mathbf{p}} C &= b_{\mathbf{p}} \\
C b_{\mathbf{p}} C &= a_{\mathbf{p}}.
\end{aligned}
$$

Let us test this definition on the KG field. Application of T yields

$$\begin{aligned} C\phi C &= \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\left(Ca_{\mathbf{p}}Ce^{-ip\cdot x}+(Cb_{\mathbf{p}}C)^{\dagger}e^{ip\cdot x}\right) \\ &= \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\left(b_{\mathbf{p}}e^{-ip\cdot x}+a^{\dagger}_{\mathbf{p}}e^{ip\cdot x}\right) \\ &= \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\left(b_{\mathbf{p}}e^{-ip\cdot x}+a^{\dagger}_{\mathbf{p}}e^{ip\cdot x}\right) \\ &= \phi^*(t,\mathbf{x}) \end{aligned}$$

as required. We note here that

$$C\phi^* C = \phi(t,\mathbf{x}).$$

This will be used later to find the transformation properties of the current.

Now, we can find the transformation properties of the current. In addition to knowing how ϕ and ϕ^* transform under P, T, and C, we need the transformation rules for the derivative ∂^μ. Parity takes $\mathbf{x}\to-\mathbf{x}$ and $t\to t$ and therefore, $\boldsymbol{\nabla}\to-\boldsymbol{\nabla}$ and $\partial_0\to\partial_0$. Time reversal takes $\mathbf{x}\to\mathbf{x}$ and $t\to -t$ and therefore, $\boldsymbol{\nabla}\to\boldsymbol{\nabla}$ and $\partial_0\to-\partial_0$. Charge conjugation does not effect the derivative. With $g^{\mu\mu}$ defined such that there is no sum we can write the transformations of the derivative in a compact way,

$$\begin{aligned} P\partial^\mu P &= g^{\mu\mu}\partial^\mu \\ T\partial^\mu T &= -g^{\mu\mu}\partial^\mu \\ C\partial^\mu C &= \partial^\mu. \end{aligned}$$

Thus, under P, T, and C, the current becomes

$$\begin{aligned} PJ^\mu(t,\mathbf{x})P &= i\left(P\phi^*(t,\mathbf{x})P\partial^\mu P\phi(t,\mathbf{x})P - \partial^\mu P\phi^*(t,\mathbf{x})PP\phi(t,\mathbf{x})P\right) \\ &= i\left(\phi^*(t,-\mathbf{x})g^{\mu\mu}\partial^\mu\phi(t,-\mathbf{x}) - \left(g^{\mu\mu}\partial^\mu\phi^*(t,-\mathbf{x})\right)\phi(t,-\mathbf{x})\right) \\ &= (-1)^\mu J^\mu(t,-\mathbf{x}) \end{aligned}$$

$$\begin{aligned} TJ^\mu(t,\mathbf{x})T &= i\left(T\phi^*(t,\mathbf{x})T\partial^\mu T\phi(t,\mathbf{x})T - T\partial^\mu T\phi^*(t,\mathbf{x})TT\phi(t,\mathbf{x})T\right) \\ &= i\left(\phi^*(-t,\mathbf{x})\left(-g^{\mu\mu}\right)\partial^\mu\phi(-t,\mathbf{x}) - \left(\left(-g^{\mu\mu}\right)\partial^\mu\phi^*(-t,\mathbf{x})\right)\phi(-t,\mathbf{x})\right) \\ &= -\left(-1\right)^\mu J^\mu(-t,\mathbf{x}) \end{aligned}$$

$$\begin{aligned} CJ^\mu(t,\mathbf{x})C &= i\left(C\phi^*(t,\mathbf{x})C\partial^\mu C\phi(t,\mathbf{x})C - \partial^\mu C\phi^*(t,\mathbf{x})CC\phi(t,\mathbf{x})C\right) \\ &= i\left(\phi(t,\mathbf{x})\partial^\mu\phi^*(t,\mathbf{x}) - \left(\partial^\mu\phi(t,\mathbf{x})\right)\phi^*(t,\mathbf{x})\right) \\ &= i\left(\phi(t,\mathbf{x})\partial^\mu\phi^*(t,\mathbf{x}) - \left(\partial^\mu\phi(t,\mathbf{x})\right)\phi^*(t,\mathbf{x})\right) \\ &= -J^\mu(t,\mathbf{x}). \end{aligned}$$

■

(c) Show that any Hermitian Lorentz-scalar local operator built from $\psi(x)$, $\phi(x)$, and their conjugates has $CPT=+1$.

Proof: A Hermitian, Lorentz invariant operator must have ϕ paired with ϕ^* and ψ paired with $\bar{\psi}$. Additionally, the Lorentz indices must be contracted to form a Lorentz scalar. Looking at table in P&S we can see that any combination of these bilinears which yields a Hermitian, Lorentz invariant will have $CPT=1$.

■

1.8 Bound states

Two spin-1/2 particles can combine to a state of total spin either 0 or 1. The wavefunctions for these states are odd and even, respectively, under the interchange of the two spins.

(a) Use this information to compute the quantum numbers under P and C of all electron-positron bound states with S, P, or D wavefunctions.

The Ps Schrodinger equation is the Schrodinger equation for a particle of electric charge e and reduced mass $\mu = m_e/2$ in the Coulomb potential $V(|\mathbf{x}|) = -\alpha/|\mathbf{x}|$ where e (< 0) is the electron charge, m_e is the electron mass, $\alpha \approx 1/137$ is the fine structure constant and $\boldsymbol{x}$ is the separation distance between the electron and positron in Ps. Instead of directly solving the Ps Schrodinger equation, the position space wave functions of Ps can be obtained by taking advantage of the similarity between the Schrodinger equation for Ps and that for hydrogen.

Specifically, the position space wave functions can be obtained by replacing the hydrogen Bohr radius with the Ps Bohr radius in the hydrogen position space wave functions. These wave functions are characterized by the principal quantum number (or energy quantum number), n, the orbital angular momentum quantum number, l, and the orbital angular momentum projection quantum number, m_l. The position space wave functions are

$$\psi_{nlm_l}(\boldsymbol{x}) = \sqrt{\left(\frac{2}{na}\right)^3 \frac{(n-l-1)!}{2n\left[(n+1)!\right]^3}} e^{-|\boldsymbol{x}|/na} \left(\frac{2|\boldsymbol{x}|}{na}\right)^l L^{2l+1}_{n-l-1}\left(\frac{2|\boldsymbol{x}|}{na}\right) Y_l^m(\theta,\phi), \tag{1.1}$$

where $a = 2/m_e\alpha$ is the Bohr radius of Ps, L^m_{n-m} are associated Laguerre polynomials and $Y_l^{m_l}$ are spherical harmonics.

Now that we have the position space wave functions of Ps, we can construct the Ps bound state. In the centre of mass frame, the Ps bound state can be expressed by

$$|\Psi_{nlm_l;s,m_s}\rangle = \sqrt{2m_{\text{Ps}}} \int \frac{d^3k}{(2\pi)^3} \psi_{nlm_l}(\boldsymbol{k}) \frac{1}{\sqrt{2m_e}} \frac{1}{\sqrt{2m_e}} |\boldsymbol{k}; -\boldsymbol{k}; s, m_s\rangle, \tag{1.2}$$

where s is the total spin of the Ps bound state and m_s is the spin projection along the z-axis. The momentum space wavefunction,

$$\psi_{nlm_l}(\boldsymbol{k}) \equiv \int d^3x e^{i\boldsymbol{k}\cdot\boldsymbol{r}} \psi_{nlm_l}(\boldsymbol{x}), \tag{1.3}$$

gives the amplitude for finding Ps in a given configuration where the electron has momentum $\boldsymbol{k}$. The free state in equation (1.2) is

$$|\boldsymbol{k}; -\boldsymbol{k}; 0, 0\rangle = \left(a_{\boldsymbol{k}}^{\frac{1}{2}\dagger} b_{-\boldsymbol{k}}^{-\frac{1}{2}\dagger} - a_{\boldsymbol{k}}^{-\frac{1}{2}\dagger} b_{-\boldsymbol{k}}^{\frac{1}{2}\dagger}\right) |0\rangle/\sqrt{2},$$

for p-Ps ($s = m_s = 0$) and

$$|\boldsymbol{k}; -\boldsymbol{k}; 1, m_s\rangle = \begin{cases} a_{\boldsymbol{k}}^{\frac{1}{2}\dagger} b_{-\boldsymbol{k}}^{\frac{1}{2}\dagger}|0\rangle & \text{for } m_s = 1, \\ \left(a_{\boldsymbol{k}}^{\frac{1}{2}\dagger} b_{-\boldsymbol{k}}^{-\frac{1}{2}\dagger} + a_{\boldsymbol{k}}^{-\frac{1}{2}\dagger} b_{-\boldsymbol{k}}^{\frac{1}{2}\dagger}\right)|0\rangle/\sqrt{2} & \text{for } m_s = 0, \\ a_{\boldsymbol{k}}^{-\frac{1}{2}\dagger} b_{-\boldsymbol{k}}^{-\frac{1}{2}\dagger}|0\rangle & \text{for } m_s = -1, \end{cases}$$

for o-Ps ($s = 1$) where $|0\rangle$ is the vacuum state.

Applying the parity operator, we obtain

$$\begin{aligned} P|\Psi_{nlm_l;00}\rangle &= \frac{1}{\sqrt{m_e}} \int \frac{d^3k}{(2\pi)^3} \psi_{nlm_l}(\boldsymbol{k}) P|\boldsymbol{k}; -\boldsymbol{k}; 00\rangle \\ &= \frac{1}{\sqrt{2m_e}} \int \frac{d^3k}{(2\pi)^3} \psi_{nlm_l}(\boldsymbol{k}) P\left(a_{\boldsymbol{k}}^{\frac{1}{2}\dagger} b_{-\boldsymbol{k}}^{-\frac{1}{2}\dagger} - a_{\boldsymbol{k}}^{-\frac{1}{2}\dagger} b_{-\boldsymbol{k}}^{\frac{1}{2}\dagger}\right) P|0\rangle \\ &= -|\eta_a|^2 \frac{1}{\sqrt{2m_e}} \int \frac{d^3k}{(2\pi)^3} \psi_{nlm_l}(-\boldsymbol{k}) P\left(a_{\boldsymbol{k}}^{\frac{1}{2}\dagger} b_{-\boldsymbol{k}}^{-\frac{1}{2}\dagger} - a_{\boldsymbol{k}}^{-\frac{1}{2}\dagger} b_{-\boldsymbol{k}}^{\frac{1}{2}\dagger}\right) P|0\rangle \end{aligned}$$

where we have assumed that the vacuum state is even under parity. To simplify the above further, note from equation (1.3), that the parity of $\psi_{nlm_l}(\boldsymbol{k})$ is the same as $\psi_{lnm_l}(\boldsymbol{x})$. Therefore, we can invert the spatial coordinate in (1.1) to obtain the parity of the momentum space wavefunction. The only part of (1.1) that is sensitive to such an inversion is the spherical harmonic, $Y_{lm}(\theta,\phi) \to (-1)^l Y_{lm}(\theta,\phi)$. Thus, p-Ps transforms as

$$P|\Psi_{nlm_l;00}\rangle = (-1)^{l+1}|\Psi_{nlm_l;00}\rangle$$

under parity. Similarity, o-Ps transforms as

$$P|\Psi_{nlm_l;1m_s}\rangle = (-1)^{l+1}|\Psi_{nlm_l;1m_s}\rangle$$

under parity and has an P eigenvalue of $(-1)^{l+1}$. Notice that the relative phase difference between the fermion and anti-fermion inversion phases, $\eta_b^* = -\eta_a$, contributes a factor of -1 to the parity of Ps (p-Ps and o-Ps) in addition to the parity of the wave function; this extra factor is called the intrinsic parity of a fermion--anti-fermion system.

The remaining C and T eigenvalues of Ps are obtained in a similar manner and are listed below in Table 1.

Discrete Transform	Ps Eigenvalue
P	$(-1)^{l+1}$
C	$(-1)^{l+s}$
T	$(-1)^{s+1}$

Table 1: P, C and T eigenvalues of the Ps state $|\Psi_{nlm_l;sm_s}\rangle$. Here, l is the orbital angular momentum quantum number and s is the spin angular momentum quantum number.

(b) Since the electron-photon coupling is given by the Hamiltonian

$$\Delta H = \int d^3x \; eA_\mu j^\mu,$$

where j^μ is the electric current, electrodynamics is invariant to p and C if the components of the vector potential have the same P and C parity as the corresponding components of j^μ. Show that this implies the following surprising fact: The spin-0 ground state of positronium can decay to 2 photon, but the spin-1 state must decay to 3 photons. Find the selection rules for the annihilation of higher positronium states, and for 1-photon transitions between positronium levels.