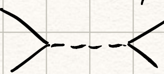
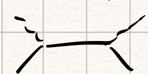
Summary of reading: EH 2.3, 2.6

§2.3 Examples from Yukawa theory

We examined the tree level 4pt amplitudes in Yukawa theory:

 and 

Many useful properties of the spinor helicity variables are discussed in the context of these simple amplitudes:

1) define spinor brackets: $\langle pq\rangle = \langle p|_{\dot a}\, |q\rangle^{\dot a}$ and $[pq] = [p_a\, q]^a$
   note the asymmetry of these brackets due to the ε-tensor: $\langle pq\rangle = -\langle qp\rangle$
   $\langle pq] = [pq\rangle = 0$

2) Converting minkowski dot products to spinor helicity variables: $(p+q)^2 = \langle pq\rangle [pq]$ for any null momenta p and q.

3) $[pq]^* = \langle pq\rangle$ for $p, q \in \mathbb{R}^{3,1}$

4) incorporating γ-matrices: $\bar u_-(p)\gamma^\mu v_+(q) \equiv \langle p|\gamma^\mu|2]$, $\bar u_+(p)\gamma^\mu v_-(q) \equiv [p|\gamma^\mu|q\rangle$,
   $\bar u_\pm \gamma^\mu v_\pm = 0$

5) $[k|\gamma^\mu|p\rangle = \langle p|\gamma^\mu|k]$ and $[k|\gamma^\mu|p\rangle^* = [p|\gamma^\mu|k\rangle$ (for $p, q \in \mathbb{R}^{3,1}$)

6) Fierz identities: $\langle 1|\gamma^\mu|2]\langle 3|\gamma_\mu|4] = 2\langle 13\rangle[24]$

7) momentum conservation in spinor helicity variables: $\sum_{i=1}^n |i\rangle[i = 0$

8) Schouten identity (follows trivially from the fact that 3 vectors in a plane cannot be linearly independent)
$$|i\rangle\langle jk\rangle + |j\rangle\langle ki\rangle + |k\rangle\langle ij\rangle = 0$$

§2.6 Little Group scaling

little group = all $\Lambda_p \in SO(d-1,1)$ that do not change the direction of the momentum p^μ. The angle and square brackets satisfy the massless Weyl equation $p|p\rangle = 0 = p|p]$ for $p^2 = 0$. The Weyl eq is invariant under the scaling: $|p\rangle \to t|p\rangle$ and $|p] \to t^{-1}|p]$ and is exactly the littlegroup transformation in spinor helicity notation. (t is a phase for $p \in \mathbb{R}^{d-1,1}$ and $t \in \mathbb{C}$ for $p \in \mathbb{C}^{d-1,1}$)

only external particles of a Feynman diagram scale under the little group:
1) scalars: no scaling
2) fermions: spinors scale as t^{-2h} where $h = \pm\frac{1}{2}$
3) bosons: polarization vectors scale as t^{-2h} for $h = \pm 1$

$\Rightarrow$ for an amplitude of only massless particles, the little group scaling is
$$A_n(\{|1\rangle, |1], h_1\}, \dots, \{t_i|i\rangle, t_i^{-1}|i], h_i\} \dots) = t_i^{-2h_i} A_n(\dots \{|i\rangle, |i], h_i\}, \dots)$$

this is a powerful constraint that will be exploited many times. For example, the little group scaling uniquely fixes the 3pt amplitude up to an overall constant

$$A_3(1^{h_1} 2^{h_2} 3^{h_3}) = c \langle 12\rangle^{h_3-h_2-h_1} \langle 13\rangle^{h_2-h_1-h_3} \langle 23\rangle^{h_1-h_2-h_3}$$

where we have used the fact that the 3pt amplitude of massless particles depends only on angle or square brackets.

on-shell

if we avoid making the assumption that 3pt amplitudes only depend on square or angle brackets we could determine the correct structure using dimensional analysis. For example, consider 3 gluon amplitude:

$$A_3(1^- 2^- 3^+) = g \underbrace{\frac{\langle 12\rangle^3}{\langle 13\rangle\langle 23\rangle}}_{\text{dimension } 1} \quad \text{or} \quad A_3(1^- 2^- 3^+) = g' \underbrace{\frac{[13][23]}{[12]^3}}_{\text{dimension } -1}$$

Compatible to $AA\partial A$

would need $AA\underbrace{\frac{\partial}{\Box}}A$

cannot appear in local $\mathcal{L}$.

note that $[g]=0$ and $[g']=2$ since both amplitudes must have same dim

In general: $\dim(A_n) = d - \frac{(d-2)}{2}n$

Exercises: 2.1, 2.3, 2.4, 2.8, 2.32, 2.34 of EH and 33.2 of Srednicki

EH 2.1

$$p^\mu = (E, E\sin\theta\cos\varphi, E\sin\theta\sin\varphi, E\cos\theta)$$

$$p_{ab} = p_\mu(\sigma^\mu)_{ab} = \begin{pmatrix} -p^0+p^3 & p^1-ip^2 \\ p^1+ip^2 & -p^0-p^3 \end{pmatrix} = 2E \begin{pmatrix} \sin^2\theta/2 & e^{-i\phi}\cos\frac{\theta}{2}\sin\frac{\theta}{2} \\ e^{i\phi}\cos\frac{\theta}{2}\sin\frac{\theta}{2} & -\cos^2\theta/2 \end{pmatrix}$$

$$p^{\dot a b} = p_\mu(\bar\sigma^\mu)^{\dot a b} = -2E\begin{pmatrix} \cos^2\theta/2 & e^{-\phi}\cos\theta/2\sin\theta/2 \\ e^{i\phi}\cos\theta/2\sin\theta/2 & \sin^2\theta/2 \end{pmatrix}$$

we are given
$$|p\rangle^{\dot a} = \sqrt{2E}\begin{pmatrix} \cos\theta/2 \\ \sin\theta/2\, e^{i\phi} \end{pmatrix}$$
and want to check that it satisfies $p_{a\dot b}|p\rangle^{\dot b} = 0$. This is easily verified by using Mathematica to compute the matrix product.

Furthermore it is simple to verify (on Mathematica) that

$$\langle p|_{\dot a} = \sqrt{2E}\left(-\sin\tfrac{\theta}{2}e^{I\phi} \quad \cos\theta/2\right)$$

$$[p|^a = \sqrt{2E}\left(\cos\tfrac{\theta}{2} \quad \sin\tfrac{\theta}{2}e^{-i\phi}\right)$$

$$|p]_a = \sqrt{2E}\left(-\sin\tfrac{\theta}{2}e^{-i\phi} \quad \cos\theta/2\right)^T$$

satisfy the Weyl equations:

$$\langle p|_{\dot a}p^{\dot a b} = 0$$

$$[p|^a p_{a\dot b} = 0$$

$$p^{\dot a b}|p]_b = 0$$

and the completeness relations:

$$p_{a\dot b} = -|p]_a\langle p|_{\dot b}$$

$$p^{\dot a b} = -|p\rangle^{\dot a}[p|^b$$

EH 2.3 Prove the Fierz identity $\langle 1|\gamma^\mu|2]\langle 3|\gamma_\mu|4] = 2\langle 13\rangle[24]$

$$\langle 1|\gamma^\mu 2]\langle 3|\gamma_\mu|4] = \bar{u}_-(p_1)\gamma^\mu v_+(p_2)\ \bar{u}_-(p_3)\gamma_\mu v(p_4)$$

$$= (0, \langle 1|_{\dot a}) \begin{pmatrix} 0 & (\sigma^\mu)_{a\dot b} \\ (\bar\sigma^\mu)^{\dot a b} & 0 \end{pmatrix} \begin{pmatrix} |2]_b \\ 0 \end{pmatrix}$$

$$\times (0, \langle 3|_{\dot c}) \begin{pmatrix} 0 & (\sigma^\mu)_{c\dot d} \\ (\bar\sigma^\mu)^{\dot c d} & 0 \end{pmatrix} \begin{pmatrix} |4]_d \\ 0 \end{pmatrix}$$

$$= (0, \langle 1|_{\dot a}) \begin{pmatrix} 0 \\ (\bar\sigma^\mu)^{\dot a b}|2]_b \end{pmatrix} (0\ \langle 3|_{\dot c}) \begin{pmatrix} 0 \\ (\bar\sigma_\mu)^{\dot c d}|4]_d \end{pmatrix}$$

$$= \langle 1|_{\dot a}\langle 3|_{\dot c}\ \underbrace{(\bar\sigma^\mu)^{\dot a b}(\bar\sigma_\mu)^{\dot c d}}_{-2\epsilon^{\dot a\dot c}\epsilon^{bd}}\ |2]_b|4]_d$$

(from appendix)

$$= -2\langle 1|_{\dot a}\ |3\rangle^{\dot a}\quad [4|^b\ |2]_b$$
$$= +2\ \langle 13\rangle\ [24]$$

EH 2.4 Show $\langle k|\gamma^\mu|k] = 2k^\mu$ and $\langle k|P|k] = 2P\cdot k$

$$\langle k|\gamma^\mu|k] = \mathrm{Tr}[\gamma^\mu|k]\langle k|]$$
$$= \tfrac{1}{2}\mathrm{Tr}[\gamma^\mu(|k]\langle k| + |k\rangle[k|)]$$
$$= -\tfrac{1}{2}\mathrm{Tr}[\gamma^\mu \not k]$$
$$= -\tfrac{1}{2}k_\nu(-4\eta^{\mu\nu})$$
$$= 2k^\mu$$

where we have used the fact that $[k|\gamma^\mu|p\rangle^* = [p|\gamma^\mu|k\rangle$ for $p,k \in \mathbb{R}^{3,1}$

$\langle k|P|k] = 2P\cdot k$ follows trivially $\blacksquare$

$$\langle k|\gamma^\mu|k] = \bar{u}_-\gamma^\mu v_+ = \bar{u}_-\gamma^\mu u_-$$

$$(\langle k|\gamma^\mu|k])^\dagger = v_+^\dagger\ \gamma^{\mu\dagger}\ \bar{u}_-^\dagger \qquad u_\pm = v_\mp$$
$$= \bar{v}_+\gamma^0\gamma^{\mu\dagger}\gamma^0 u_- \qquad \bar{u}_\pm = \bar{v}_\mp$$

$$\Rightarrow (\langle k|\gamma^\mu|k])^\dagger = \langle k|\gamma^\mu|k] \in \mathbb{R}$$

$$\langle k|\gamma^\mu|p] = \bar{u}_-(k)\gamma^\mu v_+(p)$$

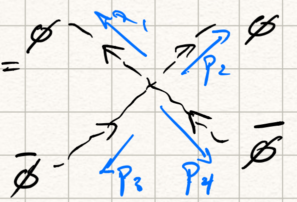
$$\langle k|\gamma^\mu|p]^\dagger = v_+^\dagger(p)\ \gamma^{\mu\dagger}\ \bar{u}_-^\dagger(k)$$
$$= \bar{v}_+(p)\gamma^\mu u_-(k)$$
$$= \bar{u}_-(p)\gamma^\mu v_+(k)$$
$$= [p|\gamma^\mu|k\rangle$$

$$\Rightarrow \langle k|\gamma^\mu|k] = [k|\gamma^\mu|k\rangle$$

EH 2.8 $\quad \mathcal{L} = i\psi^\dagger \bar\sigma^\mu \partial_\mu \psi - \partial_\mu \bar\phi\, \partial_\mu \phi + \frac{1}{2} g \phi \psi\psi + \frac{1}{2} g^* \bar\phi \psi^\dagger \psi^\dagger - \frac{1}{4}\lambda |\phi|^2$

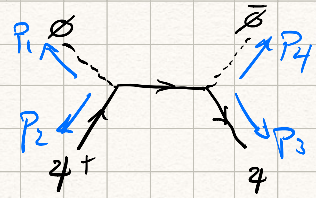
where ψ is a Weyl fermion and ϕ is a complex scalar.

$iA_4(\phi\phi\bar\phi\bar\phi) =$



$= -i\lambda$

$iA_4(\phi f^- f^+ \bar\phi) =$



(there is no $(1 \leftrightarrow 4)$ diagram b/c ϕ is a complex scalar)

$$= \frac{ig^* \; -i\bar u_3(-\not p_1 - \not p_2) v_2 \;\; ig}{(p_1+p_2)^2}$$

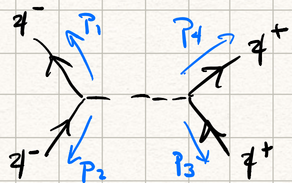
$$= i|g|^2 \frac{\bar u_3(\not p_3 + \not p_4) v_2}{(p_3+p_4)^2}$$

$$= i|g|^2 \frac{[3|(\,|3]\langle 3| + |3\rangle[3| + |4]\langle 4| + |4\rangle[4|\,)|2\rangle}{\langle 34\rangle [34]}$$

$$= i|g|^2 \frac{[34]\langle 42\rangle}{\langle 34\rangle [34]}$$

$$= -i|g|^2 \frac{\langle 24\rangle}{\langle 34\rangle}$$

$iA_4(f^- f^- f^+ f^+) =$



$$= \frac{(ig)(ig^*)(-i)}{(p_3+p_4)^2}\; \bar u_1 v_2 \; \bar u_4 v_3$$

$$= \frac{i|g|^2}{\langle 34\rangle[34]} \langle 12\rangle [43]$$

$$= -i|g|^2 \frac{\langle 12\rangle}{\langle 34\rangle}$$

EH 2.32

a) $A_5 = g_a \dfrac{[13]^4}{[12][23][34][45][51]} \xrightarrow[\text{transformation}]{\text{little group}} t_1^{-2} t_2^{2} t_3^{-2} t_4^{2} t_5^{2} A_5$

$\Rightarrow (h_1\ h_2\ h_3\ h_4\ h_5) = (1, -1, 1, -1, -1)$

The mass dim of the amplitude is $[A_5] = 4-5 = -1 \Rightarrow [g_a] = 0$

$\Rightarrow$ Yang mills theory.

b) $A_4 = g_b \dfrac{\langle 14\rangle \langle 24\rangle^2}{\langle 12\rangle \langle 23\rangle \langle 34\rangle} \longrightarrow t_1^{0} t_2^{0} t_3^{-2} t_4^{2} A_4$

$\Rightarrow (h_1\ h_2\ h_3\ h_4) = (0, 0, 1, -1)$

$[A_4] = 0 \Rightarrow [g_b] = 0$

we need 4 fields in the interaction term such that it has dim 4 so that $[g_b] = 0$.

scalar QED/YM

c) $A_4 = g_c \dfrac{\langle 12\rangle^7 [12]}{\langle 13\rangle \langle 14\rangle \langle 23\rangle \langle 24\rangle \langle 34\rangle^2} \longrightarrow t_1^{4} t_2^{4} t_3^{-4} t_4^{-4} A_4$

$\Rightarrow (h_1\ h_2\ h_3\ h_4) = (-2, -2, 2, 2)$

$[A_4] = 0 \Rightarrow [g_c] = -2$

looks like a 4-graviton interaction : $R \sim (g_{\mu\nu})^2$ so we expect such an interaction from R^2

EH 2.32

define amplitude as $A_3(1^{h_1} 2^{h_2} 3^{h_3})$ for $h_1, h_2 = \pm\frac{1}{2}$ and $h_3 = \pm 1$

under little group scaling. $A_3 \to t_1^{-2h_1} t_2^{-2h_2} t_3^{-2h_3} A_3$

$$\langle ij\rangle \to t_i t_j \langle ij\rangle \qquad [ij] \to t_i^{-1} t_j^{-1} [ij]$$

while we have not covered it in our reading yet, 3-particle kinematics is special.
For 3 massless particles,

$$|1] \propto |2] \propto |3]$$
$$\&$$
$$|1\rangle \propto |2\rangle \propto |3\rangle$$

this implies:

1) amplitudes can only depend on either $|\rangle$ or $|]$ brackets, not both.
2) 3-particle massless amplitudes are non-zero only for complex momentum.

Thus, the 3pt amplitude is completely fixed to be

$$A_3^{h_1 h_2 h_3} = g \langle 12\rangle^{h_3 - h_2 - h_1} \langle 13\rangle^{h_2 - h_1 - h_3} \langle 23\rangle^{h_1 - h_2 - h_3}$$

there are 3 classes of amplitudes we can have for a gluino - gluino - gluon amplitude:

1) $(h_1\, h_2\, h_3) = (-\frac{1}{2}\ -\frac{1}{2}\ +1)$
$$A_3^{-\frac{1}{2}-\frac{1}{2}+1} = g \frac{\langle 12\rangle^2}{\langle 13\rangle\langle 23\rangle} \qquad \text{where } [g] = 1$$

2) $(h_1\, h_2\, h_3) = (+\frac{1}{2}\ -\frac{1}{2}\ -1)$
$$A_3^{+\frac{1}{2}-\frac{1}{2}-1} = g \frac{\langle 23\rangle^2}{\langle 12\rangle} \qquad \text{where } [g] = 0$$

3) $(h_1\, h_2\, h_3) = (-\frac{1}{2}\ -\frac{1}{2}\ -1)$
$$A_3^{-\frac{1}{2}-\frac{1}{2}-1} = g \langle 13\rangle\langle 23\rangle \qquad \text{where } [g] = -1$$

what kind of Lagrangian could give us such amplitudes?

2) $\bar{\psi} A \psi$ is a natural candidate. Looking through §2.4 we see that A_3^{+--} has the same from as the QED amplitudes (which we know follow from $\mathcal{L}_{int} = \bar{\psi} A \psi$)

1) $[g] = 1$, $[\psi] = \frac{3}{2}$, $[A] = 1$ $\Rightarrow$ $g \frac{1}{\square}(\psi\bar{\psi}) A^{\sim}$ (non-local)

2) $[g] = -1$, $[\psi] = \frac{3}{2}$, $[A] = 1$ $\Rightarrow g \bar{\psi}\psi A^2$

EH 2.33

From $A_3^{h_1h_2h_3} = g \langle 12\rangle^{h_3-h_1-h_2}\langle 13\rangle^{h_2-h_3-h_1}\langle 23\rangle^{h_1-h_2-h_3}$ we find

$$A_3^{-2,-2,-2} = g\langle 12\rangle^2\langle 13\rangle^2\langle 23\rangle^2 \quad \text{for } [g]=-5$$

$$A_3^{-2,-2,2} = g\,\frac{\langle 12\rangle^6}{\langle 13\rangle^2\langle 23\rangle^2} \quad \text{for } [g]=-1$$

where $[A_3^{h_1h_2h_3}] = 4-3 = 1$

Comparing to the gluon amplitudes

$$A_3^{-1-1-1} = g\langle 12\rangle\langle 13\rangle\langle 23\rangle \quad \text{for } [g]=-2$$

$$A_3^{-1-1+1} = g\,\frac{\langle 12\rangle^3}{\langle 13\rangle\langle 23\rangle} \quad \text{for } [g]=0$$

We see that the kinematic part of the gravity amplitudes is the square of kinematic part of the gluon amplitude $\longrightarrow$ BCJ & color kin duality!

Srednicki 33.2

$$[J_i, J_j] = i\varepsilon_{ijk} J_k \qquad N_i = \tfrac{1}{2}(J_i - ik_i)$$
$$[J_i, k_j] = i\varepsilon_{ijk} k_k \qquad N_i^\dagger = \tfrac{1}{2}(J_i + ik_i)$$
$$[k_i, k_j] = -i\varepsilon_{ijk} J_k$$

$$\begin{aligned}
[N_i, N_j] &= \tfrac{1}{4}[J_i - ik_i, J_j - ik_j] \\
&= \tfrac{1}{4}\left([J_i,J_j] - i[J_i,k_j] - i[k_i,J_j] - [k_i,k_j]\right) \\
&= \tfrac{1}{4}\left(i\varepsilon_{ijk}J_k - i\,i\varepsilon_{ijk}k_k + i\,i\underbrace{\varepsilon_{jik}}_{-\varepsilon_{ijk}k_k}k_k + i\varepsilon_{ijk}J_k\right) \\
&= \\
&= \tfrac{i}{2}\varepsilon_{ijk}(J_k - ik_k) \\
&= i\varepsilon_{ijk}N_k
\end{aligned}$$

$$\begin{aligned}
[N_i^\dagger, N_j^\dagger] &= \tfrac{1}{4}\left([J_i,J_j] + i[J_i,k_j] + i[k_i,J_j] - [k_i,k_j]\right) \\
&= \tfrac{1}{4}\,2i\,\varepsilon_{ijk}(J_k + ik_j) \\
&= i\varepsilon_{ijk}N_k^\dagger
\end{aligned}$$

$$[N_i, N_j^\dagger] = \tfrac{1}{4}\Big(\cancel{\underbrace{[J_i,J_j]}_{i\varepsilon_{ijk}J_k}} + i\cancel{\underbrace{[J_i,k_j]}_{i^2\varepsilon_{ijk}k_k}} - i\cancel{\underbrace{[k_i,J_j]}_{(-i)^2\varepsilon_{jik}k_k}} + \cancel{\underbrace{[k_i,k_j]}_{-i\varepsilon_{ijk}J_k}}\Big) = 0$$

$\Rightarrow$ Lorentz group $= SU(2)_R \otimes SU(2)_L$