Phys: 731 String Theory

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Problem 1: JBBS 1.1

Part (a)

Here, we are asked to examine the non-relativistic limit of the classical action for a relativistic point particle. To go to the the non-relativistic limit we fix the gauge by setting $\tau = X^0 = t$ and take the limit $\dot{X}^i \equiv v^i \ll 1$. Then the point particle action becomes

$$\begin{aligned} S_{\text{pp}} &= -m \int d\tau \, \sqrt{-\partial_\tau X^\mu \partial_\tau X_\mu} \\ &= -m \int dt \, \sqrt{-\left(-1 + \dot{\boldsymbol{X}} \cdot \dot{\boldsymbol{X}}\right)} \\ &= -m \int dt \, \sqrt{1 - v^2} \\ &\approx \int dt \left(\frac{mv^2}{2} - m \right). \end{aligned} \tag{1}$$

The first term is the usual kinetic energy term while the second is from the rest mass. The Lagrangian can be written as the difference of kinetic and potential terms

$$L = T - V \tag{2}$$

where $T = \frac{1}{2} m v^2$ and $V = m$.

Part (b)

The Nambu-Goto string action is

$$S_{NG} = -\frac{1}{2\pi\alpha'} \int d\tau \, d\sigma \, \sqrt{-h} \tag{3}$$

where $h = \det h_{ab}$ and $h_{ab} = \partial_a X^\mu \partial_b X_\mu$. To connect with the non-relativistic limit we choose the gauge such that $\tau = X^0$. Then,

$$h_{ab} = \begin{bmatrix} -1 + v^2 & \boldsymbol{v} \cdot \boldsymbol{y} \\ \boldsymbol{v} \cdot \boldsymbol{y} & y^2 \end{bmatrix} \tag{4}$$

where $\boldsymbol{v} = \partial_\tau \boldsymbol{X}$ and $\boldsymbol{y} = \partial_\sigma \boldsymbol{X}$. The determinant, $-h$, is

$$-h = \left(1 - v^2\right) y^2 + (\boldsymbol{v} \cdot \boldsymbol{y})^2 . \tag{5}$$

In the limit that $v \ll 1$ the Nambu-Goto action becomes,

$$\begin{aligned} S_{NG} &\approx -\frac{1}{2\pi\alpha'} \int d\tau \, d\sigma \, y \left(1 - \frac{1}{2}\left(v^2 - \frac{\boldsymbol{v}\cdot\boldsymbol{y}}{y^2}\right)\right) \\ &= \frac{1}{2\pi\alpha'} \int d\tau \, d\sigma \, y \left(\frac{1}{2}\left(v^2 - \frac{\boldsymbol{v}\cdot\boldsymbol{y}}{y^2}\right) - 1\right). \end{aligned} \tag{6}$$

The first term in the action above is proportional to the string velocity squared. Therefore, we expect the first term to be "kinetic" and the second term to be "potential". In fact, the problem hints that the kinetic term will only depend on the transverse velocity. To see this, note that the vector $\boldsymbol{y}$ is tangent to the string. Therefore, to get the transverse velocity we must subtract out the tangent component of $\boldsymbol{v}$

$$\boldsymbol{v}_T = \boldsymbol{v} - \frac{(\boldsymbol{v}\cdot\boldsymbol{y})}{y^2}\boldsymbol{y}. \tag{7}$$

Squaring the transverse velocity,

$$v_T^2 = v^2 - \frac{(\boldsymbol{v}\cdot\boldsymbol{y})^2}{y^2}, \tag{8}$$

we see that the kinetic term in the action is indeed proportional to the transverse velocity squared

$$S_{NG} = \frac{1}{2\pi\alpha'} \int d\tau \, d\sigma \, y \left(\frac{v_T^2}{2} - 1\right). \tag{9}$$

Next, we examine the potential term which should be proportional to the length of the string. The length of the string is given by

$$\ell_s = \int d\sigma \, |\partial_\sigma \boldsymbol{X}| = \int d\sigma \, y. \tag{10}$$

With the above, the NG Lagrangian can be written as a kinetic term (proportional to v_T^2) minus a potential term (proportional to ℓ_s)

$$L_{NG} = T_{NG} - V_{NG} \tag{11}$$

where

$$T_{NG} = \frac{1}{2\pi\alpha'} \int d\sigma \, y \left(\frac{v_T^2}{2}\right), \tag{12}$$

$$V_{NG} = \frac{\ell_s}{2\pi\alpha'}. \tag{13}$$

Finally, we want to identify the tension, T, and the linear mass density, μ, of the string. The potential term is some constant times the length of the string. By dimensional analysis, the constant of proportionality is energy per length (i.e., the tension)

$$T = \frac{1}{2\pi\alpha'} = \mu. \tag{14}$$

Problem 2: JBBS 1.5

We are asked to evaluate the "twisted" sum

$$S = \sum_{n=1}^{\infty}(n-\theta) = \frac{1}{24} - \frac{1}{8}(2\theta-1)^2 \tag{15}$$

using the regularization scheme described in §1.3 of JBBS. In the text a similar sum is evaluated by regularizing the sum using the exponential

$$\exp\left\{-\epsilon\frac{|k_\sigma|}{\sqrt{\gamma_{\sigma\sigma}}}\right\} \tag{16}$$

where $k_\sigma = n\pi/\ell$, γ_{ab} is the the metric on the world sheet and the factor $1/\sqrt{\gamma_{\sigma\sigma}}$ is included to make this invariant under reparameterizations of σ (above $\sigma \in [0,\ell]$).

We evaluate S by evaluating the sum

$$S' = \sum_{n=1}^{\infty}(n-\theta)e^{-\epsilon\frac{k_\sigma}{\sqrt{\gamma_{\sigma\sigma}}}} \tag{17}$$

for $k_\sigma = (n-\theta)\pi/\ell$ and then passing to the limit $\epsilon \to 0$. Defining $a = \sqrt{\pi/(2p^+\ell\alpha')}$, S' becomes

$$\begin{aligned}
S' &= \sum_{n=1}^{\infty}(n-\theta)e^{-\epsilon a(n-\theta)} \\
&= -\frac{1}{a}\frac{d}{d\epsilon}\sum_{n=1}^{\infty}e^{-\epsilon a(n-\theta)} \\
&= -\frac{1}{a}\frac{d}{d\epsilon}e^{\epsilon a\theta}\sum_{n=1}^{\infty}\left(e^{-\epsilon a}\right)^n \\
&= -\frac{1}{a}\frac{d}{d\epsilon}e^{\epsilon a\theta}e^{-\epsilon a}\sum_{n=0}^{\infty}\left(e^{-\epsilon a}\right)^n \\
&= -\frac{1}{a}\frac{d}{d\epsilon}\frac{e^{\epsilon a\theta}}{e^{\epsilon a}-1} \\
&= \frac{d}{d\epsilon}\left[\frac{-1}{a^2\epsilon} + \frac{1-2\theta}{2a} - \frac{6\theta^2-6\theta+1}{12}\epsilon + \mathcal{O}(\epsilon^2)\right] \\
&= \frac{1}{a^2\epsilon^2} - \frac{6\theta^2-6\theta+1}{12} + \mathcal{O}(\epsilon) \\
&= \frac{1}{a^2\epsilon^2} + \left(\frac{1}{24} - \frac{1}{8}(2\theta-1)^2\right) + \mathcal{O}(\epsilon).
\end{aligned} \tag{18}$$

In the limit $\epsilon \to 0$ the terms of $\mathcal{O}(\epsilon)$ and higher vanish. However, the first term above is a quadratically divergent in the small ϵ limit. This divergent term is cutoff dependent and proportional to ℓ (recall $a = \sqrt{\pi/(2p^+\ell\alpha')}$). JBBS argues that this divergence can be canceled by a counterterm proportional to $\int d\tau d\sigma\sqrt{-\gamma}$ in the action. Therefore we discard the quadratically divergent term and find that

$$S = \lim_{\epsilon\to 0} S' = \frac{1}{24} - \frac{1}{8}(2\theta-1)^2 \tag{19}$$

as required.

Problem 3: JBBS 1.5

In the light cone gauge, the transverse coordinates X^μ satisfy the free wave equation

$$\partial_\tau X^\mu = c^2 \partial_\sigma^2 X^\mu. \tag{20}$$

The general solution is

$$X^\mu(\tau,\sigma) = x_0^\mu + a^\mu \tau + b^\mu \sigma + \sum_{n\neq 0}\left(A_n^\mu e^{-i\omega_n(c\tau+\sigma)} + B_n^\mu e^{-i\omega_n(c\tau-\sigma)}\right). \tag{21}$$

The Neumann boundary conditions on the $i < 25$ coordinates yield the solutions

$$X^{i<25}(\tau,\sigma) = x_0^i + \frac{p^i}{p^+}\tau + i\sqrt{2\alpha'}\sum_{n\neq 0}\frac{\alpha_n^i}{n}\exp\left(-\frac{\pi i n c\tau}{\ell}\right)\cos\left(\frac{\pi n\sigma}{\ell}\right). \tag{22}$$

For $i = 25$ we apply Dirichlet boundary conditions

$$X^{25}(\tau,\sigma=0) = 0 \tag{23}$$
$$X^{25}(\tau,\sigma=\ell) = y \tag{24}$$

to get

$$X^{25}(\tau,\sigma) = \frac{y}{\ell}\sigma + \sqrt{2\alpha'}\sum_{n\neq 0}\frac{\alpha_n^{25}}{n}\exp\left(-\frac{\pi n i c\tau}{\ell}\right)\sin\left(\frac{\pi n\sigma}{\ell}\right). \tag{25}$$

Notice that the reality of X^{25} implies that $\alpha_{-n}^{25} = \left(\alpha_n^{25}\right)^\dagger$.

Now with the mode expansion in hand we quantize the string modes and compute the mass spectrum. To do this we calculate the conjugate momenta Π^{25}. From the Polyakov Lagrangian

$$L = -\frac{\ell}{2\pi\alpha'}\gamma_{\sigma\sigma}\partial_\tau x^- + \frac{1}{4\pi\alpha'}\int_0^\ell d\sigma\left[\gamma_{\sigma\sigma}\partial_\tau X^i\partial_\tau X^i - \frac{\partial_\sigma X^i\partial_\sigma X^i}{\gamma_{\sigma\sigma}}\right], \tag{26}$$

we obtain

$$\begin{aligned}\Pi^{25} &= \frac{\delta L}{\delta \dot{X}^{25}}\\ &= \frac{1}{2\pi\alpha'}\gamma_{\sigma\sigma}\partial_\tau X^{25}\\ &= \frac{p^+}{\ell}\partial_\tau X^{25}\\ &= \frac{-i}{\sqrt{2\alpha'}\ell}\sum_{n\neq}\alpha_n^{25}\exp\left(-\frac{\pi n i c\tau}{\ell}\right)\sin\left(\frac{\pi n\sigma}{\ell}\right)\end{aligned} \tag{27}$$

where we have used $p^+/\ell = \gamma_{\sigma\sigma}/(2\pi\alpha')$ and $c/\ell = 1/(2\pi\alpha' p^+)$.

The commutation relations among the α_n^i are derived by imposing the equal τ commutation position-momentum commutation relations

$$[X^i(\sigma), \Pi^i(\sigma')] = i\delta(\sigma-\sigma'). \tag{28}$$

Substituting the explicit forms for X^{25} and Π^{25} we obtain

$$i\delta(\sigma-\sigma') = -\frac{i}{\ell}\sum_{n,m\neq 0}\frac{1}{n}\left[\alpha_n^{25},\alpha_m^{25}\right]\exp\left(-\frac{\pi(n+m)ic\tau}{\ell}\right)\sin\left(\frac{\pi n\sigma}{\ell}\right)\sin\left(\frac{\pi m\sigma'}{\ell}\right). \tag{29}$$

Since the LHS is independent of τ, the coefficient of $\exp\left(-\frac{\pi(n+m)ic\tau}{\ell}\right)$ on the RHS must vanish for $m+n\neq 0$. This restricts the sum on the RHS to configurations where $m+n=0$

$$\begin{aligned}\delta(\sigma-\sigma') &= -\frac{1}{\ell}\sum_{n\neq 0}\frac{1}{n}\left[\alpha_n^{25},\alpha_{-n}^{25}\right]\sin\left(\frac{\pi n\sigma}{\ell}\right)\sin\left(\frac{-\pi n\sigma'}{\ell}\right)\\ &= \frac{1}{\ell}\sum_{n\neq 0}\frac{1}{n}\left[\alpha_n^{25},\alpha_{-n}^{25}\right]\sin\left(\frac{\pi n\sigma}{\ell}\right)\sin\left(\frac{\pi n\sigma'}{\ell}\right)\\ &= \frac{2}{\ell}\sum_{n>0}\frac{1}{n}\left[\alpha_n^{25},\alpha_{-n}^{25}\right]\sin\left(\frac{\pi n\sigma}{\ell}\right)\sin\left(\frac{\pi n\sigma'}{\ell}\right).\end{aligned} \tag{30}$$

Multiplying by $\sin\left(\frac{\pi n'\sigma'}{\ell}\right)$ and integrating by σ' we obtain,

$$\begin{aligned}\int d\sigma'\sin\left(\frac{\pi n'\sigma'}{\ell}\right)\delta(\sigma-\sigma') &= \frac{2}{\ell}\sum_{n>0}\frac{1}{n}\left[\alpha_n^{25},\alpha_{-n}^{25}\right]\sin\left(\frac{\pi n\sigma}{\ell}\right)\int d\sigma'\sin\left(\frac{\pi n'\sigma'}{\ell}\right)\sin\left(\frac{\pi n\sigma'}{\ell}\right)\\ \sin\left(\frac{\pi n'\sigma}{\ell}\right) &= \sum_{n>0}\frac{1}{n}\left[\alpha_n^{25},\alpha_{-n}^{25}\right]\sin\left(\frac{\pi n\sigma}{\ell}\right)\delta_{nn'}\\ &= \frac{1}{n'}\left[\alpha_{n'}^{25},\alpha_{-n'}^{25}\right]\sin\left(\frac{\pi n'\sigma}{\ell}\right).\end{aligned} \tag{31}$$

From the above, we arrive at the mode commutation relations

$$\left[\alpha_n^{25},\alpha_{-n}^{25}\right] = n. \tag{32}$$

To calculate the mass spectrum we need the contribution to the Hamiltonian from the 25^{th} mode. The Hamiltonian is given by equation (1.3.19) of JBBS

$$H = \sum_{i=2}^{D-1} H^i \tag{33}$$

$$H^i = \frac{\ell}{4\pi\alpha' p^+}\int_0^\ell d\sigma\left(2\pi\alpha'\Pi^i\Pi^i + \frac{1}{2\pi\alpha'}\partial_\sigma X^i\partial_\sigma X^i\right). \tag{34}$$

Using

$$\left(\Pi^{25}\right)^2 = -\frac{1}{2\alpha'\ell^2}\sum_{n,m\neq 0}\alpha_m^{25}\alpha_n^{25}\exp\left(-\frac{\pi(m+n)ic\tau}{\ell}\right)\sin\left(\frac{\pi m\sigma}{\ell}\right)\sin\left(\frac{\pi n\sigma}{\ell}\right) \tag{35}$$

$$\partial_\sigma X^{25} = \frac{y}{\ell} + \frac{\sqrt{2\alpha'}\pi}{\ell}\sum_{n\neq 0}\alpha_n^{25}\exp\left(-\frac{\pi nic\tau}{\ell}\right)\cos\left(\frac{\pi n\sigma}{\ell}\right) \tag{36}$$

$$\left(\partial_\sigma X^{25}\right)^2 \to \left(\frac{y}{\ell}\right)^2 + \frac{2\alpha'\pi^2}{\ell^2}\sum_{m,n\neq 0}\alpha_m^{25}\alpha_n^{25}\exp\left(-\frac{\pi(m+n)ic\tau}{\ell}\right)\cos\left(\frac{\pi m\sigma}{\ell}\right)\cos\left(\frac{\pi n\sigma}{\ell}\right) \tag{37}$$

(where → denotes equality under a $d\sigma$ integral) we obtain

$$\begin{aligned}
H^{25} &= \frac{\ell}{4\pi\alpha' p^+}\int_0^\ell d\sigma\left(2\pi\alpha'\Pi^{25}\Pi^{25} + \frac{1}{2\pi\alpha'}\partial_\sigma X^{25}\partial_\sigma X^{25}\right)\\
&= \frac{1}{4\alpha' p^+\ell}\sum_{n,m\neq 0}\alpha_m^{25}\alpha_n^{25}\exp\left(-\frac{\pi(m+n)ic\tau}{\ell}\right)\int_0^\ell d\sigma\left(\cos\left(\frac{\pi m\sigma}{\ell}\right)\cos\left(\frac{\pi n\sigma}{\ell}\right) - \sin\left(\frac{\pi m\sigma}{\ell}\right)\sin\left(\frac{\pi n\sigma}{\ell}\right)\right)\\
&+ \frac{y^2}{8\pi^2\alpha'^2 p^+}.
\end{aligned} \tag{38}$$

After some algebra one can show that

$$\begin{aligned}
&\sum_{n,m\neq 0}\alpha_m^{25}\alpha_n^{25}\exp\left(-\frac{\pi(m+n)ic\tau}{\ell}\right)\int_0^\ell d\sigma\left(\cos\left(\frac{\pi m\sigma}{\ell}\right)\cos\left(\frac{\pi n\sigma}{\ell}\right) - \sin\left(\frac{\pi m\sigma}{\ell}\right)\sin\left(\frac{\pi n\sigma}{\ell}\right)\right)\\
&= 2\ell\sum_{m>0}\left(\alpha_{-m}^{25}\alpha_m^{25} + \frac{m}{2}\right)\\
&= 2\ell\left(\sum_{m>0}\alpha_{-m}^{25}\alpha_m^{25} - \frac{1}{24}\right).
\end{aligned} \tag{39}$$

Thus,

$$H^{25} = \frac{1}{2\alpha' p^+}\left(\sum_{m>0}\alpha_{-m}^{25}\alpha_m^{25} - \frac{1}{24}\right) + \frac{y^2}{8\pi^2\alpha'^2 p^+}. \tag{40}$$

From JBBS, the Hamiltonian for the modes where $i < 25$ is given by

$$H^{i<25} = \frac{p^i p^i}{2p^+} + \frac{1}{2\alpha' p^+}\left(\sum_{m>0}\alpha_{-m}^{25}\alpha_m^{25} - \frac{1}{24}\right).$$

Therefore the total Hamiltonian is

$$\begin{aligned}
H &= \sum_{i=2}^{D-1}\left[\frac{p^i p^i}{2p^+} + \frac{1}{2\alpha' p^+}\left(\sum_{m>0}\alpha_{-m}^{25}\alpha_m^{25} - \frac{1}{24}\right)\right] + \frac{1}{2\alpha' p^+}\left(\sum_{m>0}\alpha_{-m}^{25}\alpha_m^{25} - \frac{1}{24}\right) + \frac{y^2}{8\pi^2\alpha'^2 p^+}\\
&= \frac{1}{2\alpha' p^+}\left(\sum_{i=2}^{D-1}\sum_{m>0}\alpha_{-m}^i\alpha_m^i + \frac{2-D}{24}\right) + \sum_{i=2}^{D-1}\frac{p^i p^i}{2p^+} + \frac{y^2}{8\pi^2\alpha'^2 p^+}.
\end{aligned} \tag{41}$$

Now we can calculate the mass spectrum of the string

$$\begin{aligned}
M^2 &= -p_\mu p^\mu\\
&= 2p^+p^- - p^i p^i\\
&= 2p^+ H - p^i p^i\\
&= \frac{1}{\alpha'}\left(\sum_{i=2}^{D-1}\sum_{m>0}\alpha_{-m}^i\alpha_m^i + \frac{2-D}{24}\right) + \sum_{i=2}^{D-1}p^i p^i + \frac{y^2}{4\pi^2\alpha'^2} - \sum_{i=2}^{D-1}p^i p^i\\
&= \frac{N}{\alpha'} + \frac{2-D}{24\alpha'} + \frac{y^2}{4\pi^2\alpha'^2}.
\end{aligned} \tag{42}$$

where $N=\sum_{i=2}^{D-1}\sum_{m>0}\alpha_{-m}^i\alpha_m^i \equiv \sum_{i=2}^{D-1}\sum_{m>0} mN_{i,m}$ is the level.

For the theory to be non-tachyonic, we require $M^2|_{N=0} \geq 0$. This implies

$$y^2 \geq \frac{\alpha' \pi^2}{6}(D-2). \tag{43}$$

If $y=0$ then both ends of the string are attached to the same D_{24} brane. The massless particles correspond to the states where one of the $D-2$ oscillators are excited. The invariant "photon" mass is

$$M_1^2 = \frac{2-D}{24\alpha'} + \frac{1}{\alpha'} = \frac{26-D}{24\alpha^2}.$$

To ensure that the photon is indeed massless, we require $D=26$. Here the modes for $i<25$ oscillate parallel to the D_{24} brane and correspond to fields on the 24-dimensional hypersurface. The $i=25$ modes are perpendicular to the brane and correspond to fluctuations of the brane.

Problem 4

We are asked to find the mode expansion for an open string with one end satisfying a Neumann condition and the other end lying on a D_p-brane having $X^{p+1},\dots,X^{25}=0$. For $i \leq p$, the solutions are given by (22). For $i \geq p+1$, we apply the boundary conditions

$$X^{i\geq p+1}(\tau,\sigma=0) = 0 \tag{44}$$

$$\partial_\sigma X^{i\geq p+1}(\tau,\sigma=\ell) = 0 \tag{45}$$

to the general solution (21) to obtain

$$X^{i\geq p+1}(\tau,\sigma) = \sqrt{2\alpha'}\sum_{n\in\mathbb{Z}+\frac{1}{2}} \frac{\alpha_n^i}{n}\exp\left(-\frac{in\pi c\tau}{\ell}\right)\sin\left(\frac{\pi\sigma}{\ell}\right). \tag{46}$$

Notice that the linear terms in $\tau$and σ along with the constant term are not permitted with the conditions (44) and (45). Furthermore, the reality of X^i implies that $\alpha_{-n}^i = \alpha_n^{i\dagger}$. For $i\geq p+1$, the canonical momentum is

$$\begin{aligned}\Pi^{i\geq p+1}(\tau,\sigma) &= \frac{p^+}{\ell}\partial_\tau X^{i\geq p+1}\\ &= -\frac{i}{\sqrt{2\alpha'}\ell}\sum_{n\in\mathbb{Z}+\frac{1}{2}}\alpha_n^i\exp\left(-\frac{in\pi c\tau}{\ell}\right)\sin\left(\frac{\pi\sigma}{\ell}\right).\end{aligned} \tag{47}$$

Notice that (46) and (47) are obtained from (25) and (27) by setting $y=0$ and letting n run over the half-integers instead of the whole integers.

To quantize the modes on the string we impose the canonical commutation relations

$$[X^i(\sigma),\Pi^i(\sigma')] = i\delta(\sigma-\sigma'). \tag{48}$$

It is easy to show that for all i the Fourier modes satisfy the following commutation relation

$$\left[\alpha_n^i,\alpha_{-n}^i\right] = n. \tag{49}$$

The contributions to the Hamiltonian from the $i \geq p+1$ modes are

$$
\begin{aligned}
H^{i\geq p+1} &= \frac{1}{4\alpha' p^+ \ell} \sum_{n,m\in\mathbb{Z}+\frac{1}{2}} \alpha^i_m \alpha^i_n \exp\left(-\frac{\pi(m+n)ic\tau}{\ell}\right) \\
&\quad \times \int_0^\ell d\sigma \left(\cos\left(\frac{\pi m\sigma}{\ell}\right)\cos\left(\frac{\pi n\sigma}{\ell}\right) - \sin\left(\frac{\pi m\sigma}{\ell}\right)\sin\left(\frac{\pi n\sigma}{\ell}\right)\right) \\
&= \frac{1}{2\alpha' p^+} \sum_{m\in\mathbb{Z}+\frac{1}{2}} \left(\alpha^i_{-m}\alpha^i_m + \frac{m}{2}\right) \\
&= \frac{1}{2\alpha' p^+} \left(\sum_{m\in\mathbb{Z}+\frac{1}{2}} \alpha^i_{-m}\alpha^i_m + \frac{1}{2}\sum_{m>0}\left(m-\frac{1}{2}\right)\right) \\
&= \frac{1}{2\alpha' p^+} \left(\sum_{m\in\mathbb{Z}+\frac{1}{2}} \alpha^i_{-m}\alpha^i_m + \frac{1}{2}\left(\frac{1}{24} - \frac{1}{8}\left(2\left(\frac{1}{2}\right)-1\right)^2\right)\right) \\
&= \frac{1}{2\alpha' p^+} \left(\sum_{m\in\mathbb{Z}+\frac{1}{2}} \alpha^i_{-m}\alpha^i_m + \frac{1}{48}\right) \qquad (50)
\end{aligned}
$$

where we have used the result from problem 2 to complete the sum in the second term. The full Hamiltonian is becomes

$$
\begin{aligned}
H &= \sum_{i=2}^{p}\left[\frac{p^ip^i}{2p^+} + \frac{1}{2\alpha' p^+}\left(\sum_{m>0}\alpha^i_{-m}\alpha^i_m - \frac{1}{24}\right)\right] + \frac{1}{2\alpha' p^+}\sum_{i=p+1}^{D-1}\left(\sum_{m\in\mathbb{Z}+\frac{1}{2}}\alpha^i_{-m}\alpha^i_m + \frac{1}{48}\right) \\
&= \frac{1}{2\alpha' p^+}\left(\sum_{i=2}^{p}\sum_{m>0}\alpha^i_{-m}\alpha^i_m + \sum_{i=p+1}^{D-1}\sum_{m\in\mathbb{Z}+\frac{1}{2}}\alpha^i_{-m}\alpha^i_m\right) + \frac{1}{2\alpha' p^+}\left(-\sum_{i=2}^{p}\frac{1}{24} + \sum_{i=p+1}^{D-1}\frac{1}{48}\right) + \sum_{i=2}^{p}\frac{p^ip^i}{2p^+} \\
&= \frac{1}{2\alpha' p^+}\left(\sum_{i=2}^{p}\sum_{m>0}\alpha^i_{-m}\alpha^i_m + \sum_{i=p+1}^{D-1}\sum_{m\in\mathbb{Z}+\frac{1}{2}}\alpha^i_{-m}\alpha^i_m\right) + \frac{1}{2\alpha' p^+}\left(-\frac{p-1}{24} + \frac{D-1-p}{48}\right) + \sum_{i=2}^{p}\frac{p^ip^i}{2p^+} \\
&= \frac{N}{2\alpha' p^+} + \frac{1}{2\alpha' p^+}\frac{D-3p+1}{48} + \sum_{i=2}^{p}\frac{p^ip^i}{2p^+} \qquad (51)
\end{aligned}
$$

where

$$
N = \sum_{i=2}^{p}\sum_{m>0}\alpha^i_{-m}\alpha^i_m + \sum_{i=p+1}^{D-1}\sum_{m\in\mathbb{Z}+\frac{1}{2}}\alpha^i_{-m}\alpha^i_m. \qquad (52)
$$

The mass spectrum is then

$$
M^2 = \frac{N}{\alpha'} + \frac{D-3p+1}{48\alpha'}. \qquad (53)
$$

For the theory to be tachyonic, we require

$$
\frac{D-3p+1}{48\alpha'} < 0 \qquad \Longrightarrow p > \frac{D+1}{3}. \qquad (54)
$$