Sep 2, 2005

Physics 340 lecture 1

The first question to ask is...

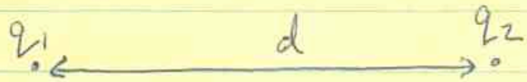
Why study electromagnetism?

A fundamental part of physics – one of the four forces
It is ubiquitous in everyday life
natural phenomena in
geo/astrophysics
"Advanced physics" have to deal with
abstract concepts (fields)
mathematical techniques

Today, the plan is to introduce some basic ideas as well as to give an <u>overview</u> of the whole course.

In this course, we will think about electric and magnetic <u>fields</u> rather than <u>forces</u> between charges and currents.

Force between two charges



Coulomb's law

$$F = \frac{q_1 q_2}{4\pi\varepsilon_0 d^2}$$

charge q has units of Coulombs (C)
the charge of an electron is $-1.6 \times 10^{-19}\,C = -e$

a useful thing to remember is that $\dfrac{1}{4\pi\varepsilon_0} = 10^{-7} c^2$

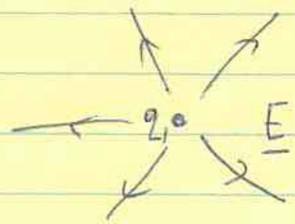
$\varepsilon_0 = 8.85 \times 10^{-12} \; \frac{C^2}{Nm^2}$ "permittivity of free space"

~~Now subtly change our thinking — the charge~~

We can write the force on a test charge as $\underline{F} = q\,\underline{E}$, where $\underline{E}$ is the electric field.
In other words, $\underline{E}$ is the force per unit charge (units NC^{-1} or Vm^{-1})

The idea is to think of the first charge producing an electric field

$$\underline{E} = \frac{q_1}{4\pi\varepsilon_0 d^2}\,\hat{\underline{r}}$$

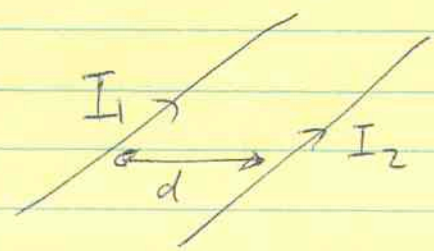


which then acts on the second charge, $\underline{F} = q_2\,\underline{E}$

Force between two currents

what about permanent magnets? They also contain microscopic currents

For magnetic forces, the basic object is a current, eg. consider two parallel current-carrying wires



I = current = charge per second passing a given point on the wire
units are Cs^{-1}
or A (amperes) (amps)

Note that a positive current can be either positively charged particles moving in the direction of the current, or negativelycharged particles moving opposite to the current direction.

The force between two current carrying wires is

$$F = \frac{\mu_0 I_1 \cdot I_2}{2\cancel{4}\pi d^{\cancel{2}}}$$

(attractive if the currents are aligned)

(this is the force per unit length)

μ_0 is the "permeability of free space"

$$\frac{\mu_0}{4\pi} = 10^{-7} \, \frac{\mathrm{Tm}}{\mathrm{A}}$$

[Note that $\frac{1}{\mu_0 \varepsilon_0} = c^2$ or $c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}$ the speed of light is already appearing.

We'll see later that light emerges as a wave-like solution of Maxwell's equations.]

Now do the same thing as before and think about <u>fields</u>. The current produces a magnetic field, the magnetic field acts on the second current.

First, the force is $\underline{F} = q\,(\underline{v} \times \underline{B})$

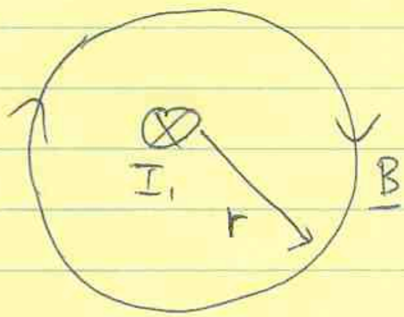
on a charge q moving with velocity $\underline{v}$.

So if we think about two parallel currents

$$\underset{I_1}{\otimes} \longrightarrow \text{ATTRACT} \longleftarrow \underset{I_2}{\otimes}$$

$\underline{v}$ is into the page

then the magnetic field at the position of I_2 must point <u>downwards</u> so that $\underline{v} \times \underline{B}$ points to the left.



B due to a wire is

$$\underline{B} = \frac{\mu_0 I}{2\pi r} \hat{e}_\phi$$

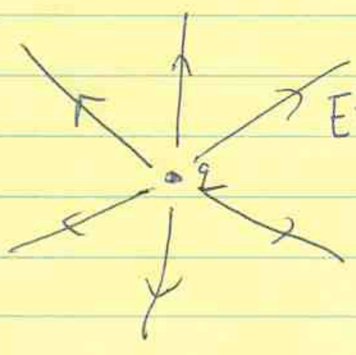
[check $F = N_2 q v B$ $\quad$ $N =$ number of charges per unit length

$$= N_2 q v \frac{\mu_0 I_1}{2\pi d} \qquad I_2 = N_2 q v$$

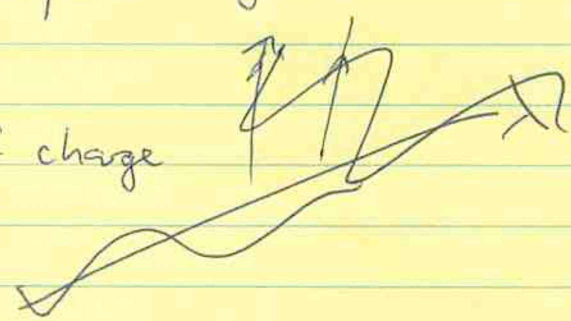
$$= \frac{\mu_0 I_1 I_2}{2\pi d}$$]

<u>Vector calculus gives a way to describe these fields</u>

we've seen that the electric field of a point charge is

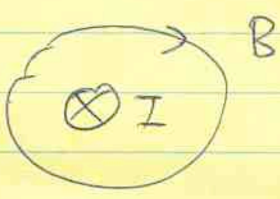


$E = \frac{q}{4\pi\varepsilon_0 r^2}$ or a line of charge



$$E = \frac{\lambda}{2\pi\varepsilon_0 r}$$

for a current

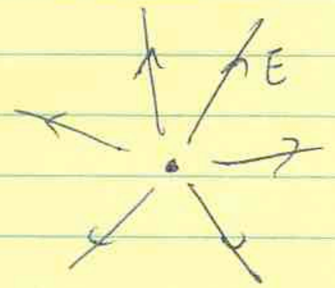


Note these fields look very different:

$\underline{E}$ "diverges" whereas $\underline{B}$ has "loops"

electric field lines begin and end on charges
magnetic field lines close on themselves to form loops.

Maxwell's equations are a way of describing this behaviour using vector calculus.



$\underline{\nabla} \cdot \underline{E} = \rho/\epsilon_0$

$\underline{\nabla} \times \underline{B} = \mu_0 \underline{J}$

$\underline{\nabla} \times \underline{E} = 0$

$\underline{\nabla} \cdot \underline{B} = 0$

(Maxwell's equations for time-independent fields)

We'll review the maths of divergence and curl next time, and then first for $\underline{E}$ then for $\underline{B}$, start with the observed force law and derive Maxwell's equations for the fields.

Lectures 2 +3 | Sep 7 +9, 2005 | Notes on div, grad, curl etc.

Last time, we discussed forces between charges and currents, and what the corresponding electric and magnetic fields look like.

Today, we'll talk about the different mathematical tools we can use to describe vector fields. (We'll cover most of Griffiths chapter 1, except section 1.5 delta functions which we'll cover next week).

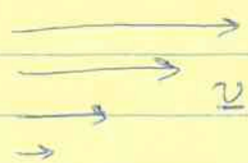
Scalar and vector fields

Examples of scalar fields: temperature $T(x,y,z)$; electric potential $V(x,y,z)$; or gravitational potential, chemical potential...

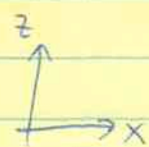
Examples of vector fields: velocity of a fluid $\underline{v}(x,y,z)$; electric field $\underline{E}(x,y,z)$

To sketch a vector field, draw representative vectors whose length is prpportional to the magnitude of the field at each point.

eg. a shearing fluid flow



$$\underline{v} = z\hat{x}$$



Derivatives of scalar and vector fields

gradient operator

$$\underline{\nabla} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right)$$

a vector

see inside front cover * of Griffiths for div, grad, curl in spherical and cylindrical coordinates

gradient of a scalar field

$$\text{grad}\, \phi = \underline{\nabla}\phi = \left(\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z} \right)$$

scalar field $\phi(x,y,z)$

$\underline{\nabla}\phi$ points in the direction of the maximum gradient

The gradient in the direction of unit vector $\hat{\underline{n}}$ is $\hat{\underline{n}} \cdot \underline{\nabla}\phi$

eg. if $\hat{\underline{n}} = \hat{\underline{x}}$, $\hat{\underline{x}} \cdot \underline{\nabla}\phi = \frac{\partial \phi}{\partial x}$ gradient in the x-direction

There are two ways to operate $\underline{\nabla}$ on a vector field $\underline{A}$

① divergence $\underline{\nabla} \cdot \underline{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$ (a scalar)

② curl $\underline{\nabla} \times \underline{A} = \begin{vmatrix} \hat{\underline{x}} & \hat{\underline{y}} & \hat{\underline{z}} \\ \partial/\partial x & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$ (a vector)

$$= \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}, \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}, \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right)$$

We briefly discussed the geometric interpretation of each of these last time. More on that later today.

The second derivative $\underline{\nabla} \cdot \underline{\nabla}\phi = \nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2}$

is called the Laplacian.

Volume, surface, and line integrals

① Volume integral eg. $\int dV\, \underline{\Phi}(\underline{r}) = \int d^3\underline{r}\; \underline{\Phi}(\underline{r})$

Cartesian coords $\int dx\,dy\,dz\;\underline{\Phi}(x,y,z)$

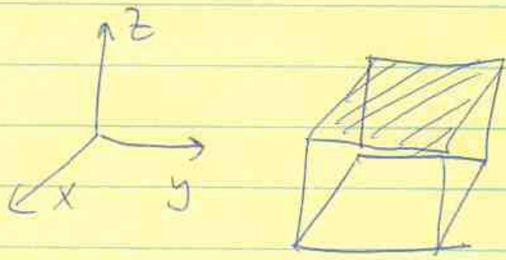
Spherical coords $\int r^2\,dr\,\sin\theta\,d\theta\,d\phi\;\underline{\Phi}(r,\theta,\phi)$

② Surface integral $\int_S \underline{v}\cdot d\underline{A}$

$d\underline{A} = \hat{\underline{n}}\,|d\underline{A}|$

$\hat{\underline{n}}$ is the unit vector normal to the surface at each point

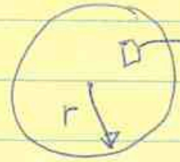
eg. Cartesian coords



$$\underline{v} = (v_x, v_y, v_z)$$

$$\int_{\substack{\text{top of}\\ \text{cube}}} \underline{v}\cdot d\underline{A} = \int_0^a\!\!\int_0^a v_z(x,y,z=a)\,dx\,dy$$

eg. spherical coords



sphere

$d\underline{A} = \hat{\underline{r}}\; r^2 \sin\theta\, d\theta\, d\phi$

$$\int_{\text{sphere}} \underline{v}\cdot d\underline{A} = \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta\, d\theta\;\; v_r(r=a)\, a^2$$

Procedure for doing surface integrals:

1) write down $d\underline{A}$
2) calculate $\underline{v} \cdot d\underline{A}$
3) integrate, remembering to apply conditions on the surface (eg. z = constant or r = constant)

③ Line integrals

$$\int_{\text{path}} \underline{v} \cdot d\underline{l}$$

$$d\underline{l} = dx\,\hat{\underline{x}} + dy\,\hat{\underline{y}} + dz\,\hat{\underline{z}}$$

$$\underline{v} \cdot d\underline{l} = v_x\,dx + v_y\,dy + v_z\,dz$$

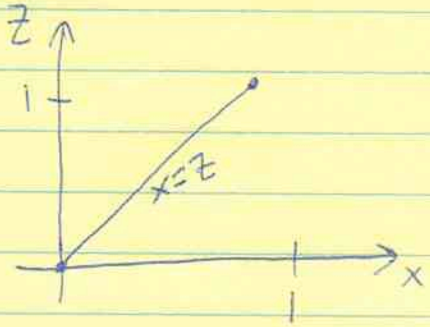
$$\Rightarrow \int \underline{v} \cdot d\underline{l} = \int v_x\,dx + \int v_y\,dy + \int v_z\,dz$$

where v_x, v_y, v_z are functions of (x, y, z)

The tricky part is to specify the path, so in

$\int dx\, v_x(x, y, z)$ you must specify $y(x)$ and $z(x)$ along the path.

eg.



let's integrate along a path $x = z$ in the $y = 0$ plane.

$$\int \underline{v} \cdot d\underline{l} = \int_0^1 v_x(x=z, y=0)\,dx + \int_0^1 v_z(x=z, y=0)\,dz$$

$$+ \underbrace{\int_0^0 dy\; v_y(x=z, y=0)}_{\text{this term is zero because path length in } y\text{-direction} = 0.}$$

Note that the procedure is similar for line integrals as for surface integrals

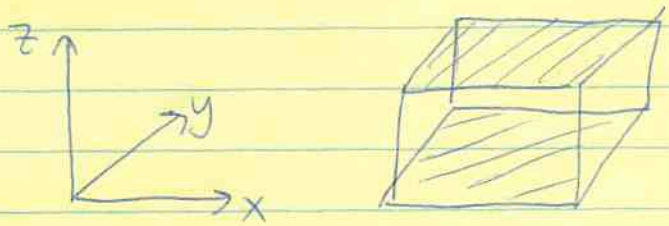
1) write down $d\underline{l}$
2) calculate $\underline{v} \cdot d\underline{l}$
3) integrate each term, remembering to relate the coordinates to each other depending on the specified path.

The divergence theorem

Consider a fluid with velocity $\underline{v}$ and density ρ, both functions of position.

Define the mass flux $\underline{U} = \rho \underline{v}$ (units $= \text{kg m}^{-2}\text{s}^{-1}$)

the amount of mass flowing per unit area per unit time at each point.



cube of side a

how much mass is flowing out of the top and bottom of the cube?
$$\int_0^a \int_0^a dx\,dy \left[U_z(z=a) - U_z(z=0) \right]$$

but $$U_z(z=a) - U_z(z=0) = \int_0^a dz \frac{\partial U_z}{\partial z}$$

$\Rightarrow$ the mass loss through the top and bottom is $\int dx\,dy\,dz \frac{\partial U_z}{\partial z}$

the total mass loss is
$$\int \underline{U} \cdot d\underline{A} = \int dx\,dy\,dz \left(\frac{\partial U_z}{\partial z} + \frac{\partial U_x}{\partial x} + \frac{\partial U_y}{\partial y} \right)$$

$$\boxed{\int_S \underline{U} \cdot d\underline{A} = \int_V (\underline{\nabla} \cdot \underline{U})\, dV}$$

This is the divergence theorem, and holds for and volume V with surface S *

For the fluid, note that the total mass in volume V is
$$M = \int \rho\, dV$$

therefore

$$\frac{dM}{dt} = \frac{d}{dt}\int \rho \, dV = \int \frac{\partial \rho}{\partial t}\, dV$$

$$= -\int \underline{u}\cdot d\underline{A} = -\int (\underline{\nabla}\cdot\underline{u})\, dV$$

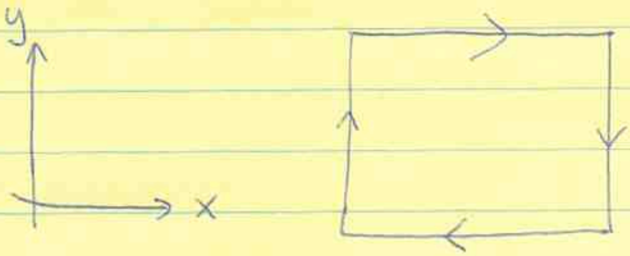
$$\Rightarrow \quad \boxed{\frac{\partial \rho}{\partial t} = -\,\underline{\nabla}\cdot(\rho\underline{v})}$$

equation of continuity for a fluid

we'll see something similar for electric currents later

Stokes' theorem

Now do a line integral of the velocity around a loop.



$$\oint \underline{v}\cdot d\underline{\ell} = \int_0^1 dx\; v_x(y=1) + \int_1^0 dx\; v_x(y=0) + \begin{pmatrix}\text{vertical}\\ \text{terms}\end{pmatrix}$$

$$= \int_0^1 dx\,\left[v_x(y=1) - v_x(y=0)\right] + \begin{pmatrix}\text{vertical}\\ \text{terms}\end{pmatrix}$$

$$= \int_0^1\!\!\int_0^1 dx\,dy\;\frac{\partial v_x}{\partial y} + \begin{pmatrix}\text{vertical}\\ \text{terms}\end{pmatrix}$$

$$= \int_0^1\!\!\int_0^1 dx\,dy\left(\frac{\partial v_x}{\partial y} - \frac{\partial v_y}{\partial x}\right) = -\int_0^1\!\!\int_0^1 dx\,dy\,(\underline{\nabla}\times\underline{v})_z$$

$$\boxed{\oint \underline{v}\cdot d\underline{\ell} = \int_S d\underline{A}\cdot(\underline{\nabla}\times\underline{v})}$$

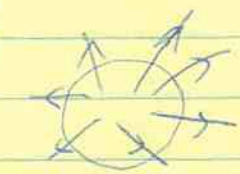
Stokes' theorem *

true for any surface bounded by the loop

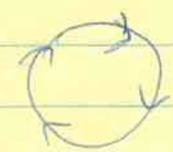
Note the direction of $d\underline{A}$ is given by the right hand rule based on the integration path.
In the previous exaple, we integrated clockwise $\Rightarrow$ $d\underline{A}$ must point into the page $\Rightarrow$ $d\underline{A} = -\hat{\underline{z}}$

$$\Rightarrow \int d\underline{A} \cdot (\underline{\nabla} \times \underline{v}) = - \int dx\, dy\, (\underline{\nabla} \times \underline{v})_z \quad ✓$$

The divergence and Stoke's theorems show us that a vector field with $\underline{\nabla} \cdot \underline{v} > 0$ has a net flux accross a closed surface at that point – it's vectors are diverging (or converging)



A vector field with $\underline{\nabla} \times \underline{v} > 0$ has "loops" – a line integral around a loop at that point has a finite contribution.



As we argued ~~was~~ in the first lecture, the first case corresponds to electrostatic fields; the second to magnetostatic fields.

PHYS 340 lecture 4 Sep 12th 2005

THIS WEEK

1. $\nabla \cdot \mathbf{E} = \rho/\varepsilon_0$
   $\nabla \times \mathbf{E} = 0$
2. How to calculate $\mathbf{E}$ given a distribution of charge

ELECTROSTATICS

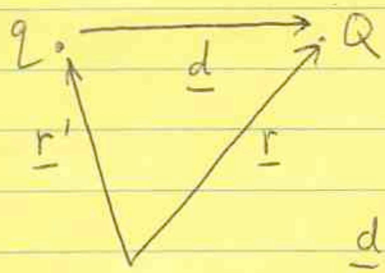
Our approach to both electrostatics and magnetostatics will be very similar. We start with the experimentally confirmed force law and then constrain the properties of the field.

Two charges repel or attract according to Coulomb's law

$$F_Q = \frac{qQ}{4\pi\varepsilon_0 d^2}$$

force on Q

in vector notation

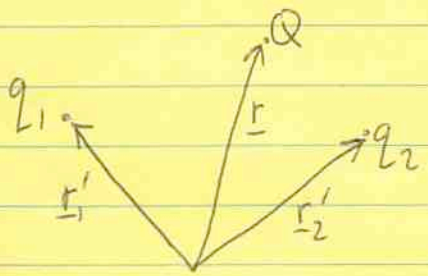


$\mathbf{d} = \mathbf{r} - \mathbf{r}'$

$$\mathbf{F}_Q = \frac{Qq}{4\pi\varepsilon_0 |\mathbf{d}|^2}\,\hat{\mathbf{d}} = \frac{Qq}{4\pi\varepsilon_0}\,\frac{\mathbf{d}}{|\mathbf{d}|^3}$$

$$\mathbf{F}_Q = \frac{Qq}{4\pi\varepsilon_0}\,\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}$$

Principle of superposition



$$\mathbf{F}_Q = \frac{Q}{4\pi\varepsilon_0}\left[q_1 \frac{\mathbf{r}-\mathbf{r}_1'}{|\mathbf{r}-\mathbf{r}_1'|^3} + q_2 \frac{\mathbf{r}-\mathbf{r}_2'}{|\mathbf{r}-\mathbf{r}_2'|^3} \right]$$

If we generalize to N charges

$$\underline{F}_Q = \frac{Q}{4\pi\varepsilon_0} \sum_{i=1}^{N} q_i \frac{\underline{r}-\underline{r}_i'}{|\underline{r}-\underline{r}_i'|^3}$$

The next step is to define the electric field as

$$\underline{F}_Q = Q\,\underline{E} \qquad Q \text{ is a "test charge"}$$

$$\Rightarrow \boxed{\underline{E}(\underline{r}) = \frac{1}{4\pi\varepsilon_0} \sum_{i=1}^{N} q_i \frac{\underline{r}-\underline{r}_i'}{|\underline{r}-\underline{r}_i'|^3}}$$

the electric field of a collection of point charges.

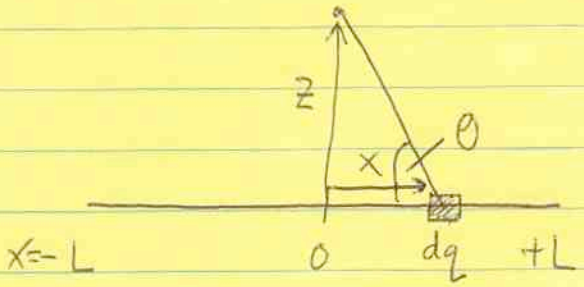
It's straightforward to generalize this to a continuous charge distribution

$$\underline{E}(\underline{r}) = \frac{1}{4\pi\varepsilon_0} \int dq(\underline{r}') \frac{\underline{r}-\underline{r}'}{|\underline{r}-\underline{r}'|^3}$$

where the element of charge dq can be written in different ways depending on the geometry

$dq = \rho(\underline{r}')\,dV$	volume charge density	ρ	C/m^3
$\sigma(\underline{r}')\,dA$	surface charge density	σ	C/m^2
$\lambda(\underline{r}')\,dx$	line charge density	λ	C/m

Example: line charge (example 2.1 from Griffiths)

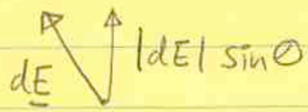


λ = constant

$dq = \lambda\, dx$

The first step is to $\boxed{\text{take advantage of the symmetry.}}$ Because we are calculating the field at $x=0$, the x-components cancel: $E_x = 0$.

The z-component is



$$\Rightarrow \quad E_z = 2\int_0^L |dE| \sin\theta = 2\int_0^L \frac{dq}{4\pi\varepsilon_0}\,\frac{1}{x^2+z^2}\,\sin\theta$$

But $\sin\theta = \dfrac{z}{\sqrt{x^2+z^2}}$

$$\Rightarrow \quad E_z = 2\int_0^L \frac{\lambda}{4\pi\varepsilon_0}\,dx\,\frac{1}{x^2+z^2}\,\frac{z}{\sqrt{x^2+z^2}}$$

$$= \frac{\lambda z}{2\pi\varepsilon_0}\int_0^L \frac{dx}{(x^2+z^2)^{3/2}}$$

$$\boxed{E_z = \frac{\lambda}{2\pi\varepsilon_0}\,\frac{L}{z}\,\frac{1}{\sqrt{L^2+z^2}}}$$

do the integral with the substitution

$x = z\tan y$

and use $dx = z\sec^2 y\, dy$

$1+\tan^2 y = \sec^2 y$

you'll need to prove $\sin\left[\tan^{-1}\frac{L}{x}\right] = \frac{L}{\sqrt{L^2+z^2}}$

Now take limits to check our result:

$z \gg L \qquad E = \dfrac{2\lambda L}{4\pi\varepsilon_0 z^2}$ } same as a point charge

$z \ll L \qquad E = \dfrac{\lambda}{2\pi\varepsilon_0 z}$ } the line charge is effectively infinite

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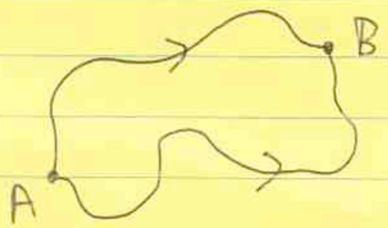
<u>The curl-free nature of $\underline{E}$</u>

Consider moving our test charge Q from A to B in an electrostatic field.



What is the work done (by <u>us</u>)?

$$dW = -d\underline{l} \cdot Q\underline{E}(\underline{r})$$

$$\Rightarrow \boxed{W = -\int_A^B Q\,\underline{E}(\underline{r}) \cdot d\underline{l}}$$

The work done W must be <u>path independent</u>, otherwise we could extract energy by moving the charge in the right way!

$\Rightarrow$ $\underline{E}$ is a <u>conservative</u> field.

$\int_A^B \underline{E} \cdot d\underline{l}$ is independent of path

$$\Rightarrow \oint \underline{E} \cdot d\underline{l} = 0$$

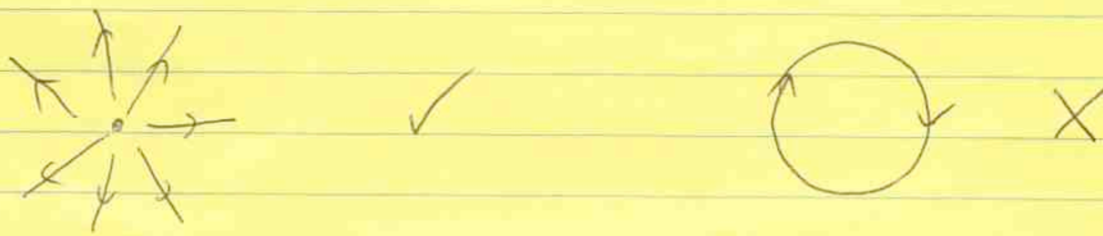
Now apply Stokes' theorem

$$\Rightarrow \int \underline{\nabla} \times \underline{E} \cdot d\underline{A} = 0$$

$$\Rightarrow \boxed{\underline{\nabla} \times \underline{E} = 0}$$ since the size and shape of the loop is arbitrary.

This is the first field equation of electrostatics, and tells us that the electrostatic field is <u>curl-free</u> or <u>irrotational</u>.

What does this mean physically? Think about lines of force



There are no loops of $\underline{E}$ $\Rightarrow$ the field lines must begin and end somewhere – on charges! This behavior is described by Gauss' law $\boxed{\underline{\nabla} \cdot \underline{E} = \rho / \varepsilon_0}$ which we'll derive next time.

PHYS 340 lecture 5 Sep 14th 2005

One consequence of the fact that $\underline{\nabla} \times \underline{E} = 0$ is that we can define the electric potential. Any curl-free field can be written as the gradient of a scalar field.

Define $\underline{E} = -\underline{\nabla} V(\underline{r})$

[Check $\underline{\nabla} \times \underline{E} = -\underline{\nabla} \times \underline{\nabla} V$

$$= -\begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \frac{\partial V}{\partial x} & \frac{\partial V}{\partial y} & \frac{\partial V}{\partial z} \end{vmatrix} = 0$$

eg. x-component $\frac{\partial}{\partial y}\left(\frac{\partial V}{\partial z}\right) - \frac{\partial}{\partial z}\left(\frac{\partial V}{\partial y}\right) = 0$]

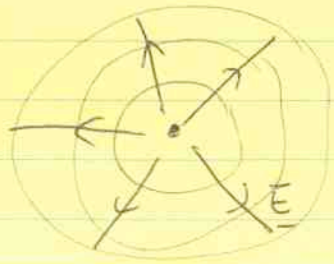
A few points about electric potential $V(\underline{r})$:

1) units of V are volts or JC^{-1}

$$[E] = \left[\frac{\text{Newtons}}{C}\right]$$

$$[V] = \left[\frac{\text{Newtons} \cdot m}{C}\right] = \left[J/C\right]$$

2) lines of constant V are "<u>equipotentials</u>"



electric field lines are perpendicular to the equipotential surfaces.

3) V is determined up to a constant

Because $\underline{E}$ is the gradient of V, you can add a constant without changing $\underline{E}$.

4) The work done in moving charge q from A to B is

$$W = -\int_A^B q\,\underline{E}\cdot d\underline{l} = \int_A^B q\,\underline{\nabla}V\cdot d\underline{l} = q\,(V_B - V_A)$$

$$= q\,\Delta V$$

5) V is a very useful quantity because it's often simpler to calculate. Once we've found V, we can then calculate $\underline{E}$ using $\underline{E} = -\underline{\nabla}V$.

To calculate V, we note that

$$\frac{\underline{r}-\underline{r}'}{|\underline{r}-\underline{r}'|^3} = -\underline{\nabla}\frac{1}{|\underline{r}-\underline{r}'|} \qquad \left(\text{compare } \frac{\partial}{\partial x}\left(\frac{1}{x}\right) = -\frac{1}{x^2}\right)$$

use this to write

$$\underline{E}(\underline{r}) = \int \frac{dq(\underline{r}')}{4\pi\varepsilon_0}\frac{\underline{r}-\underline{r}'}{|\underline{r}-\underline{r}'|^3} = -\int \frac{dq(\underline{r}')}{4\pi\varepsilon_0}\left[\underline{\nabla}\frac{1}{|\underline{r}-\underline{r}'|}\right]$$

But $\underline{\nabla}$ only operates on $\underline{r}$ not $\underline{r}'$

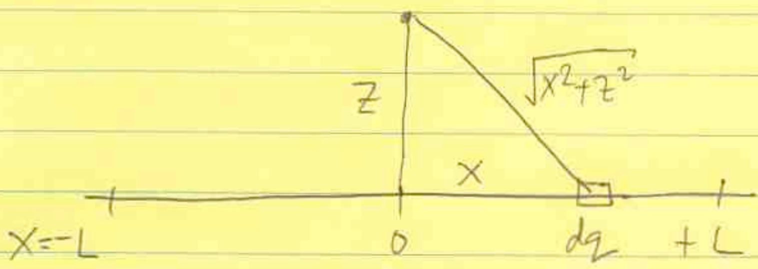
$$\Rightarrow \quad \underline{E} = -\underline{\nabla} V$$

where $\boxed{V(\underline{r}) = \int \frac{dq(\underline{r}')}{4\pi\varepsilon_0 |\underline{r}-\underline{r}'|}}$

Note that V is determined only up to a constant since $E = -\nabla V$ you can add a constant without changing $\underline{E}$

usually a simpler integral than the one for $\underline{E}(\underline{r})$

Example: let's return to the line charge



$$V(z) = \int \frac{dq}{4\pi\varepsilon_0}\frac{1}{|\underline{r}-\underline{r}'|} = \int_{-L}^{L} \frac{\lambda}{4\pi\varepsilon_0}\frac{dx}{\sqrt{x^2+z^2}}$$

$$\Rightarrow \boxed{V(z) = \frac{\lambda}{4\pi\varepsilon_0}\ln\left\{\frac{L+\sqrt{z^2+L^2}}{-L+\sqrt{z^2+L^2}}\right\}}$$

$E_x = 0$ because of symmetry. $\quad E_z = -\frac{\partial V}{\partial z} = \frac{\lambda}{2\pi\varepsilon_0}\frac{L}{z}\frac{1}{\sqrt{L^2+z^2}}$ ✓

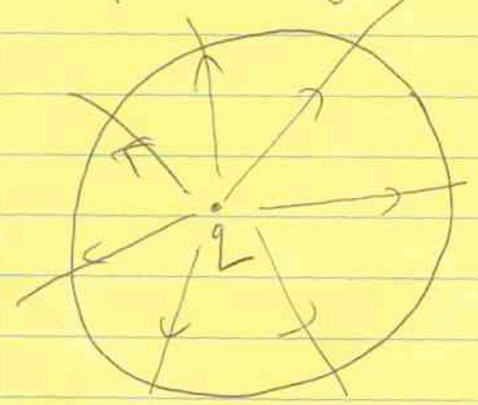
Gauss' Law

The second field equation is Gauss' law $\underline{\nabla} \cdot \underline{E} = \rho/\varepsilon_0$.

To derive this, we consider the quantity

$$\boxed{\Phi = \int \underline{E} \cdot d\underline{A}}$$ the electric flux

First, calculate Φ by integrating over a sphere around a point charge.



$$\underline{E} = \frac{q}{4\pi\varepsilon_0 r^2}\,\hat{\underline{r}}$$

$$d\underline{A} = r^2 \sin\theta \, d\theta \, d\phi \; \hat{\underline{r}} = r^2 d\Omega \, \hat{\underline{r}}$$

$$\Rightarrow \underline{E} \cdot d\underline{A} = d\Phi$$

$$= \frac{q}{4\pi\varepsilon_0 \cancel{r^2}} \cancel{r^2} \, d\Omega$$

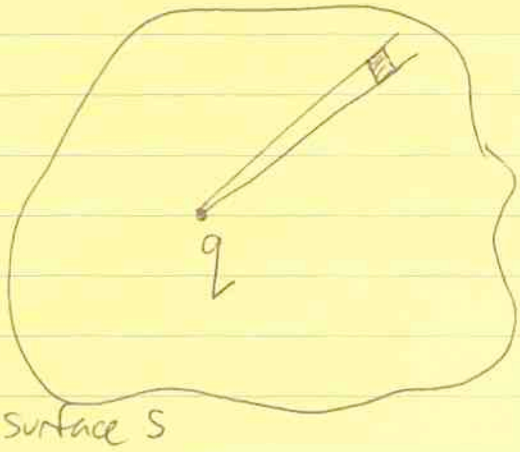
the r^2 factors cancel

This is a remarkable result — $d\Phi$ is independent of distance from the point charge. The total flux is

$$\Phi = \int \underline{E} \cdot d\underline{A} = \frac{q}{\varepsilon_0} \frac{\int d\Omega}{4\pi} = \frac{q}{\varepsilon_0}$$

PHYS 340 lecture 6 Sep 16th 2005

What about integrating over a more complicated surface?



$$d\underline{A} = \frac{\hat{\underline{n}}\, r^2 d\Omega}{|\hat{\underline{n}}\cdot\hat{\underline{r}}|}$$

the size of the area element is the projected area

$$\Rightarrow \quad d\Phi = \underline{E}\cdot d\underline{A}$$

$$= \frac{q}{\varepsilon_0}\,\frac{d\Omega}{4\pi}\,\frac{\hat{\underline{n}}\cdot\hat{\underline{r}}}{|\hat{\underline{n}}\cdot\hat{\underline{r}}|}$$

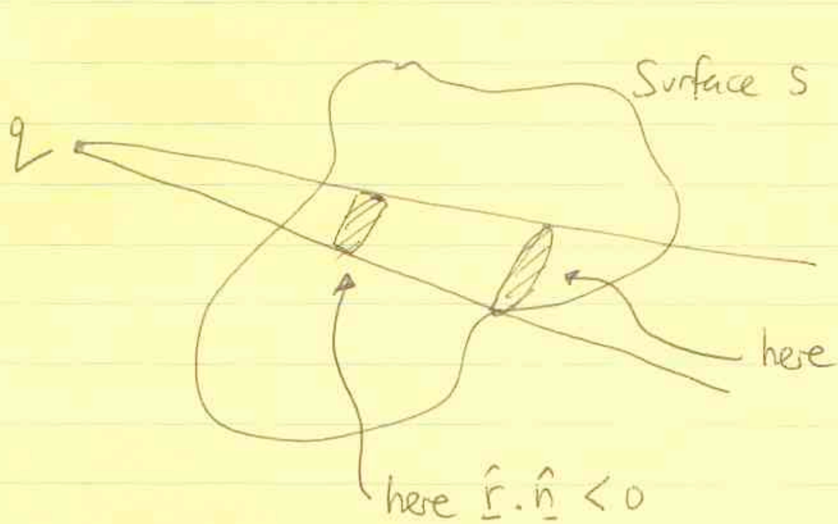
(the last factor is ± 1 depending on whether $\underline{E}$ points into or out of the volume)

Integrating over $d\Omega$ covers the entire surface S

$$\Rightarrow \quad \boxed{\int_S \underline{E}\cdot d\underline{A} = q/\varepsilon_0}$$

if there are several charges, $q \to \sum_i q_i$

What if the charge is outside the volume?



the two contributions cancel!

* charges outside do not contribute *

$\Rightarrow$ Gauss' Law $$\boxed{\int_S \underline{E} \cdot d\underline{A} = \frac{Q_{enclosed}}{\varepsilon_0}}$$

Applying the divergence theorem gives the differential form of Gauss' law

LHS $$\int_S \underline{E} \cdot d\underline{A} = \int \underline{\nabla} \cdot \underline{E} \, dV$$

RHS $$\frac{Q_{enc}}{\varepsilon_0} = \frac{1}{\varepsilon_0} \int \rho \, dV$$

$$\Rightarrow \quad \boxed{\underline{\nabla} \cdot \underline{E} = \rho / \varepsilon_0}$$

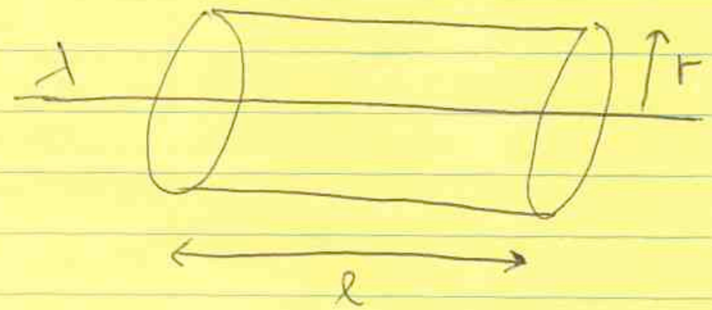
As with $\underline{\nabla} \times \underline{E} = 0$, we've derived this result by arguing from a physical perspective. A more mathematical proof is to take the divergence of the integral expression for $\underline{E}$ — we'll do this next week.

Applications of Gauss' Law

charge distributions with a high degree of symmetry

1) infinite line charge

because the line charge has a length $\gg r$ the field is radial



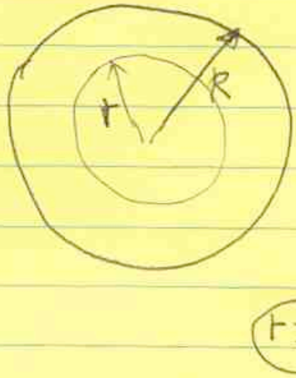
$\Rightarrow$ integrate over the surface of a Gaussian cylinder

$$\int \underline{E} \cdot d\underline{A} = E(r)\, 2\pi r \ell = \frac{\lambda \ell}{\varepsilon_0}$$

$$\Rightarrow \boxed{E(r) = \frac{\lambda}{2\pi r \varepsilon_0}}$$

as we found previously for a line charge with $r \ll L$

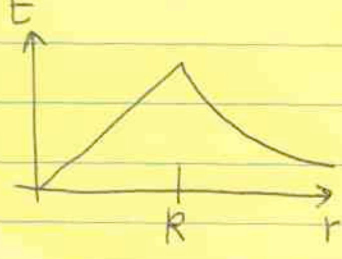
2) uniformly-charged sphere



$r < R$ $\quad 4\pi r^2 E = \rho \frac{4\pi r^3}{3} \frac{1}{\varepsilon_0}$

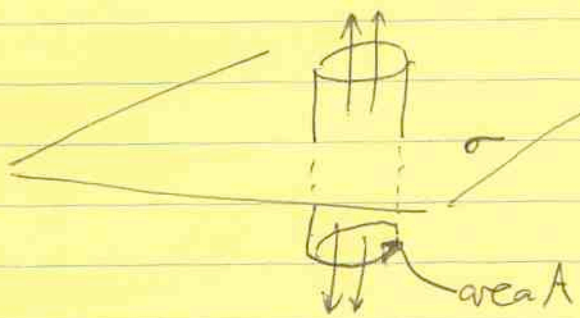
$$\boxed{E = \frac{\rho r}{3\varepsilon_0}}$$

$r > R$ $\quad 4\pi r^2 E = \rho \frac{4\pi R^3}{3} \frac{1}{\varepsilon_0}$



$$\boxed{E = \frac{\rho R^3}{3\varepsilon_0 r^2} = \frac{Q_{total}}{4\pi \varepsilon_0 r^2}}$$

3) uniformly charged plane



by symmetry, the field is vertical if the sheet of charge is very large.

then $2EA = \dfrac{\sigma A}{\varepsilon_0}$ or $\boxed{E = \dfrac{\sigma}{2\varepsilon_0}}$

Notice that spherical symmetry $E \propto \dfrac{1}{r^2}$ at large distances

cylindrical $E \propto \dfrac{1}{r}$

plane $E \simeq$ constant

Last week, we showed that $\underline{\nabla} \times \underline{E} = 0$ and $\underline{\nabla} \cdot \underline{E} = \rho/\varepsilon_0$ by arguing from a physical perspective – either that the work done in moving from point A to B is independent of path, or by integrating the electric flux $\int \underline{E} \cdot d\underline{A}$ and relating it to the interior charge.

But we can also derive these results by starting with

$$\underline{E}(\underline{r}) = \int \frac{dq(\underline{r}')}{4\pi\varepsilon_0} \frac{\underline{r}-\underline{r}'}{|\underline{r}-\underline{r}'|^3} \qquad \text{—— } (*)$$

First, we showed in HW #2 that

$$\frac{\underline{r}-\underline{r}'}{|\underline{r}-\underline{r}'|^3} = -\underline{\nabla}\left(\frac{1}{|\underline{r}-\underline{r}'|}\right)$$

so that we can write $\underline{E} = -\underline{\nabla} V$

where $V(\underline{r}) = \int \dfrac{dq(\underline{r}')}{4\pi\varepsilon_0 |\underline{r}-\underline{r}'|}$

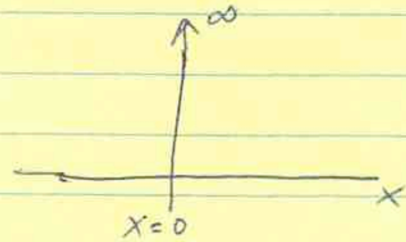
This immediately implies $\boxed{\underline{\nabla} \times \underline{E} = 0}$.

What about Gauss' law? We can derive it by taking the divergence of $(*)$ directly. First though, a reminder about the Dirac delta function.

Dirac delta function (see Griffiths 1.5)

defining properties:

1) $$\delta(x) = \begin{cases} \infty & x = 0 \\ 0 & x \neq 0 \end{cases}$$

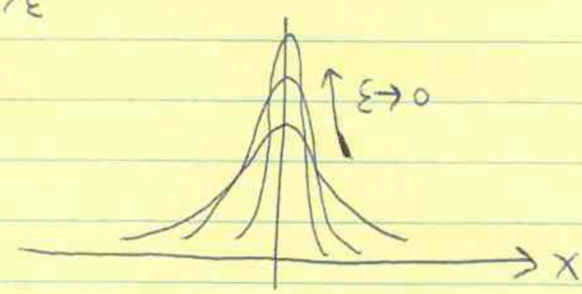


2) $$\int dx\ \delta(x) = 1$$

3) $$\int dx\ \delta(x)\ f(x) = f(0)$$

One way to think of it is as a limit of a function such as

$$\delta(x) = \lim_{\varepsilon \to 0} \frac{1}{\sqrt{\pi \varepsilon}} e^{-x^2/\varepsilon}$$



Note that $\delta(x)$ only makes sense under an integral sign.

$$\int_{a-\varepsilon}^{a+\varepsilon} dx\ \delta(x-a)\ f(x) = f(a)$$

the limits can span any range, but must include a!

This might remind you of our calculation of $\int \underline{E} \cdot d\underline{A}$ for a point charge. The size of the sphere doesn't matter, but it must enclose Q!

Direct calculation of $\underline{\nabla}\cdot\underline{E}$

Start with $\underline{E}(\underline{r}) = \int \frac{\rho(\underline{r}')\, dV'}{4\pi\varepsilon_0} \frac{\underline{r}-\underline{r}'}{|\underline{r}-\underline{r}'|^3}$

$$\Rightarrow \quad \underline{\nabla}\cdot\underline{E} = \int \frac{\rho(\underline{r}')\, dV'}{4\pi\varepsilon_0}\, \underline{\nabla}\cdot\left(\frac{\underline{r}-\underline{r}'}{|\underline{r}-\underline{r}'|^3}\right) \qquad -(1)$$

(because $\underline{\nabla}$ acts on $\underline{r}$ but not $\underline{r}'$, so we can take it inside).

Consider first $\underline{\nabla}\cdot\left(\frac{\underline{r}}{|\underline{r}|^3}\right)$ (ie. shift the origin by a constant vector $\underline{r}'$).

Write $\underline{v} = \frac{\underline{r}}{|\underline{r}|^3}$

$$\underline{\nabla}\cdot\underline{v} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 v_r\right) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(\hat{r}\right) = 0\,!$$

The divergence vanishes! But if we integrate over a sphere centred at the origin, we get

$$\int(\underline{\nabla}\cdot\underline{v})\, dV = \int \underline{v}\cdot d\underline{A} = \int \frac{1}{R^2} R^2 \sin\theta\, d\theta\, d\phi$$

$$= \underline{4\pi} \qquad \text{independent of } R!$$

So even though $\underline{\nabla}\cdot\underline{v} = 0$, $\int_{\text{sphere}} \underline{v}\cdot d\underline{A} = 4\pi$.

This strange behavior arises because $\underline{v}$ diverges at the origin.

In fact, $$\underline{\nabla} \cdot \left(\frac{\underline{r}}{|\underline{r}|^3} \right) = 4\pi\, \delta^3(\underline{r})$$

A 3D delta function, defined so that

$$\int_{\text{all space}} f(\underline{r})\, \delta^{(3)}(\underline{r}-\underline{a})\, dV = f(\underline{a}).$$

So now go back to equation (1).

$$\underline{\nabla} \cdot \underline{E} = \int \frac{\rho(\underline{r}')\, dV'}{4\pi\varepsilon_0}\, \underline{\nabla} \cdot \left(\frac{\underline{r}-\underline{r}'}{|\underline{r}-\underline{r}'|^3} \right)$$

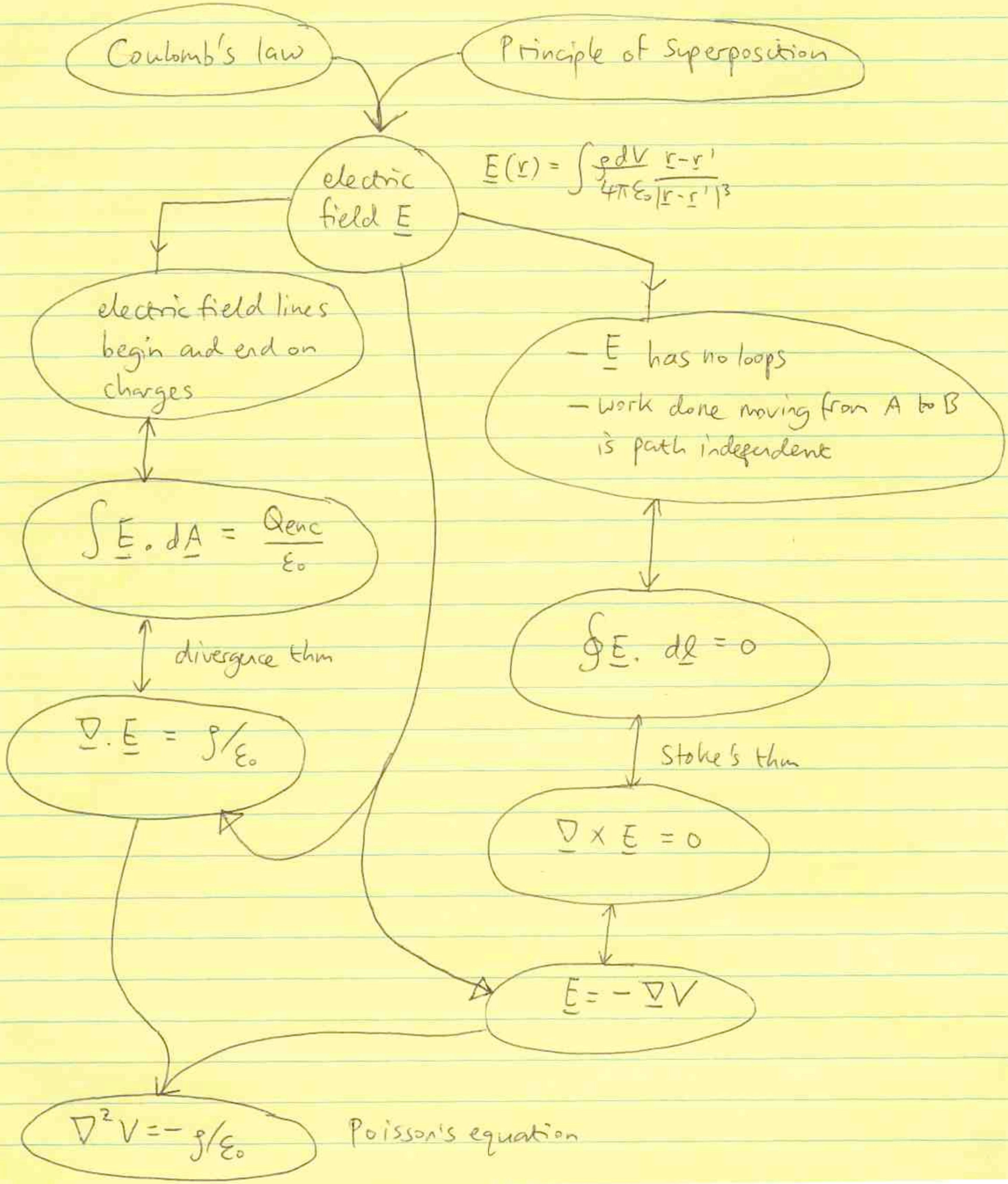
$$= \int \frac{\rho(\underline{r}')\, dV'}{4\pi\varepsilon_0}\, 4\pi\, \delta^3(\underline{r}-\underline{r}')$$

$$= \rho(\underline{r})/\varepsilon_0$$

$$\Rightarrow \quad \boxed{\underline{\nabla} \cdot \underline{E} = \rho(\underline{r})/\varepsilon_0}$$

This is a good place to stop and give a summary so far:

ELECTROSTATICS



PHYS 340 lecture 8 | Sep 21st 2005

Today, we'll talk about <u>electrostatic energy</u>.

Recall that the work done in moving a particle from A to B is

$$W = -\int_A^B q\,\underline{E}\cdot d\underline{l} = +\int_A^B q\,\underline{\nabla} V\cdot d\underline{l}$$

$$= q\left[V(B) - V(A)\right]$$

$$\Rightarrow \quad \underline{W = q\,\Delta V}$$

eg. what is the potential at the surface of a uniformly-charged sphere?

Gauss' Law $\Rightarrow$ $E(r) = \dfrac{Q}{4\pi\varepsilon_0 r^2}$

$$\underline{E} = -\underline{\nabla} V \Rightarrow -\int_R^\infty \frac{Q}{4\pi\varepsilon_0 r^2}\,dr = V(\infty) - V(R)$$

Since we are free to add a constant to V, we can choose $V(\infty) = 0$

$$\Rightarrow \quad \boxed{V(R) = \frac{Q}{4\pi\varepsilon_0 R}}$$

The work done to bring a charge from ∞ to $r = R$ is

$$W = q\,\Delta V = \underline{q\,V(R)}.$$

(if $q < 0$ then W is negative – the potential energy is released as the charge falls in).

Given this result, we can ask, how much energy does it take to build up the sphere?

build it up shell by shell

$$W = \int_0^R dq \, V(r)$$

$$= \int_0^R 4\pi r^2 dr \, \rho \; \frac{4\pi r^3 \rho}{3} \frac{1}{4\pi\varepsilon_0 r}$$

$$= \frac{4\pi\rho^2}{3\varepsilon_0} \int_0^R r^4 dr = \frac{4\pi\rho^2 R^5}{15\,\varepsilon_0}$$

$$= \left(\frac{4\pi}{3} R^3 \rho\right)^2 \frac{3}{4\pi\varepsilon_0} \frac{1}{5R} = \frac{3}{5} \frac{Q^2}{4\pi\varepsilon_0 R}$$

$$\boxed{W = \frac{3}{5} \frac{Q^2}{4\pi\varepsilon_0 R}}$$

Energy of a collection of point charges

Imagine we assemble n charges

$q_1 \xleftrightarrow{d_{12}} q_2$

$$W = \frac{q_1}{4\pi\varepsilon_0 d_{12}} q_2 = V_1 q_2 = U_{12}$$

q_1 q_2 q_3

$$W = U_{12} + U_{13} + U_{32}$$

q_1 q_2 q_3 q_4

$$W = U_{12} + U_{13} + U_{32}$$
$$+ U_{41} + U_{42} + U_{43}$$

For n charges, $$W = \frac{1}{2}\sum_{i=1}^{n}\left(\sum_{\substack{j=1\\ j\neq i}}^{n} U_{ij}\right)$$

$$= \frac{1}{2}\sum_{i=1}^{n}\sum_{\substack{j=1\\ j\neq i}}^{n} \frac{q_i q_j}{4\pi\varepsilon_0 d_{ij}}$$

$$\boxed{W = \frac{1}{2}\sum_{i=1}^{n} q_i V(\underline{r}_i)}$$

where $V(\underline{r}_i) = \sum_{j\neq i} \frac{q_j}{4\pi\varepsilon_0 d_{ij}}$

the potential due to all the other charges

Generalize this to a continuous distribution:

$$W = \frac{1}{2}\int \frac{d^3\underline{r}\, d^3\underline{r}'\ \rho(\underline{r})\,\rho(\underline{r}')}{4\pi\varepsilon_0\,|\underline{r}-\underline{r}'|}$$

$$\Rightarrow \quad \boxed{W = \frac{1}{2}\int d^3\underline{r}\ \rho(\underline{r})\, V(\underline{r})} \quad -(*)$$

Energy in the field

We can rewrite (*) using Gauss' law.

$$W = \frac{1}{2}\int d^3\underline{r}\ (\varepsilon_0 \underline{\nabla}.\underline{E})\, V(\underline{r})$$

Now integrate by parts. To do this, note that

$$\underline{\nabla}.(\underline{E}V) = V\underline{\nabla}.\underline{E} + \underline{E}.\underline{\nabla}V$$

$$\Rightarrow \quad W = \frac{1}{2}\int d^3\underline{r}\ \varepsilon_0\left[\underline{\nabla}\cdot(\underline{E}V) - \underline{E}\cdot\underline{\nabla}V\right]$$

$$= \frac{\varepsilon_0}{2}\left[\int d^3\underline{r}\ \underline{\nabla}\cdot(\underline{E}V) - \int d^3\underline{r}\ \underline{E}\cdot\underline{\nabla}V\right]$$

use divergence theorem — use $\underline{E} = -\underline{\nabla}V$

$$\Rightarrow \quad \boxed{W = \frac{\varepsilon_0}{2}\left[\int_S d\underline{A}\cdot\underline{E}V + \int d^3\underline{r}\,|\underline{E}|^2\right]}$$

take $S \to \infty$ where $V \to 0$

$$\Rightarrow \quad \boxed{W = \int d^3\underline{r}\ \frac{1}{2}\varepsilon_0|\underline{E}|^2} \longrightarrow (**)$$

The energy density in the electric field is $\frac{1}{2}\varepsilon_0|\underline{E}|^2$.

When you build up a collection of charges, can think of the work done being stored in the electric field.

Note that $(**)$ implies $W > 0$, always positive. But we started with the energy for two point charges

$$U_{12} = \frac{q_1 q_2}{4\pi\varepsilon_0 d_{12}}$$ which can be positive or negative.

As Griffiths discusses, this is because the self-energy of the point charges is not included in U_{12}. In the homework you show that the total energy for a system of two point charges can be written

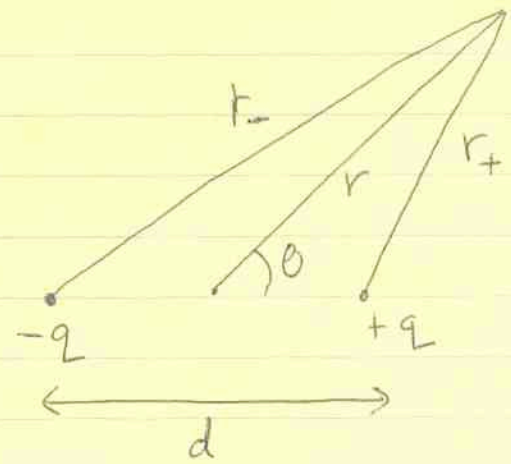
$$W = \underbrace{W_{11}}_{\infty} + \underbrace{W_{22}}_{\infty} + \underbrace{W_{12}}_{\frac{q_1 q_2}{4\pi\varepsilon_0 d_{12}}}$$

selfenergy

PHYS 340 lecture 9 — Sep 23rd, 2005.

Electric dipoles

Charges $+q$ and $-q$ separated by distance d.



Cosine rule or vector addition

$$\Rightarrow \quad r_+^2 = r^2 + d^2 - 2r\frac{d}{2}\cos\theta$$

$$r_-^2 = r^2 + d^2 - 2r\frac{d}{2}\cos(\pi - \theta)$$

$$\Rightarrow \quad r_+^2 = r^2 + d^2 - rd\cos\theta$$
$$r_-^2 = r^2 + d^2 + rd\cos\theta$$

The potential is $V(r) = \dfrac{q}{4\pi\varepsilon_0}\left(\dfrac{1}{r_+} - \dfrac{1}{r_-}\right)$

Now go very far away $r \gg d$

$$r_+ = r\left[1 + \left(\frac{d}{r}\right)^2 - \frac{d}{r}\cos\theta\right]^{1/2}$$

$$r_- = r\left[1 + \left(\frac{d}{r}\right)^2 + \frac{d}{r}\cos\theta\right]^{1/2}$$

Recall that $(1+x)^{-1/2} \simeq 1 - \dfrac{x}{2} + \dfrac{1}{2}\left(-\dfrac{1}{2}\right)\left(-\dfrac{3}{2}\right)x^2 + O(x^3)$

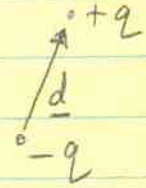
(for $x \ll 1$)

$$\Rightarrow \quad \frac{1}{r_\pm} \simeq \frac{1}{r}\left(1 \pm \frac{d}{2r}\cos\theta + O\left(\left(\frac{d}{r}\right)^2\right)\right)$$

$$\therefore \quad V(r) \simeq \frac{q}{4\pi\varepsilon_0 r}\left(1 + \frac{d}{2r}\cos\theta - \left(1 - \frac{d}{2r}\cos\theta\right) + O\left(\frac{d}{r}\right)^2\right)$$

$$\simeq \frac{qd\cos\theta}{4\pi\varepsilon_0 r^2}$$

define the dipole moment $\underline{p} = q\,\underline{d}$



$$\boxed{V(\underline{r}) = \frac{\underline{p}\cdot\hat{\underline{r}}}{4\pi\varepsilon_0 r^2}}$$

notice that it is $\propto \frac{1}{r^2}$

for a single point charge $V \propto \frac{1}{r}$

The electric field of a dipole is given by

$$\underline{E} = -\underline{\nabla} V$$

$$\Rightarrow \quad E_r = -\frac{\partial V}{\partial r} = \frac{2\,\underline{p}\cdot\hat{\underline{r}}}{4\pi\varepsilon_0 r^3}$$

$$E_\theta = -\frac{1}{r}\frac{\partial V}{\partial \theta} = \frac{1}{4\pi\varepsilon_0 r^3}\, p\sin\theta$$

$$E_\phi = 0$$

$$\Rightarrow \quad \boxed{\underline{E}(r,\theta) = \frac{p}{4\pi\varepsilon_0 r^3}\left(2\cos\theta\,\hat{\underline{r}} + \sin\theta\,\hat{\underline{\theta}}\right)}$$

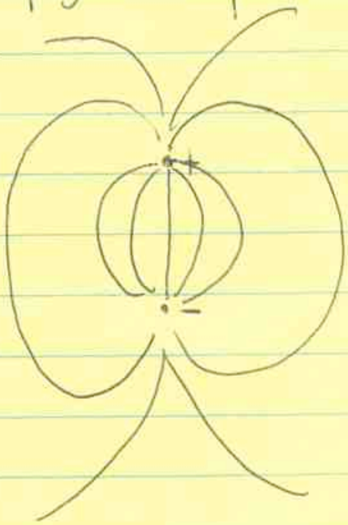
[In vector notation, this can be written

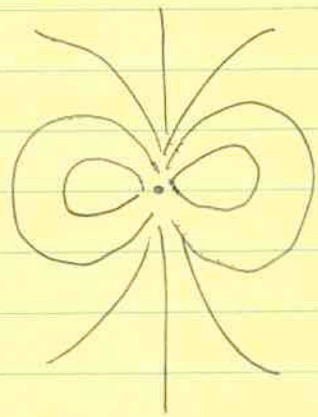
$$\underline{E} = \frac{1}{4\pi\varepsilon_0 r^3}\left[3(\underline{p}\cdot\hat{\underline{r}})\hat{\underline{r}} - \underline{p}\right]$$

]

A "pure" dipole has $d \to 0$ with p held finite

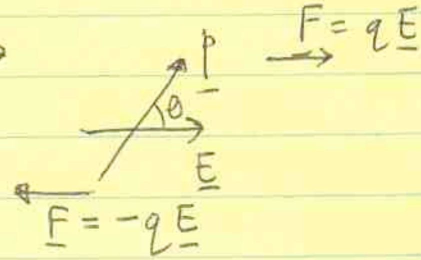
"physical dipole" "pure dipole"





Force on a dipole

a) uniform field $\xrightarrow{\underline{E}}$



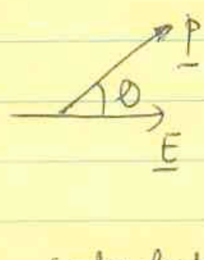
— no net force

— torque $|\underline{N}| = |\underline{p}||\underline{E}| \sin\theta$

or $\boxed{\underline{N} = \underline{p} \times \underline{E}}$

the torque is such as to align the dipole with the field.

— can define a potential energy

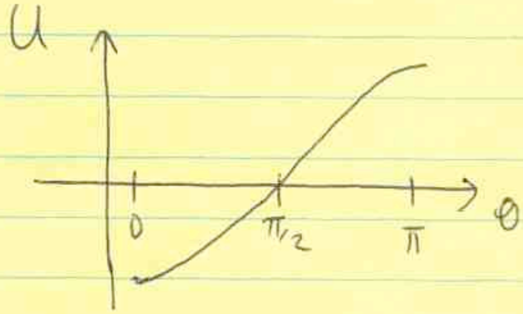


$U = -\underline{p} \cdot \underline{E}$

to see this, calculate the work done

$$W = \int_{\pi/2}^{\theta} N \, d\theta' = \int_{\pi/2}^{\theta} pE \sin\theta' \, d\theta'$$

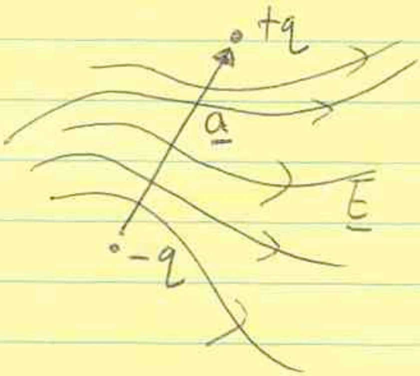
$$= -pE\cos\theta = -\underline{p}\cdot\underline{E}$$



the torque vanishes at $\theta = 0$ and $\theta = \pi$
but the $\theta = \pi$ point is unstable to small perturbations $\theta \to \theta + \delta\theta$

$\Rightarrow$ the dipole aligns with the field

b) non-uniform field



there is now a net force

Consider the x-component

$$E_x^+ = E_x^- + a_x \frac{\partial E_x}{\partial x} + a_y \frac{\partial E_x}{\partial y} + a_z \frac{\partial E_x}{\partial z}$$

But the net force is $q(E_x^+ - E_x^-) = F_x$

and $q\underline{a} = \underline{p}$

$\Rightarrow \quad F_x = (\underline{p}\cdot\underline{\nabla})E_x$

$\Rightarrow \quad \boxed{\underline{F} = (\underline{p}\cdot\underline{\nabla})\underline{E}}$

note that you can read "$\underline{p}\cdot\underline{\nabla}$" as the derivative along the direction of $\underline{p}$. So if the electric field varies along $\underline{p}$, then there is a net force on the dipole.

If $\underline{p}$ is constant, can write $\underline{F} = \underline{\nabla}(\underline{p}\cdot\underline{E})$

$$= -\underline{\nabla} U$$

The force on a dipole in a non-uniform field explains the ability of a charged comb to pick up neutral pieces of paper: the paper has an induced dipole which experiences a force. The force is attractive because when the paper moves closer, $\underline{p}$ and $\underline{E}$ are maximized which reduces $U = -\underline{p}\cdot\underline{E}$.

PHYS 340 lecture 10 — Sep 26th, 2005

Multipole expansion

As $r \to \infty$, we know that the potential looks like

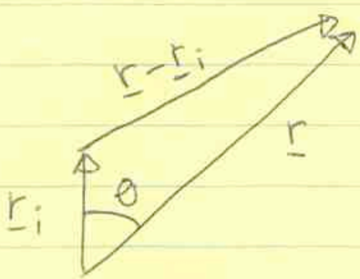
$$V(r) \to \frac{Q}{4\pi\varepsilon_0 r}$$ where Q is the total charge.

But what if $Q = 0$?

Consider a collection of charges $\{q_i\}$ $i = 1, N$

The potential is $V(\underline{r}) = \sum_{i=1}^{N} \frac{q_i}{4\pi\varepsilon_0 |\underline{r} - \underline{r}_i|}$

Expand this in powers of $\frac{1}{r}$



$$|\underline{r} - \underline{r}_i|^2 = r^2 + r_i^2 - 2 r_i r \cos\theta$$

$$\Rightarrow \frac{1}{|\underline{r} - \underline{r}_i|} = \frac{1}{r\sqrt{1+\varepsilon}}$$

where $\varepsilon = \left(\frac{r_i}{r}\right)^2 - \frac{2 r_i}{r}\cos\theta$

When $r \gg r_i$, $\varepsilon \ll 1$ so that

$$(1+\varepsilon)^{-1/2} \simeq 1 - \tfrac{1}{2}\varepsilon + \tfrac{3}{8}\varepsilon^2 + O(\varepsilon^3)$$

$$\Rightarrow \frac{1}{|\underline{r} - \underline{r}_i|} \simeq \frac{1}{r}\left[1 + \frac{r_i}{r}\cos\theta - \frac{1}{2}\left(\frac{r_i}{r}\right)^2 + \frac{3}{2}\left(\frac{r_i}{r}\right)^2\cos^2\theta + \dots\right]$$

$$\Rightarrow \frac{1}{|\underline{r}-\underline{r}_i|} \simeq \frac{1}{r} + \frac{r_i \cos\theta}{r^2} + \frac{1}{2}\left(\frac{r_i}{r}\right)^2 \frac{(3\cos^2\theta - 1)}{r} + O\left(\frac{1}{r^4}\right)$$

But $\cos\theta = \hat{\underline{r}} \cdot \hat{\underline{r}}_i$, so we can write

$$V(\underline{r}) \simeq \sum_{i=1}^{N} \frac{q_i}{4\pi\varepsilon_0} \left[\frac{1}{r} + \frac{\hat{\underline{r}} \cdot \underline{r}_i}{r^2} + \frac{1}{2} \frac{3(\hat{\underline{r}} \cdot \underline{r}_i)^2 - |\underline{r}_i|^2}{r^3} \right]$$

the first term is the "monopole" term

$$V(r) = \frac{Q}{4\pi\varepsilon_0 r} \quad \text{where } Q = \sum_{i=1}^{N} q_i$$

is the <u>total charge</u>.

The "dipole" term is $\dfrac{\hat{\underline{r}} \cdot \underline{p}}{4\pi\varepsilon_0 r^2}$ where $\underline{p} = \sum q_i \underline{r}_i$

is the <u>dipole moment</u>

The "quadrupole" term is

$$\sum_{i=1}^{N} \frac{q_i}{4\pi\varepsilon_0} \frac{3(\hat{\underline{r}} \cdot \underline{r}_i)^2 - r_i^2}{2r^3}$$

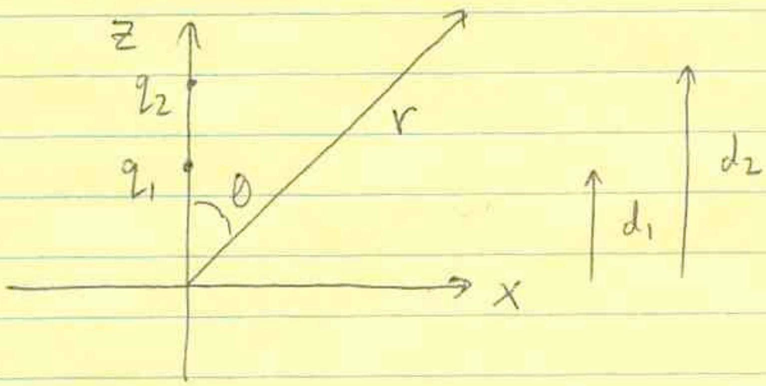
We can write it in terms of a quadrupole moment <u>tensor</u>

$$\frac{\hat{\underline{r}} \cdot \underline{\underline{Q}}_2 \cdot \hat{\underline{r}}}{4\pi\varepsilon_0 r^3} \quad \text{where} \quad \underline{\underline{Q}}_2 = \sum_{i=1}^{N} \frac{q_i}{2} \left[3\underline{r}_i \underline{r}_i - r_i^2 \underline{\underline{I}} \right]$$

but this is beyond the scope of this course. We'll just mention that $\underline{p}$ and $\underline{\underline{Q}}_2$ are somewhat analogous to the center of mass and moment of inertia tensor for a collection of particles, $\underline{R} = \frac{\sum m_i \underline{r}_i}{\sum m_i}$, $\underline{\underline{I}}_M = \sum m_i \left(r_i^2 \underline{\underline{I}} - \underline{r}_i \underline{r}_i \right)$

use subscript M to avoid confusion with unit tensor $\underline{\underline{I}} = \delta_{ij}$

Let's calculate an example



the total charge is $Q = q_1 + q_2$
the dipole moment is $\underline{p} = (q_1 d_1 + q_2 d_2)\,\hat{\underline{z}}$

what about the quadrupole term? We can write $\hat{\underline{r}} = \hat{\underline{x}} \sin\theta + \hat{\underline{z}} \cos\theta$

$\Rightarrow \quad \hat{\underline{r}} \cdot \underline{r}_i = d_i \cos\theta$

$\Rightarrow$ quadrupole part is

$$\frac{3(\hat{\underline{r}} \cdot \underline{r}_i)^2 - r_i^2}{2r^3}$$

$$= \frac{3 d_i^2 \cos^2\theta - d_i^2}{2r^3}$$

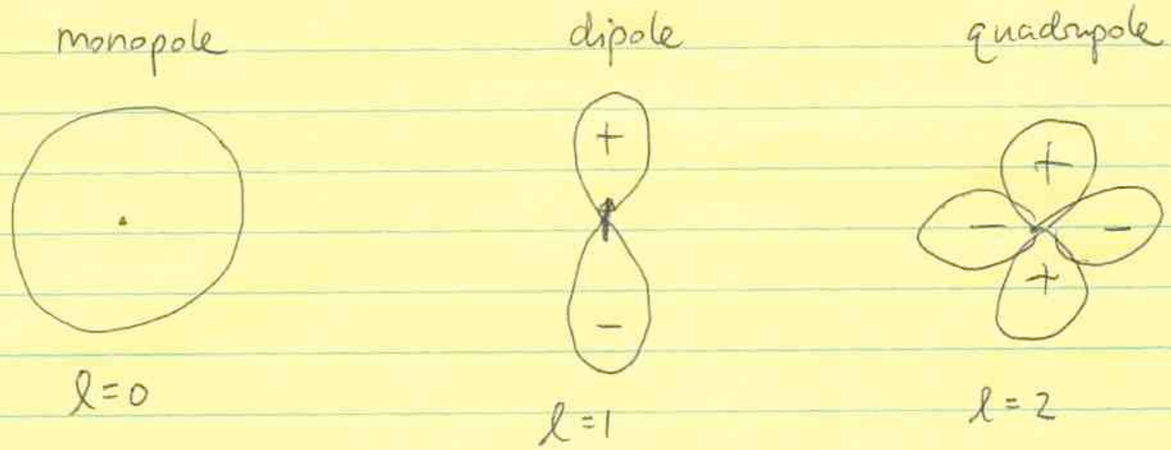
$$= \frac{d_i^2}{2r^3}(3\cos^2\theta - 1)$$

$\Rightarrow$ the potential is

$$V(r) \simeq \frac{(q_1+q_2)}{4\pi\varepsilon_0 r} + \frac{(q_1 d_1 + q_2 d_2)\cos\theta}{4\pi\varepsilon_0 r^2}$$

$$+ \frac{(q_1 d_1^2 + q_2 d_2^2)}{4\pi\varepsilon_0 r^3}\frac{(3\cos^2\theta - 1)}{2}$$

What do these angular dependences look like?



this should remind you of atomic orbitals with different l's
— we'll see why later when we solve Laplace's equation.

Note that the angular dependence can be quite different for a group of charges — for example the clock question in HW 4. The clock face of charges has a quadrupole term that doesn't depend on θ.

Finally, we can generalize the results to continuous charge distributions:

$$Q = \int \rho(\underline{r}')\, d^3\underline{r}'$$

$$\underline{p} = \int \rho(\underline{r}')\, \underline{r}'\, d^3\underline{r}'$$

and something similar for $\underline{\underline{Q}}_2$.

Note that in general $\underline{p}$ and $\underline{\underline{Q}}_2$ depend on the origin of coordinates.

<u>But</u> $\underline{p}$ is independent of origin as long as $Q = 0$.

<u>Proof</u> let $\underline{r}' \to \underline{r}' + \underline{r}_0$

then $\underline{p} \longrightarrow \underline{p}' = \int (\underline{r}' + \underline{r}_0)\, \rho(\underline{r}')\, d^3\underline{r}'$

$$= \underline{p} + \underline{r}_0 Q = \underline{p} \text{ when } Q = 0.$$

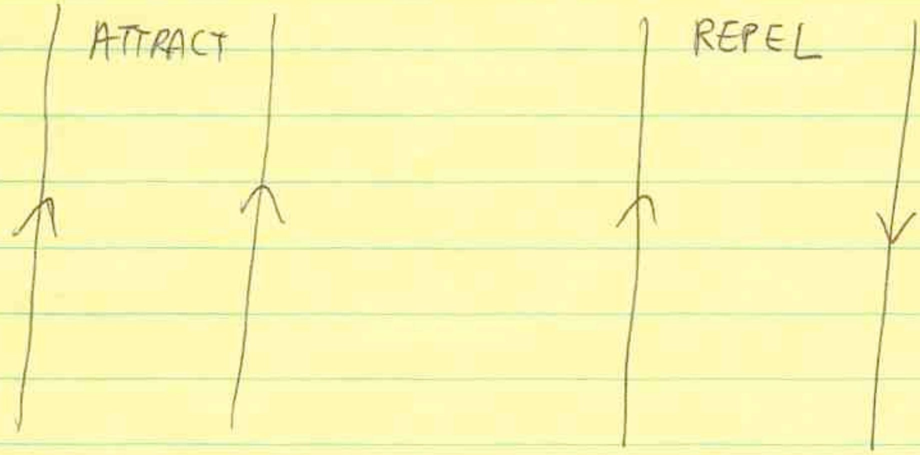
PHYS 340 lecture 11

Sep 28th, 2005

Magnetostatics

We now move on to magnetic fields. The most familiar source of magnetic fields are permanent magnets, but in fact the fundamental source is electric current (permanent magnetism arises from dynamics of atomic electrons as we shall see - a form of microscopic current).

The simplest laboratory example of magnetic forces is the force between current carrying wires

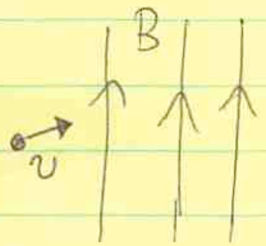


[magnetostatics
- steady currents
- B constant in time]

We'll first talk about magnetic forces, and then how to calculate the magnetic field arising from a given current configuration.

Force on a moving charge

Consider a charged particle in a uniform magnetic field



What happens?

The force is $\boxed{\underline{F} = q\, \underline{v} \times \underline{B}}$

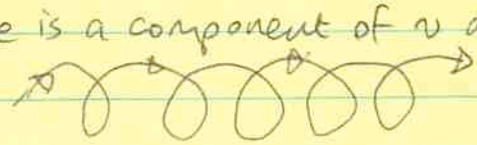
Perpendicular to the field and the velocity vector, so the charge moves in a circle around B.
the force balance is

$$\frac{mv^2}{R} = qvB$$

momentum

radius $R = \dfrac{mv}{qB} = \dfrac{p}{qB}$

angular frequency $\omega = \dfrac{v}{R} = \dfrac{qB}{m}$ cyclotron frequency.

— what if there is a component of v along B?
then 

Some comments:

① B has units of Tesla (T) in SI. ($1T \equiv 1 N A^{-1} m^{-1}$)

surface of the Earth 10^{-4} T
largest fields in laboratory with electromagnets 1 T
superconducting magnets 10 T
interstellar medium 10^{-10} T
neutron stars $10^{8} - 10^{11}$ T

② The magnetic force does no work.
Rate of work (power) $\dfrac{dW}{dt} = \underline{F} \cdot \dfrac{d\underline{x}}{dt} = \underline{F} \cdot \underline{v}$

But $\underline{F} \cdot \underline{v} = q\,(\underline{v} \times \underline{B}) \cdot \underline{v} = 0$

The force always acts perpendicular to the motion $\Rightarrow$ no work!

(3) Combined electric and magnetic fields

Lorentz force $\boxed{\underline{F} = q(\underline{E} + \underline{v} \times \underline{B})}$

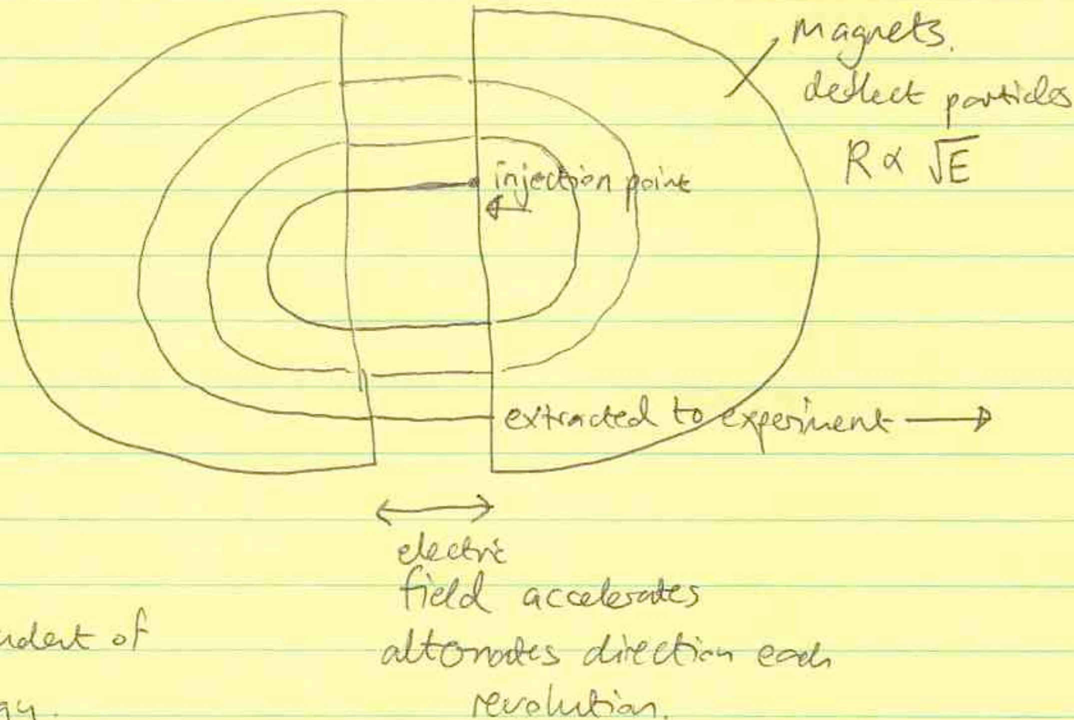
<u>Applications of cyclotron motion</u>

(1) cyclotron

Can reach energies of ~ 100 MeV

only works for non-relativistic particles because you need

$\omega = \frac{qB}{m}$ independent of energy.



(2) $\underline{E} \times \underline{B}$ drift

consider a particle initially at rest in uniform

$$\underline{E} = E\hat{y}, \quad \underline{B} = B\hat{z}.$$

equation of motion $\quad m\frac{d\underline{v}}{dt} = q(\underline{E} + \underline{v} \times \underline{B})$

$$\Rightarrow \boxed{\begin{aligned} \frac{dv_x}{dt} &= \omega v_y \qquad\qquad \frac{dv_z}{dt} = 0 \\ \frac{dv_y}{dt} &= -\omega v_x + \frac{\omega E}{B} \end{aligned}}$$

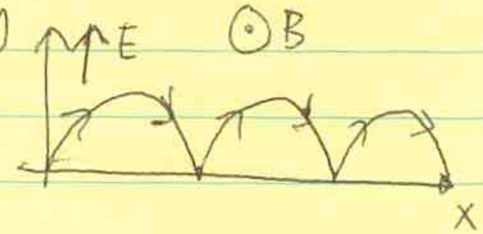
$$\Rightarrow \quad \frac{d^2 v_x}{dt^2} + \omega^2 v_x = \frac{\omega^2 E}{B}$$

solution is $\begin{pmatrix}\text{solution to} \\ \text{homogeneous equation}\end{pmatrix} + \begin{pmatrix}\text{particular} \\ \text{integral}\end{pmatrix}$

$$v_x(t) = A_1 \cos \omega t + A_2 \sin \omega t + \frac{E}{B}$$

initial conditions $\underline{v} = 0$

$$\Rightarrow \quad A_2 = -E/B \qquad A_1 = 0$$



the trajectory is $\quad x(t) = \dfrac{E}{\omega B}(\omega t - \sin \omega t)$

$$y(t) = \frac{E}{\omega B}(1 - \cos \omega t)$$

There is a drift in the x direction with velocity $u = \dfrac{E}{B}$.

In general, we can write $\boxed{\underline{u} = \dfrac{\underline{E} \times \underline{B}}{B^2}}$

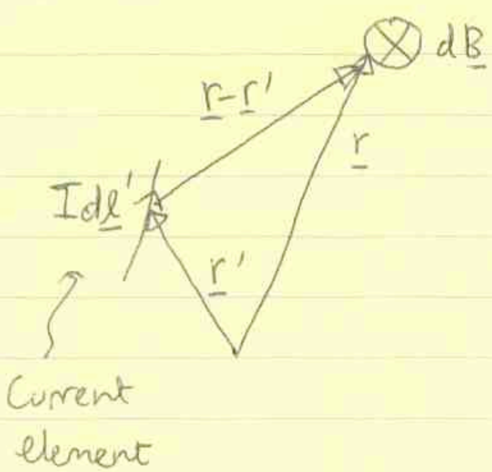
Note that the drift is charge independent! So in a plasma, there is no net current.

There are many other interesting examples of charged particle motions – mass spectrometer, focusing magnets for particle beams, particles trapped in the van Allen belts...

PHYS 340 lecture 12
Sep 30th 2005

Last time, we talked about the effect of a magnetic field on moving charges. Now discuss how to calculate $\underline{B}$ for a given distribution of currents.

The magnetic field of a steady current is given by the <u>Biot-Savart law</u>

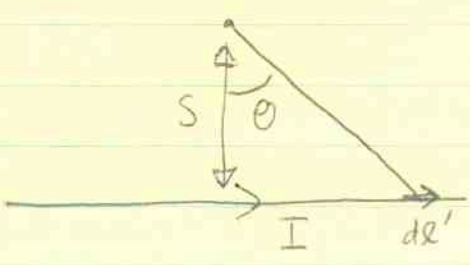


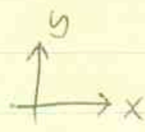
$$d\underline{B} = \frac{\mu_0 I}{4\pi} \frac{d\underline{l}' \times (\underline{r}-\underline{r}')}{|\underline{r}-\underline{r}'|^3}$$

Integrating,

$$\underline{B}(\underline{r}) = \frac{\mu_0 I}{4\pi} \int \frac{d\underline{l}' \times (\underline{r}-\underline{r}')}{|\underline{r}-\underline{r}'|^3}$$

<u>Example 1</u> a long straight wire





$$d\underline{l}' = dx\,\hat{\underline{x}}$$

$$|\underline{r}-\underline{r}'| = \sqrt{x^2+s^2}$$

$$\underline{r}-\underline{r}' = \sqrt{x^2+s^2}\,(\cos\theta\,\hat{\underline{y}} + \sin\theta\,\hat{\underline{x}})$$

$$\Rightarrow \quad d\underline{l}' \times (\underline{r}-\underline{r}') = \sqrt{x^2+s^2}\,dx\,\cos\theta\,\underbrace{\hat{\underline{x}}\times\hat{\underline{y}}}_{\hat{z}}$$

But $\cos\theta = \dfrac{s}{\sqrt{s^2+x^2}}$

$$\Rightarrow \quad d\underline{l}' \times (\underline{r} - \underline{r}') = s\,dx\,\hat{\underline{z}}$$

$$\Rightarrow \quad \boxed{\underline{B}(s) = \hat{\underline{z}}\,\frac{\mu_0 I}{4\pi}\int_{-\infty}^{\infty}\frac{s\,dx}{(s^2+x^2)^{3/2}}}$$

to integrate, use a substitution $x = s\tan\theta$

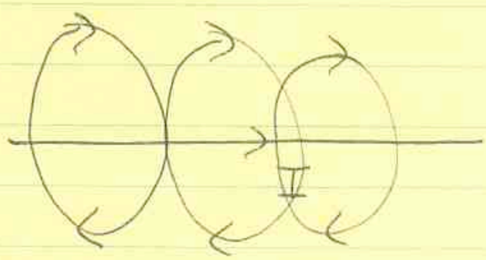
$$dx = s\sec^2\theta\,d\theta$$

$$s^2 + x^2 = s^2(1+\tan^2\theta) = s^2\sec^2\theta$$

$$\Rightarrow \quad \underline{B}(s) = \hat{\underline{z}}\,\frac{\mu_0 I s^2}{4\pi}\int_{-\pi/2}^{\pi/2}\frac{\sec^2\theta\,d\theta}{s^3\sec^3\theta}$$

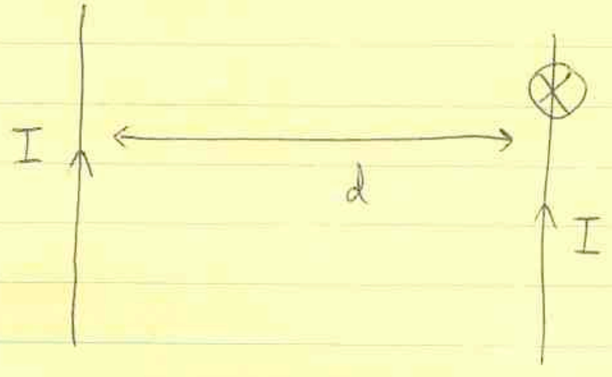
$$= \hat{\underline{z}}\,\frac{\mu_0 I}{4\pi s}\underbrace{\int_{-\pi/2}^{\pi/2}\cos\theta\,d\theta}_{=2}$$

$$\boxed{\underline{B}(s) = \hat{\underline{z}}\,\frac{\mu_0 I}{2\pi s}}$$



the field loops around the wire (right hand rule gives the direction).

We can now ask what is the force between two current-carrying wires?



$B = \frac{\mu_0 I}{2\pi d}$ ← the magnetic field from the left wire at the position of the wire at the right

Each current-carrying particle feels a force $q\underline{v} \times \underline{B}$.
The current is $I = \lambda v$, where λ = charge per unit length.

$$\therefore \text{ force per unit length } = \lambda \underline{v} \times \underline{B}$$
$$= \underline{I} \times \underline{B}$$

(differential form: $d\underline{F} = I\, d\underline{l} \times \underline{B}$)

force from current element $I\,d\underline{l}$

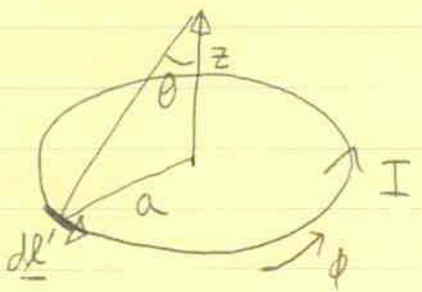
$\Rightarrow$ the force is attractive, and equal to $\underline{\underline{\dfrac{\mu_0 I_1 I_2}{2\pi d}}}$ per unit length.

(note that opposites REPEL in this case, where as opposite charges attract).

PHYS 340 lecture 13 October 3rd, 2005

Biot-Savart law example 2

a circular loop



what is the field along the z axis?

$$d\underline{l}' = a\,d\phi\,\hat{\underline{\phi}}$$

$$|\underline{r}-\underline{r}'| = \sqrt{a^2+z^2}$$

By symmetry, $\underline{B}$ must be vertical because the horizontal components cancel when we integrate around the loop.

The vertical component of $d\underline{l} \times (\underline{r}-\underline{r}')$

$$= a\,d\phi\,\sqrt{a^2+z^2}\,\sin\theta$$

$$= a^2\,d\phi$$

$$\Rightarrow \quad \underline{B}(z) = \hat{\underline{z}}\,\frac{\mu_0 I}{4\pi}\int_0^{2\pi}\frac{a^2\,d\phi}{(a^2+z^2)^{3/2}}$$

$$= \hat{\underline{z}}\,\frac{\mu_0 I}{4\pi}\,\frac{a^2}{(a^2+z^2)^{3/2}}\int_0^{2\pi} d\phi$$

$$\boxed{\underline{B}(z) = \hat{\underline{z}}\,\frac{\mu_0 I}{2}\,\frac{a^2}{(a^2+z^2)^{3/2}}} \quad \text{— } (*)$$

Notice how easy the integral is! The hard part is getting the directions right.

Physical derivation of magnetostatic field equations

For equation (*)

In the limit $z \gg a$, notice that $B \propto \dfrac{1}{z^3}$

This is reminiscent of the electric dipole scaling. In fact, a small current loop is a magnetic dipole, with dipole moment $m = I \pi a^2$.

In general

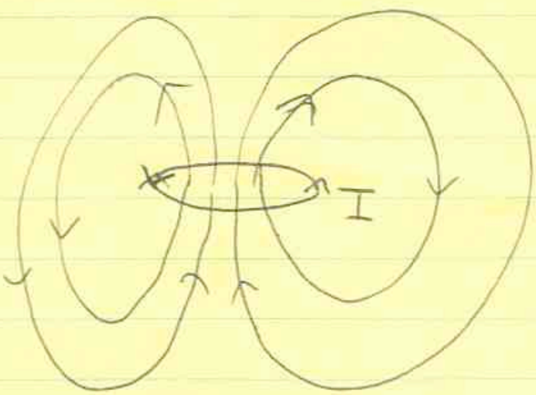
$\underline{m} = I \, d\underline{A}$

$$\underline{B} = \frac{\mu_0 m}{4\pi r^3} \left(2\cos\theta \, \underline{\hat{r}} + \sin\theta \, \underline{\hat{\theta}} \right)$$

We'll derive this result later.

The field looks like

notice the similarity to the expression for $\underline{E}$ for an electric dipole



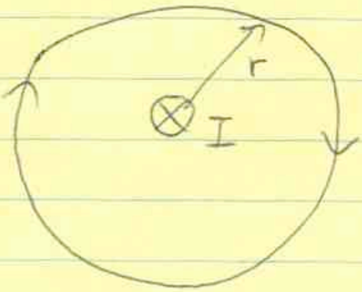
The crucial point is that a current loop is the simplest object we can construct – the equivalent of a point charge in electrostatics. But its field is a dipole field – there are no magnetic monopoles which could act as sources of magnetic field lines.

Mathematically, we can write this as $\underline{\nabla} \cdot \underline{B} = 0$

This is the first field equation of magnetostatics, we'll derive it starting from the Biot-Savart law next time.

First, let's look at $\underline{\nabla} \times \underline{B}$, or in integral form $\oint \underline{B} \cdot d\underline{\ell}$.

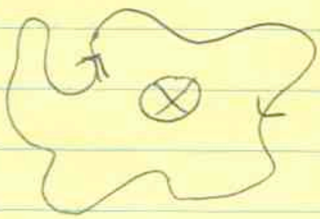
For a straight line current $\quad \underline{B} = \frac{\mu_0 I}{2\pi r} \hat{\underline{\phi}}$



integrate around a circular loop

$$\oint \underline{B} \cdot d\underline{\ell} = \frac{\mu_0 I}{2\pi r} \cdot 2\pi r = \mu_0 I$$

in fact, any shaped loop gives the same answer



$$\oint \underline{B} \cdot d\underline{\ell} = \int (r B_\phi \, d\phi + \cancel{B_r \, dr})$$

0 because $B_r = 0$

$$= \int B_\phi \, r \, d\phi$$

$$= \int \frac{\mu_0 I}{2\pi r} r \, d\phi = \mu_0 I$$

if we add more currents inside the loop, the answer is $\mu_0 I_{enc}$

total current enclosed

Currents outside make no contribution

$$\oint \underline{B} \cdot d\underline{\ell} = \int_0^\theta \mu_0 I \, d\phi + \int_\theta^0 \mu_0 I \, d\phi$$

$$= 0$$

We've considered infinite straight line currents only, but in fact it is true in general that

$$\boxed{\oint \underline{B} \cdot d\underline{l} = \mu_0 I_{enc}}$$

Ampère's Law

Stoke's theorem gives us a differential version of this law

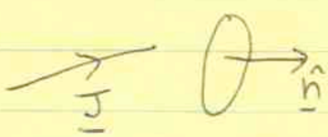
The LHS is

$$\oint \underline{B} \cdot d\underline{l} = \underbrace{\int}_{\text{any surface bounded by the loop}} \underline{\nabla} \times \underline{B} \cdot d\underline{A}$$

But the RHS is

$$\mu_0 I_{enc} = \mu_0 \underbrace{\int}_{\text{surface bounded by the loop}} \underline{J} \cdot d\underline{A}$$

where $\underline{J}$ is the current density (units $A\,m^{-2}$)



current through area $d\underline{A} = dA\,\hat{\underline{n}}$ is $dI = \underline{J} \cdot d\underline{A}$

Since Ampère's law holds for any size loop, we conclude that

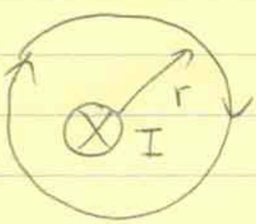
$$\boxed{\underline{\nabla} \times \underline{B} = \mu_0 \underline{J}}$$

the second field equation of magnetostatics.

Applications of Ampère's Law

As with Gauss' law, in cases with high symmetry, we can use Ampere's law to derive $\underline{B}$.

① infinite straight current-carrying wire



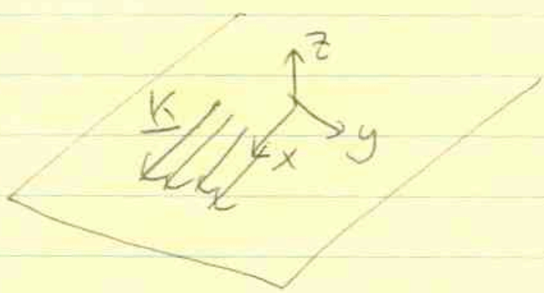
from symmetry, $\underline{B} = B_\phi \hat{\underline{\phi}}$

$$\oint \underline{B} \cdot d\underline{l} = 2\pi r B_\phi = \mu_0 I$$

$$\Rightarrow \boxed{B_\phi = \frac{\mu_0 I}{2\pi r}}$$

a result we derived earlier using the Biot-Savart law.

② A planar current sheet

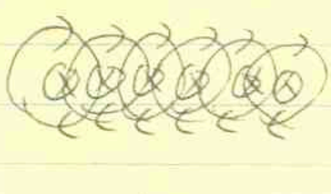


surface current density $\underline{K} = K\hat{\underline{x}}$

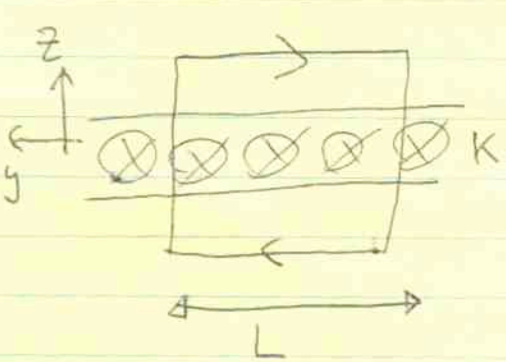
(current per unit perpendicular length)

$K \Rrightarrow \updownarrow dx \qquad dI = K\,dx$

First, think of the surface current as many parallel currents



$$\Rightarrow \quad \underline{B} = \begin{cases} -B(z)\,\hat{\underline{y}} & z > 0 \\ B(z)\,\hat{\underline{y}} & z < 0 \end{cases}$$

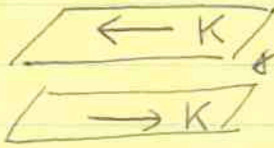


$$\oint \underline{B} \cdot d\underline{l} = 2BL = \mu_0 K L$$

$$\Rightarrow \boxed{B = \frac{\mu_0 K}{2}}$$

independent of z

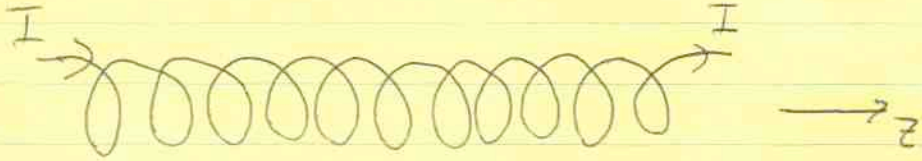
An application is a "busbar" used in power stations



$B = \mu_0 K$ between the plates

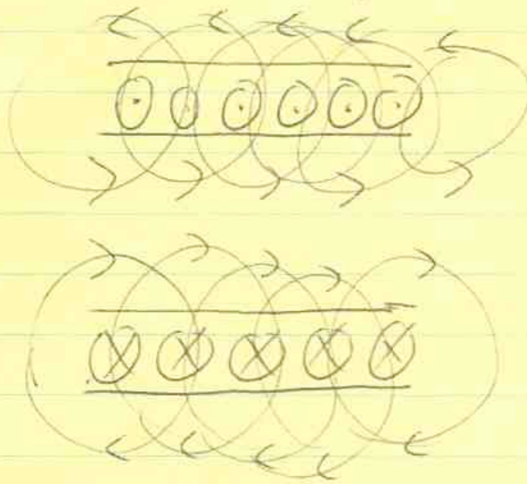
$B = 0$ outside

③ Solenoid

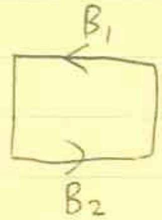


n turns per unit length

Again, from symmetry $\underline{B} = B\hat{\underline{z}}$, where B is a function of distance from the axis



First, draw a loop outside



since there is no current inside

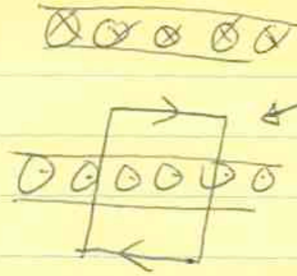
$\Rightarrow B_1 = -B_2$

But the position of the loop is arbitrary $\Rightarrow B =$ constant outside

But we have a finite current (and therefore magnetic energy)

$\Rightarrow \underline{B = 0}$ outside

inside, draw a loop



only the top part contributes

$\Rightarrow BL = \mu_0 n L I$

$\Rightarrow \boxed{B = \mu_0 n I}$

This result holds for any cross-section, as long as the cross-section doesn't vary along the length of the solenoid.

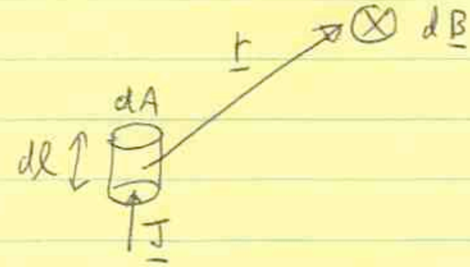
PHYS 340 lecture 15 | Oct 7th, 2005

We'll now derive the field equations $\underline{\nabla} \cdot \underline{B} = 0$ and $\underline{\nabla} \times \underline{B} = \mu_0 \underline{J}$ from the Biot-Savart law.

The derivation is quite involved, so we'll assume some results along the way.

First, rewrite the Biot-Savart law in terms of current density

$$d\underline{B} = \frac{\mu_0 I}{4\pi} \frac{d\underline{l}' \times \hat{\underline{r}}}{r^2}$$



$$\Rightarrow \quad d\underline{B} = \frac{\mu_0}{4\pi} \frac{\underline{J}\, dA\, dl \times \hat{\underline{r}}}{r^2}$$

$$\Rightarrow \quad \boxed{\underline{B}(\underline{r}) = \frac{\mu_0}{4\pi} \int \frac{\underline{J}(\underline{r}') \times (\underline{r} - \underline{r}')}{|\underline{r} - \underline{r}'|^3}\, d^3\underline{r}'}$$

Remember that we showed that $\underline{\nabla} \times \underline{E} = 0$ by writing $\underline{E} = -\underline{\nabla} V$. Here we'll do something similar:

use the result $$\frac{\underline{r} - \underline{r}'}{|\underline{r} - \underline{r}'|^3} = -\underline{\nabla}\left(\frac{1}{|\underline{r} - \underline{r}'|}\right)$$

$$\Rightarrow \quad \underline{B}(\underline{r}) = \frac{\mu_0}{4\pi} \int \underbrace{\underline{\nabla}\left(\frac{1}{\underline{r} - \underline{r}'}\right) \times \underline{J}(\underline{r}')}_{\text{this is } \underline{\nabla} \times \left(\frac{\underline{J}(\underline{r}')}{|\underline{r} - \underline{r}'|}\right)}\, d^3\underline{r}'$$

(using $\underline{\nabla} \times (\phi \underline{A}) = \phi\, \underline{\nabla} \times \underline{A} \cancel{+} - \underline{A} \times \underline{\nabla}\phi$)

$$\Rightarrow \quad B(\underline{r}) = \frac{\mu_0}{4\pi} \underline{\nabla} \times \left(\int \frac{\underline{J}(\underline{r}')}{|\underline{r}-\underline{r}'|} d^3\underline{r}' \right)$$

or $$\boxed{\underline{B} = \underline{\nabla} \times \underline{A}}$$

where $\underline{A} = \frac{\mu_0}{4\pi} \int \frac{\underline{J}(\underline{r}')}{|\underline{r}-\underline{r}'|} d^3\underline{r}'$ is the VECTOR POTENTIAL

We noted with electric potential that the restriction $\underline{\nabla} \times \underline{E} = 0$ meant that we could write the electric field (3 components at each point in space) in terms of a scalar (1 component). Here, it looks like we're writing $\underline{B}$ in terms of another vector field $\underline{A}$, but in fact all 3 components of $\underline{A}$ are not needed to determine $\underline{B}$ because we can always add any curl-free field to $\underline{A}$ and still get the same $\underline{B}$.

eg. $\underline{A} \rightarrow \underline{A} + \underline{\nabla}\lambda = \underline{A}'$ $\qquad \underline{\nabla} \times \underline{A} = \underline{\nabla} \times \underline{A}'$.

We often use this freedom to choose $\underline{A}$ such that $\underline{\nabla} \cdot \underline{A} = 0$
this is called the Coulomb gauge.
(if $\underline{A}$ does not satisfy $\underline{\nabla} \cdot \underline{A} = 0$, add a term $\underline{\nabla}\lambda$ which satisfies $\nabla^2 \lambda = -\underline{\nabla} \cdot \underline{A}$)
So $\underline{A}$ only has 2 degrees of freedom at each point in space.

Now we are in a position to derive the field equations:

1) $\underline{\nabla} \cdot \underline{B} = \underline{\nabla} \cdot (\underline{\nabla} \times \underline{A}) = 0$ because the divergence of a curl is zero.

2) $\underline{\nabla} \times \underline{B} = \underline{\nabla} \times (\underline{\nabla} \times \underline{A}) = -\nabla^2 \underline{A} + \underline{\nabla}(\underbrace{\cancel{\underline{\nabla}} \cdot \underline{A}}_{0})$ Coulomb gauge

$$\Rightarrow \quad \underline{\nabla} \times \underline{B} = -\nabla^2 \underline{A}$$

$$= -\frac{\mu_0}{4\pi} \int J(\underline{r}')\, d^3\underline{r}' \; \underline{\nabla}^2 \left(\frac{1}{|\underline{r}-\underline{r}'|} \right)$$

But we showed in previous notes that

$$\underline{\nabla} \cdot \left(\frac{\underline{r}}{|\underline{r}|^3} \right) = -\nabla^2 \left(\frac{1}{|\underline{r}|} \right) = 4\pi\, \delta^3(\underline{r})$$

$$\Rightarrow \quad \underline{\nabla} \times \underline{B} = -\frac{\mu_0}{4\pi} \int J(\underline{r}')\, d^3\underline{r}' \left(-4\pi\, \delta^3(\underline{r}-\underline{r}') \right)$$

$$\Rightarrow \quad \boxed{\underline{\nabla} \times \underline{B} = \mu_0 \underline{J}} \quad \text{Ampère's law.}$$

Conservation of charge

What is the total current out of a volume?

$$I = \int_{\text{surface}} \underline{J} . d\underline{A} = -\frac{dQ}{dt}$$ ← rate of change of charge contained in the volume

$$\Rightarrow \int d^3\underline{r} \, (\underline{\nabla} . \underline{J}) = -\frac{d}{dt}\int \rho(\underline{r}) \, d^3\underline{r}$$

$$= -\int \frac{\partial \rho}{\partial t} \, d^3 r$$ for a fixed volume

$$\Rightarrow \boxed{\underline{\nabla} . \underline{J} = -\frac{\partial \rho}{\partial t}}$$ conservation of charge.

(we derived a similar result for a fluid when we discussed the divergence theorem at the beginning of the course).

Now consider a steady-state situation for which $\frac{\partial \rho}{\partial t} = 0$

$$\Rightarrow \underline{\nabla} . \underline{J} = 0.$$

1) For 1D currents in a wire, $\underline{\nabla} . \underline{J} = 0 \Rightarrow$ <u>I must be constant along the wire</u>

2) This result also tells us that <u>Ampère's law is only valid for magnetostatics</u> because

$$\underline{\nabla} \times \underline{B} = \mu_0 \underline{J}$$

$$\Rightarrow \mu_0 \underline{\nabla} . \underline{J} = \underline{\nabla} \cdot (\underline{\nabla} \times \underline{B}) = 0$$ ↗ only true for steady state!

PHYS 340 lecture 16 Oct 12th, 2005

Last time, we derived the equations of magnetostatics.

Electrostatics	Magnetostatics
$\underline{\nabla} \cdot \underline{E} = \rho/\varepsilon_0$	$\underline{\nabla} \times \underline{B} = \mu_0 \underline{J}$
$\underline{\nabla} \times \underline{E} = 0$	$\underline{\nabla} \cdot \underline{B} = 0$
$\underline{E} = -\underline{\nabla} V$	$\underline{B} = \underline{\nabla} \times \underline{A}$
where $V(\underline{r}) = \int \frac{d^3\underline{r}'\, \rho(\underline{r}')}{4\pi\varepsilon_0 \lvert \underline{r}-\underline{r}'\rvert}$	where $\underline{A}(\underline{r}) = \int \frac{\mu_0}{4\pi} \frac{d^3\underline{r}'\, \underline{J}(\underline{r}')}{\lvert \underline{r}-\underline{r}'\rvert}$
(scalar) electric potential	magnetic vector potential

The multipole expansion of $V(\underline{r})$ is

$$V(\underline{r}) = \frac{Q}{4\pi\varepsilon_0 r} + \frac{\underline{p}\cdot\hat{\underline{r}}}{4\pi\varepsilon_0 r^2} + \begin{pmatrix}\text{quadrupole}\\ \text{term}\end{pmatrix} + \cdots$$

what is the corresponding expansion for $\underline{A}(\underline{r})$?

use the expansion $\frac{1}{\lvert\underline{r}-\underline{r}'\rvert} = \frac{1}{r} + \frac{\hat{\underline{r}}\cdot\underline{r}'}{r^2} + O\left(\frac{1}{r^3}\right) + \cdots$

and write $\underline{A}$ in terms of a line current

$$\underline{A}(\underline{r}) = \frac{\mu_0 I}{4\pi}\int \frac{d\underline{\ell}'}{\lvert\underline{r}-\underline{r}'\rvert} \simeq \frac{\mu_0 I}{4\pi r}\int d\underline{\ell}' + \frac{\mu_0 I}{4\pi r^2}\int d\underline{\ell}'(\hat{\underline{r}}\cdot\underline{r}')$$

The first term vanishes because in steady-state, the current loops are closed.

The second term can be manipulated into the form

$$\boxed{\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi r^2}\,\mathbf{m} \times \mathbf{r}'}$$ dipole vector potential

where $\mathbf{m}$ is the dipole moment $\mathbf{m} = \frac{1}{2}\int \mathbf{r}' \times I\,d\mathbf{l}'$

or $\frac{1}{2}\int \mathbf{r}' \times \mathbf{J}(\mathbf{r}')\, d^3\mathbf{r}'$

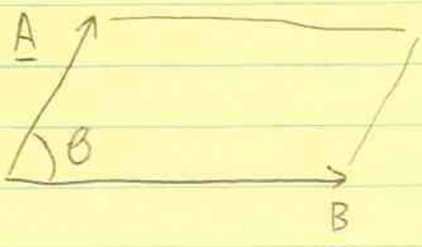
For a current loop,

$$\mathbf{m} = \frac{1}{2}\oint \mathbf{r}' \times d\mathbf{l}'\, I$$

$$= \int I\, d\mathbf{A}$$

$$\Rightarrow \boxed{\mathbf{m} = I\mathbf{A}}$$ true regardless of the shape.

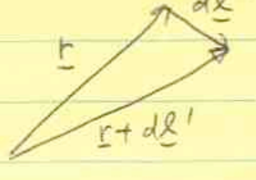
[here we used the result that $\mathbf{A} \times \mathbf{B}$ is the area of the parallelogram



area $= |\mathbf{A} \times \mathbf{B}|$

$= |\mathbf{A}|\,|\mathbf{B}| \sin\theta$

$$\Rightarrow \frac{1}{2}\mathbf{r}' \times d\mathbf{l}' = d\mathbf{A}$$

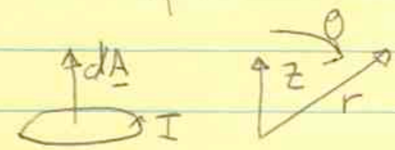


]

The field of a magnetic dipole

$\underline{B} = \underline{\nabla} \times \underline{A}$ gives the field. If we align the z-axis with $\underline{m}$,

and work in spherical coordinates, $\underline{A} = \dfrac{\mu_0 m \sin\theta}{4\pi r^2}\,\underline{\hat{\phi}}$



$$\Rightarrow \quad B_r = \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta A_\phi\right) \longleftarrow \left(r\text{-component of } \underline{\nabla}\times\underline{A}\right)$$

$$= \frac{\mu_0 m}{4\pi r^3}\, 2\cos\theta$$

(θ component of $\underline{\nabla}\times\underline{A}$)

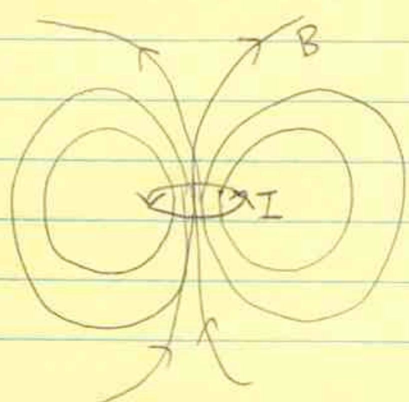
$$B_\theta = -\frac{1}{r}\frac{\partial}{\partial r}\left(rA_\phi\right) = \frac{\mu_0 m}{4\pi r^3}\sin\theta$$

$$\Rightarrow \quad \boxed{\underline{B} = \frac{\mu_0 m}{4\pi r^3}\left(2\cos\theta\,\underline{\hat{r}} + \sin\theta\,\underline{\hat{\theta}}\right)}$$

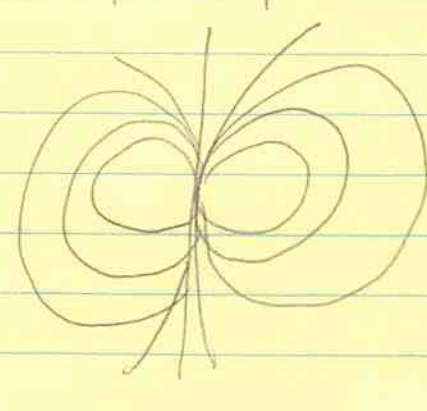
compare with $\underline{E} = \dfrac{p}{4\pi\varepsilon_0 r^3}\left(2\cos\theta\,\underline{\hat{r}} + \sin\theta\,\underline{\hat{\theta}}\right)$ for an electric dipole

As with electric dipoles, we can distinguish between

physical dipole

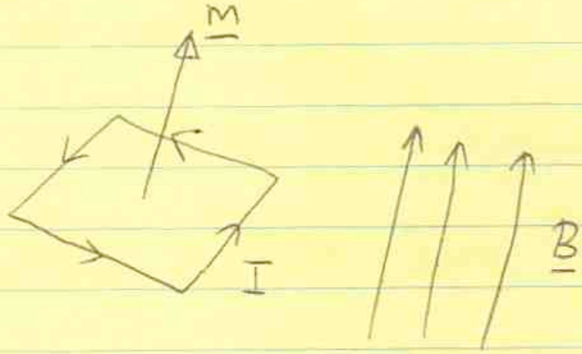


"pure" dipole

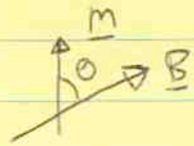


Torque and force on a dipole

Consider a square loop



there is a torque if $\underline{m}$ is misaligned with $\underline{B}$



the torque is $\quad |\underline{N}| = aF\sin\theta$

$$= a(aIB)\sin\theta$$

$$= mB\sin\theta \qquad \text{because } m = Ia^2$$

$$\Rightarrow \quad \boxed{\underline{N} = \underline{m} \times \underline{B}}$$

this is a general result, similar to the torque on an electric dipole.

$$\underline{N} = \underline{p} \times \underline{E}$$

In a non-uniform field, there is a force

$$\boxed{\underline{F} = \underline{\nabla}(\underline{m} \cdot \underline{B})}$$

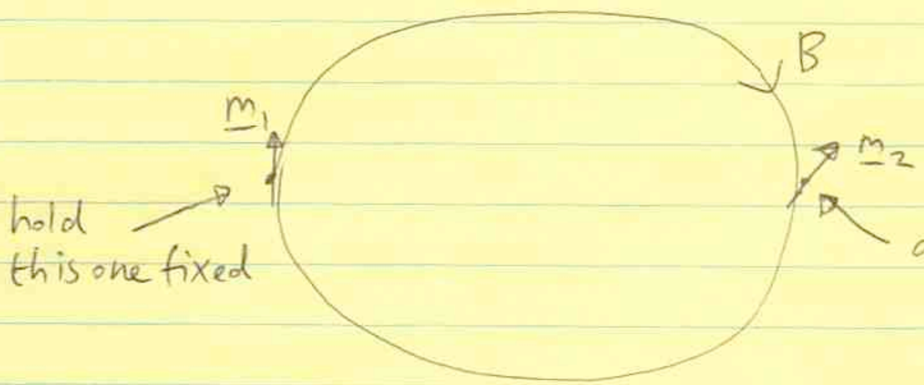
(see Griffiths prob 6.4 for a derivation)

and associated with the torque, we can write an energy

$$\boxed{U = -\underline{m} \cdot \underline{B}}$$

the force is $\underline{F} = -\underline{\nabla} U$

eg. interaction between two dipoles



$\underline{m}_2$ will align with the local field.

at this point, $B_\theta = \frac{\mu_0 m_1}{4\pi r^3}$

the dipoles want to anti-align $\uparrow \downarrow$

Comparison of electric and magnetic dipoles

Electric	Magnetic
$V(\underline{r}) = \frac{\underline{p} \cdot \hat{\underline{r}}}{4\pi\varepsilon_0 r^2}$	$\underline{A}(\underline{r}) = \frac{\mu_0 \underline{m} \times \underline{r}'}{4\pi r^2}$
$\underline{E}(\underline{r}) = \frac{p}{4\pi\varepsilon_0 r^3}\left(2\cos\theta\,\hat{\underline{r}} + \sin\theta\,\hat{\underline{\theta}}\right)$	$\underline{B}(\underline{r}) = \frac{\mu_0 m}{4\pi r^3}\left(2\cos\theta\,\hat{\underline{r}} + \sin\theta\,\hat{\underline{\theta}}\right)$
torque $\underline{N} = \underline{p} \times \underline{E}$	$\underline{N} = \underline{m} \times \underline{B}$ wants to align with local field
$\underline{p} = \sum q_i \underline{r}_i$	$\underline{m} = I\underline{A}$
force $\underline{F} = (\underline{p} \cdot \underline{\nabla})\underline{E}$	$\underline{F} = \underline{\nabla}(\underline{m} \cdot \underline{B})$
energy $U = -\underline{p} \cdot \underline{E}$	$U = -(\underline{m} \cdot \underline{B})$

Electric conductors (Griffiths §2.5)

So far we've looked at how to calculate the electric field of fixed charges, ie. given $\rho(\underline{r})$, we can either

1) calculate $\underline{E}$ by integrating $\underline{E}(\underline{r}) = \int \frac{\rho(\underline{r}')\,d^3\underline{r}'}{4\pi\varepsilon_0} \frac{\underline{r}-\underline{r}'}{|\underline{r}-\underline{r}'|^3}$

or by first finding the potential

$$V(\underline{r}) = \int \frac{\rho(\underline{r}')\,d^3\underline{r}'}{4\pi\varepsilon_0|\underline{r}-\underline{r}'|}$$

2) use Gauss' law if there is enough symmetry.

However, in many materials the electrons are free to <u>move</u>!

Properties of conductors

1) $\underline{E}=0$ inside when in electrostatic equilibrium. If this were not true, charges would move until $\underline{E}=0$ is satisfied.

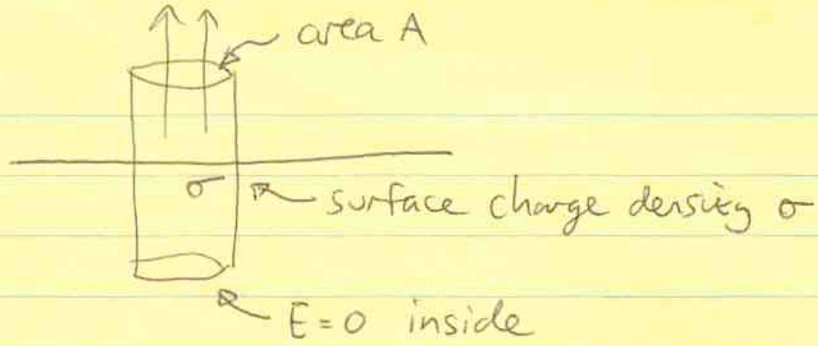
2) Inside the conductor, $\rho = 0$. All the charge resides on the surface. Because there are no electric field lines inside the conductor ($\underline{E}=0$) there can be no charge there.

3) Because $\underline{E} = -\underline{\nabla} V$, it must be that V = constant in a conductor — a conductor is an equipotential.

4) Because the conductor is an equipotential, $\underline{E}$ must be perpendicular to the surface.

We can get the electric field at the surface from

Gauss' law:

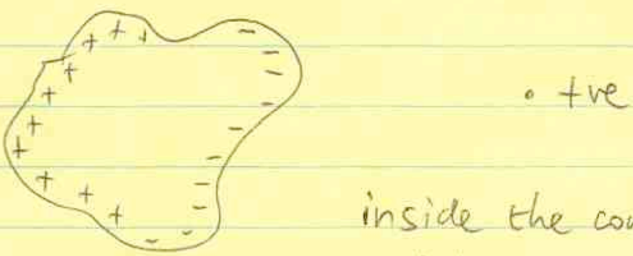


$$EA = \frac{\sigma A}{\varepsilon_0}$$

$$\Rightarrow \boxed{E = \sigma/\varepsilon_0}$$

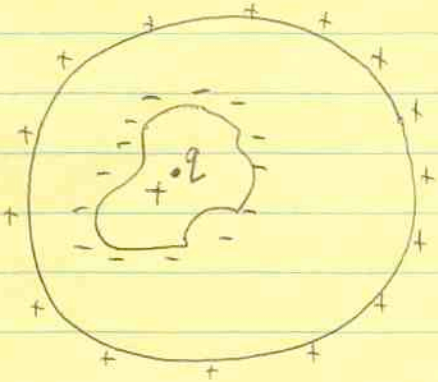
(this is twice the value of a charged plane where the charges are held fixed – because in that case, electric field lines come out from the top and bottom).

As an example, consider what happens when a point charge is brought close to a conductor with zero net charge.



inside the conductor, the electrons can move around to ensure that $\underline{E} = 0$.
The result is a net surface charge, negative on one side and positive on the other.
There is a net force on the charge (to zeroth order this is the force due to the induced dipole moment).

Cavities inside conductors



A charge q is placed inside a cavity (any shape) inside a conducting sphere.

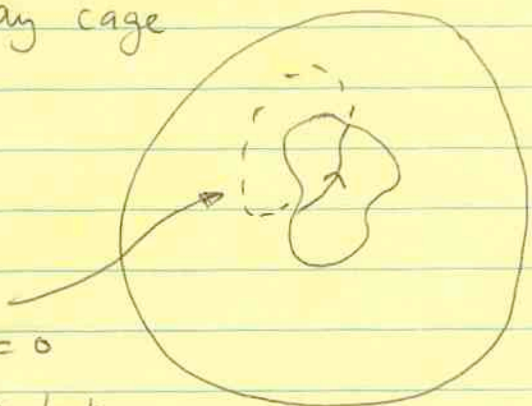
What is the external field?

A charge $-q$ is induced on the surface of the cavity, so that $\underline{E} = 0$ inside the conductor.

On the surface, there is a charge $+q$ which distributes itself uniformly

$\Rightarrow$ outside the sphere $\underline{E} = \hat{\underline{r}} \dfrac{q}{4\pi\varepsilon_0 r^2}$

Faraday cage



$\oint \underline{E} \cdot d\underline{l} = 0$ around this loop

$\Rightarrow$ $\underline{E} = 0$ inside the cavity.

if the charge inside the cavity is zero, the field inside is also zero.

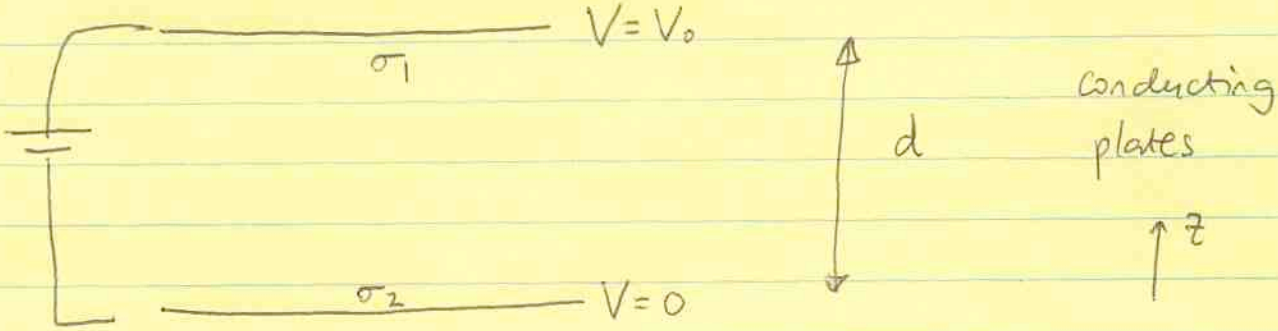
The same principle lies behind a Faraday cage that shields external fields.

Some of the most sensitive tests of the $1/r^2$ law come from trying to detect $\underline{E}$ inside a conducting cavity.

Parallel plate capacitor

Because V = constant in a conductor, we can use this as a boundary condition for either Laplace's equation $\nabla^2 V = 0$ or Poisson's equation $\nabla^2 V = -\rho/\varepsilon_0$ outside the conductor.

Consider a parallel plate capacitor

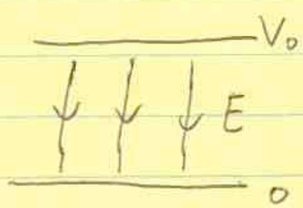


Solve Laplace's equation $\nabla^2 V = \frac{d^2 V}{dz^2} = 0$

$$\Rightarrow \quad V = A + Bz$$

$$V = 0 \text{ at } z = 0 \quad \Rightarrow \quad A = 0$$
$$V = V_0 \text{ at } z = d \quad \Rightarrow \quad B = \frac{V_0}{d}$$

$$\Rightarrow \quad \boxed{\begin{array}{l} V = \dfrac{V_0 z}{d} \\ E = -\dfrac{V_0}{d}\,\hat{\underline{z}} \end{array}}$$



What is the surface charge density?

$E = \sigma/\varepsilon_0$ for a conductor

$$\Rightarrow \quad \sigma_1 = \varepsilon_0 E = \frac{\varepsilon_0 V_0}{d}$$
$$\sigma_2 = -\varepsilon_0 E = -\frac{\varepsilon_0 V_0}{d}$$

The <u>capacitance</u> is defined as the charge stored per volt

$$C = \frac{Q}{\Delta V} = \frac{\frac{\varepsilon_0 V_0}{d} A}{V_0} = \varepsilon_0 A/d. \quad \left(\begin{array}{l}\text{plates have} \\ \text{area } A\end{array}\right)$$

What is the energy stored?

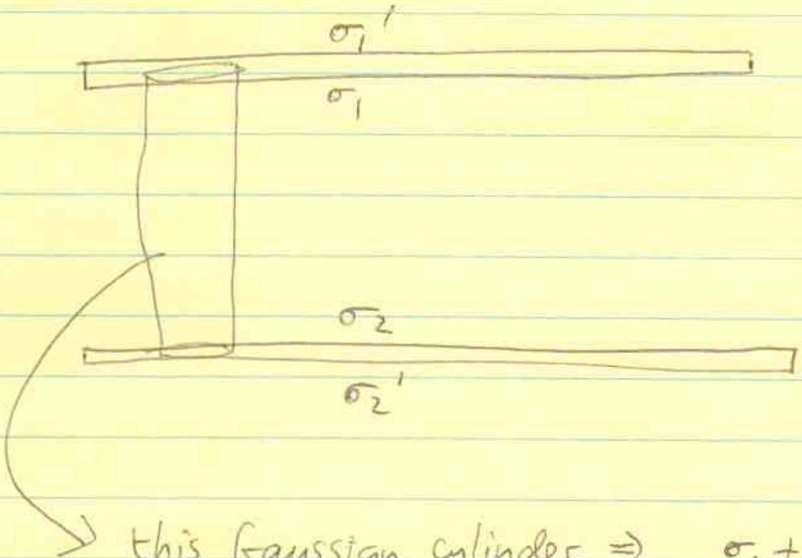
$$U = \frac{1}{2}\int \rho V \, d^3\underline{r} = \frac{1}{2}\int \sigma V \, dA$$

$$= \frac{1}{2} Q V_0$$

$$= \frac{1}{2} C V_0^2 = \frac{1}{2} Q^2/C$$

This result is independent of geometry, true for any capacitor

Let's think about the parallel plate capacitor more carefully — how is the charge distributed over the plates?



this Gaussian cylinder $\Rightarrow$ $\sigma_1 + \sigma_2 = 0$ $\Rightarrow$ $\boxed{\sigma_1 = -\sigma_2}$
because $\underline{E} = 0$ inside the conductors

The electric fields from the planes of charge σ_1' and σ_2' must also cancel, to make sure that $\underline{E} = 0$ inside the plates.

$$\Rightarrow \quad \boxed{\sigma_2' = \sigma_1'}$$

write $\sigma_+ = \sigma_1 + \sigma_1'$

$$\sigma_- = \sigma_2 + \sigma_2'$$

$$\Rightarrow \quad \sigma_+ = \sigma_1 + \sigma_1' = \sigma_1 + \sigma_2' = \sigma_1 + \sigma_- - \sigma_2$$

$$= 2\sigma_1 + \sigma_-$$

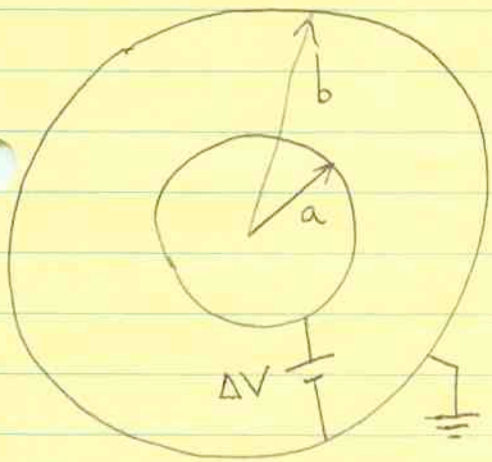
$$\Rightarrow \quad \boxed{\begin{aligned} \sigma_1 &= \frac{\sigma_+ - \sigma_-}{2} \\ \sigma_1' &= \frac{\sigma_+ + \sigma_-}{2} \end{aligned}}$$

for our parallel plate capacitor, the total charges on the plates were equal and opposite, $\sigma_+ = -\sigma_- = \sigma$

$$\Rightarrow \quad \sigma_1 = \sigma = -\sigma_2$$
$$\sigma_1' = 0 = \sigma_2'$$

$\Rightarrow$ equal and opposite charge densities on the insides of the plates, no charge on outside edges.

Capacitance of two conducting spheres



inner sphere at potential ΔV
outer sphere is grounded $V = 0$

if the charge on the inner sphere is Q

then $\underline{E}(\underline{r}) = \dfrac{Q\,\hat{\underline{r}}}{4\pi\varepsilon_0 r^2} \quad a<r<b$

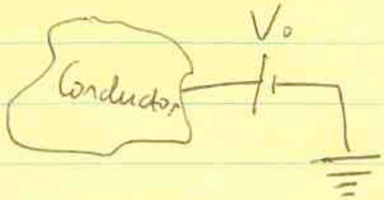
$$\Rightarrow \quad V(r) = -\int_b^r \frac{Q}{4\pi\varepsilon_0 r^2}\,dr$$
$$= \frac{Q}{4\pi\varepsilon_0}\left[\frac{1}{r} - \frac{1}{b}\right]$$

$$\Rightarrow \quad \Delta V = \frac{Q}{4\pi\varepsilon_0}\left[\frac{1}{a} - \frac{1}{b}\right]$$

$$\Rightarrow \quad C = \frac{Q}{\Delta V} = \frac{4\pi\varepsilon_0\, ab}{(b-a)} \quad //$$

A comment about "Earth".

A conductor connected to Earth is said to be "grounded" or zero potential by definition.
The idea is that the Earth has a huge capacitance relative to the conductor we're considering. It can act as a source or sink of electrons without changing its voltage $\left(\Delta V = \frac{\Delta Q}{C} \text{ very small} \right)$



conductor has $V = V_0$.

In problems, can specify potential of a conductor, or its total charge.

Units of capacitance

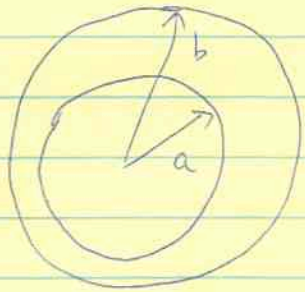
The units of capacitance are CV^{-1} or Farads (F).

Typical values are very small in these units, eg. parallel plates with area $1 cm^2$ and separation 3mm

$$C = \frac{\varepsilon_0 A}{d} = 3 \times 10^{-13} \text{ F} = \underline{\underline{0.3 \text{ pF}}}.$$

PHYS 340 lecture 19
Oct 21, 2005

We can solve the spherical capacitor another way, which is to use the conducting surfaces as boundary conditions for $\nabla^2 V = 0$.



Because of the spherical symmetry, the potential is only a function of distance from the center of the sphere. $V(r)$.

potential difference ΔV

Because V is a function of r only,

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) = 0$$

(other terms involving $\frac{\partial V}{\partial \theta}$, $\frac{\partial V}{\partial \phi}$ vanish)

$$\Rightarrow \quad \boxed{V = A + \frac{B}{r}} \quad \text{where } A \text{ and } B \text{ are constants.}$$

(this should look familiar: the potential of a point charge is $\propto \frac{1}{r}$, and we can always add a constant to V)

the first term $V = \text{constant}$ has $\frac{\partial V}{\partial r} = 0$ and so obviously satisfies $\nabla^2 V = 0$.

the second term has $r^2 \frac{\partial V}{\partial r} = \text{constant}$ because $\frac{\partial}{\partial r}\left(\frac{B}{r}\right) \propto \frac{1}{r^2}$

therefore $\nabla^2 V \propto \frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) \propto \frac{\partial}{\partial r}(\text{constant}) = 0$

Note that the solution is not ~~as~~ easy if V is a function of more than one coordinate, e.g. $V(\theta, r)$ $\rightarrow$ more on that later.

Now apply boundary conditions

$$V=0 \text{ at } r=b \Rightarrow \quad A+\frac{B}{b}=0$$

$$V=\Delta V \text{ at } r=a \Rightarrow \quad \Delta V = A+\frac{B}{a}$$

$$= B\left(\frac{1}{a}-\frac{1}{b}\right)$$

$$= \frac{B}{ab}(b-a)$$

$$\therefore \quad V(r) = A+\frac{B}{r}$$

$$= -\frac{B}{b}+\frac{B}{r}$$

$$= \frac{B}{br}(b-r) = \Delta V \frac{a}{r}\frac{(b-r)}{(b-a)}$$

$$\boxed{V(r) = \frac{\Delta V\, a}{b-a}\left(\frac{b}{r}-1\right)}$$

What is the surface charge density? (at $r=a$)

$$\sigma = -\varepsilon_0 \left.\frac{\partial V}{\partial r}\right|_{r=a} = +\frac{\varepsilon_0\, \Delta V\, a}{b-a}\left.\frac{b}{r^2}\right|_{r=a} = \left(\frac{\varepsilon_0\, \Delta V}{b-a}\right)\left(\frac{b}{a}\right)$$

total charge $\quad Q = 4\pi a^2 \sigma = \dfrac{4\pi\varepsilon_0\, ab\, \Delta V}{b-a}$

$\Rightarrow$ capacitance $\quad \boxed{C = \dfrac{Q}{\Delta V} = \dfrac{4\pi\varepsilon_0\, ab}{b-a}}$

Note that <u>when</u> $b \gg a$, $\boxed{C = 4\pi\varepsilon_0 a}$ independent of b.

An important result is the $\boxed{\text{Uniqueness Theorem}}$

If a function $V(\underline{r})$ satisfies $\nabla^2 V = -\rho/\varepsilon_0$ and the boundary conditions, it is a unique solution.

Proof 1

To prove this, consider two solutions V_1 and V_2. We will show that $V_* = V_2 - V_1 = 0$

V_* satisfies $\nabla^2 V_* = \nabla^2 V_2 - \nabla^2 V_1 = 0$ which is Laplace's equation.

Griffiths points out that the solutions to $\nabla^2 V = 0$ are "boring" in the sense that they have no local maxima or minima. This makes sense because there are no charges ($\rho = 0$) which would produce local potential wells.

Since V_1 and V_2 have the same values on the boundary, $V_* = 0$ on the boundary.

But if there are no maxima or minima $\Rightarrow$ $\underline{\underline{V_* = 0 \text{ everywhere}}}$

$\Rightarrow V_1 = V_2$

Proof 2

A more mathematical proof is to look at

$$\underline{\nabla} \cdot (V_* \underline{\nabla} V_*) = |\underline{\nabla} V_*|^2 + V_* \cancelto{0}{\nabla^2 V_*}$$

if we integrate this over the volume, we have

$$\text{LHS} = \int \underline{\nabla} \cdot (V_* \underline{\nabla} V_*) \, d^3\underline{r} = \int d\underline{A} \cdot (V_* \underline{\nabla} V_*)$$

$$= 0$$

vanishes on the boundary.

$$RHS = \int |\underline{\nabla} V_*|^2 \, d^3\underline{r}$$

which is > 0 unless $|\underline{\nabla} V_*| = 0$ everywhere

$\Rightarrow$ $V_* =$ constant

if $V_* = 0$ on the boundaries, $\underline{\underline{V_* = 0 \text{ everywhere}}}$.

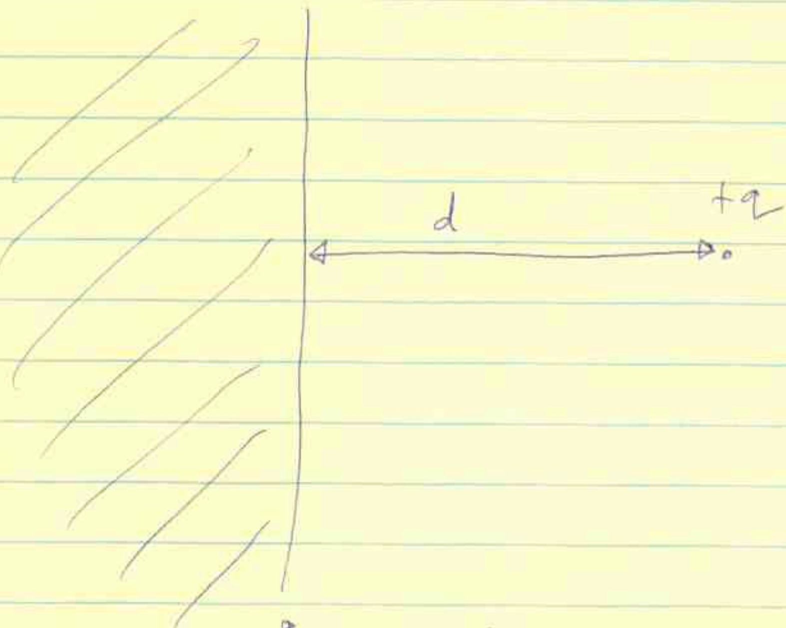
In fact, the boundary conditions can be
Dirichlet boundary conditions V specified
Neumann boundary conditions $\underline{\nabla} V \cdot \underline{\hat{n}}$ specified
given on a closed surface (could be at infinity)

The power of the uniqueness theorem is that if you can come up with a solution, no matter which method (eg. a guess) then it is the solution.
(as long as it satisfies $\nabla^2 V = -\rho/\varepsilon_0$ and the boundary conditions).

(eg. HW 6 Q2)

An immediate application of the uniqueness theorem is the
<u>METHOD OF IMAGES</u>

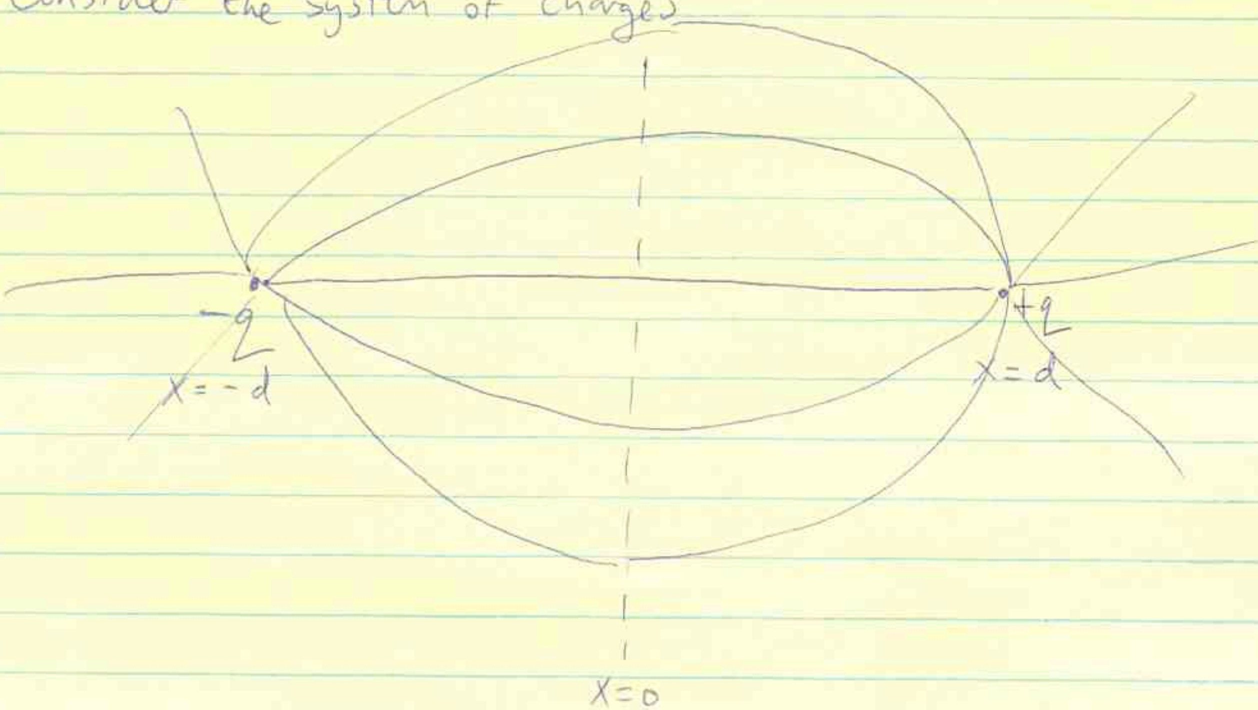
Consider a point charge next to an semi-infinite conductor



we want a solution
that has
$V = 0$ for $x = 0$
$V \to \infty$ for $x \to \infty$
in the right hand
plane

assume the conductor has $V=0$

The method of images consists of replacing the conductor by an image charge.
Consider the system of charges



The potential is

$$V(\underline{r}) = \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{|\underline{r}-\underline{r}_+|} - \frac{1}{|\underline{r}-\underline{r}_-|}\right]$$

$$= \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{\sqrt{z^2+y^2+(x-d)^2}} - \frac{1}{\sqrt{z^2+y^2+(x+d)^2}}\right]$$

This potential is a solution of $\nabla^2 V = 0$, and satisfies $V=0$ for $x=0$.

Therefore, by the uniqueness theorem, it must also be the solution in the right hand plane in the first case also.

The surface charge density on the conductor is

$$\sigma = -\varepsilon_0 \left.\frac{\partial V}{\partial x}\right|_{x=0}$$

$$= -\frac{q}{4\pi\varepsilon_0}\left[\frac{-\frac{1}{2}\cdot 2(x-d)}{((x-d)^2+y^2+z^2)^{3/2}} + \frac{\frac{1}{2}\,2\,(x+d)}{((x+d)^2+y^2+z^2)^{3/2}}\right]_{x=0}$$

$$\Rightarrow \boxed{\sigma = -\frac{2qd}{4\pi(z^2+y^2+d^2)^{3/2}}}$$

The total charge on the conductor is

$$Q = \int \sigma\, dA = \int 2\pi r\, dr \left(-\frac{2qd}{4\pi}\right)\frac{1}{(z^2+y^2+d^2)^{3/2}}$$

let $r' = r/d$ and $\cancel{z}^2 + y^2 = r^2$

$$\Rightarrow \quad Q = -q \int_0^{\infty} \frac{r' dr'}{(r'^2 + 1)^{3/2}}$$

solve with the substitution $r' = \tan\theta$

$$\Rightarrow \quad Q = -q \int_0^{\pi/2} \frac{\tan\theta \sec^2\theta \, d\theta}{\sec^3\theta}$$

$$= -q \int_0^{\pi/2} \sin\theta \, d\theta = \underline{\underline{-q}} \quad \checkmark$$

as expected, a charge $-q$ is induced on the conducting surface.

What is the force on the charge?

The electric field at the charge due to the image charge is

$$\underline{E} = \frac{-q}{4\pi\varepsilon_0 (2d)^2} \hat{\underline{x}}$$

$$\Rightarrow \quad \text{force} = \frac{-q^2 \hat{\underline{x}}}{16\pi\varepsilon_0 d^2} \quad (\text{attractive})$$

The work done in bringing the charge to distance d is

$$W = -\int \underline{F} \cdot d\underline{x} = +\int_{\infty}^{d} \frac{q^2}{16\pi\varepsilon_0} \frac{dx}{x^2}$$

$$= \underline{\underline{-\frac{q^2}{16\pi\varepsilon_0 d}}}$$

[At each stage, we place an image charge at distance $2x$ from the charge]

But what is the energy of the image charge system?

$$U = \frac{1}{2} \sum q_i V_i = -\frac{1}{2}\left[2 \cdot \frac{q^2}{4\pi\varepsilon_0 (2d)}\right]$$

$$= \underline{\underline{-\frac{q^2}{8\pi\varepsilon_0 d}}} \longrightarrow \text{Twice as large!}$$

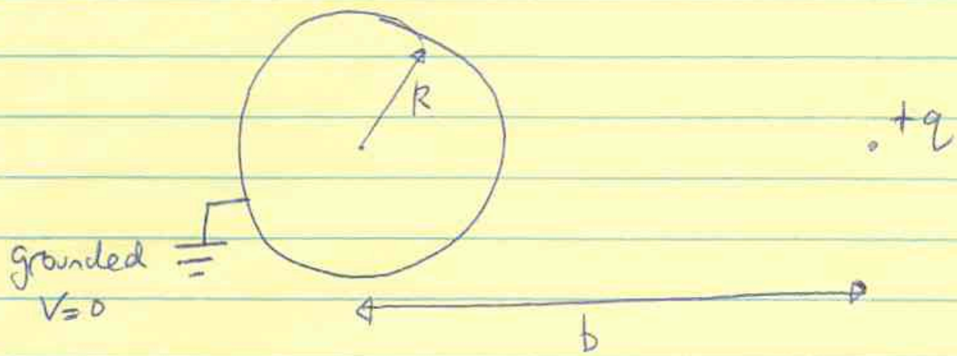
This is because moving charge around on the surface takes no work because the conductor is an equipotential.

The factor of 2 is obvious if we consider the total energy in the electric field $\longrightarrow$ for the case with the conductor, $E=0$ in the left half plane, so the integral $\int d^3\underline{r}\ \frac{1}{2}\varepsilon_0 E^2$ is $\frac{1}{2}$ of the same integral for the image charge system.

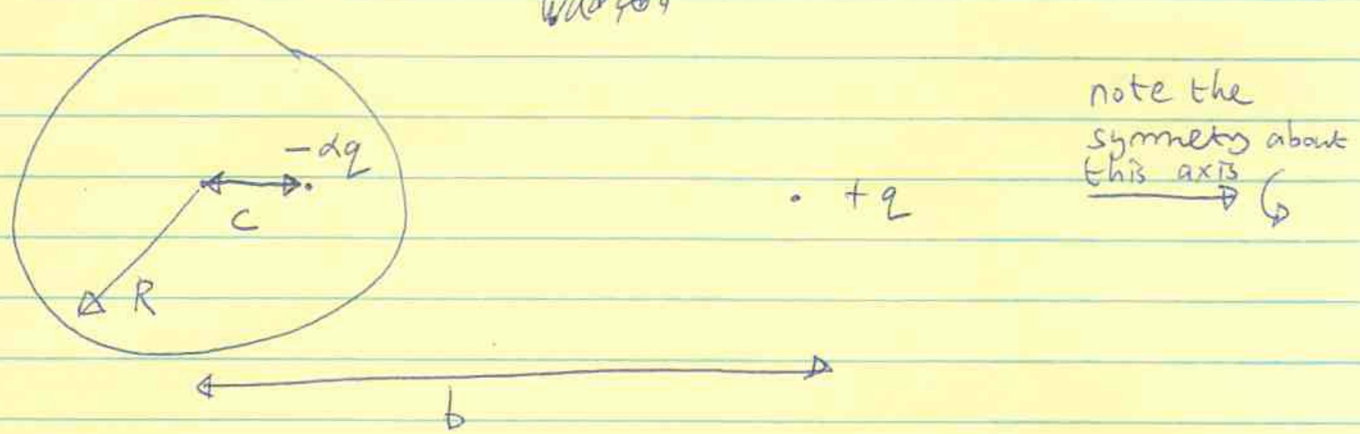
PHYS 340 lecture 21
Oct 26, 2005

Conducting sphere plus external charge

In this example, we consider a conducting sphere with an external charge



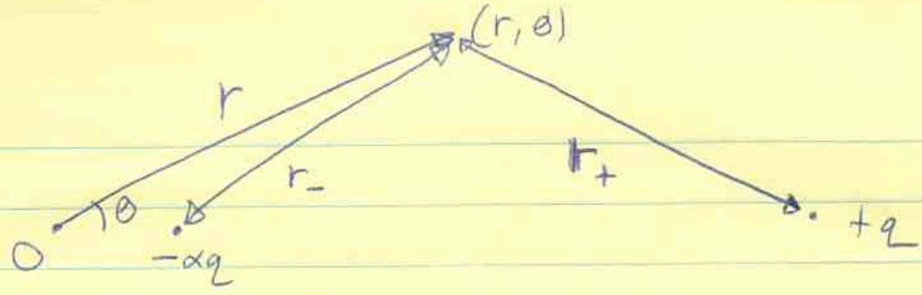
We need an image charge (s) which will give a spherical equipotential. Consider an image charge $-\alpha q$ distance c from the center of the sphere. Note that the image charge must lie inside the sphere ($c < R$) otherwise we change the original problem.



Use (r, θ) coordinates centred on the centre of the sphere.

$$V(r,\theta) = \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{\sqrt{r^2+b^2-2rb\cos\theta}} - \frac{\alpha}{\sqrt{r^2+c^2-2rc\cos\theta}}\right]$$

(see diagram on the next page).



$$r_-^2 = r^2 + c^2 - 2rc\cos\theta$$
$$r_+^2 = r^2 + b^2 - 2rb\cos\theta$$

Boundary condition: $V=0$ on the surface of the sphere $(r=R)$

$$\Rightarrow \quad 0 = \frac{1}{\sqrt{R^2+b^2-2Rb\cos\theta}} - \frac{\alpha}{\sqrt{R^2+c^2-2Rc\cos\theta}}$$

$$R^2+c^2-2Rc\cos\theta = \alpha^2\left(R^2+b^2-2Rb\cos\theta\right)$$

This must be true for all θ

$$\Rightarrow \quad R^2+c^2 = \alpha^2(R^2+b^2)$$
$$-2Rc = -2\alpha^2 Rb$$

$$\Rightarrow \quad \boxed{c = \alpha^2 b}$$

and $\quad R^2 + c^2 = \frac{c}{b}\left(R^2+b^2\right)$

$$b(R^2+c^2) = c(R^2+b^2)$$

Solutions are $c=b$ or $\boxed{cb = R^2}$

trivial solution, charges on top of one another and cancel!

this is the one we want $(c<b)$

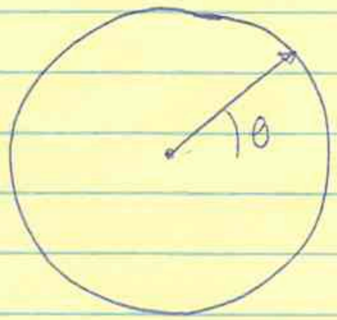
$\Rightarrow$ The image charge has a strength $-\alpha q = -\left(\frac{R}{b}\right)q$

and position $\quad c = \frac{R^2}{b} = R\left(\frac{R}{b}\right)$ //

What is the charge density on the surface of the sphere?

$$\sigma = -\varepsilon_0 \frac{\partial V}{\partial r}\bigg|_{r=R} = \frac{-q}{4\pi R} \frac{b^2 - R^2}{(b^2 + R^2 - 2Rb\cos\theta)^{3/2}}$$

(see next page for details)

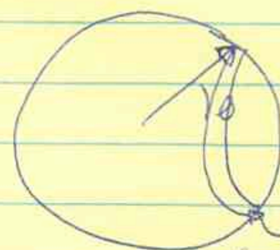


$\theta = 0$ $$\sigma = -\frac{q}{4\pi R} \frac{b^2 - R^2}{(b-R)^3}$$

$$= -\frac{q}{4\pi R} \frac{b+R}{(b-R)^2}$$

$\theta = \pi$ $$\sigma = -\frac{q}{4\pi R} \frac{b-R}{(b+R)^2}$$

$\theta = \pi/2$ $$\sigma = \frac{-q}{4\pi R} \frac{b^2 - R^2}{(b^2+R^2)^{3/2}}$$



area
$(2\pi R \sin\theta)(R\,d\theta)$

Total charge on the sphere is

$$Q = \int_0^\pi 2\pi R^2 \sin\theta \, d\theta \; \sigma(\theta)$$

$$= 2\pi R^2 \left(-\frac{q}{4\pi R}\right) (b^2 - R^2) \int_{-1}^{1} \frac{d(\cos\theta)}{(b^2 + R^2 - 2Rb\cos\theta)^{3/2}}$$

$$= \underline{\underline{-q\frac{R}{b}}}$$

(see HW 6 question 4c for this integral)

Same as the image charge!
This charge is supplied from ground to maintain the sphere at $V=0$.

Calculation of

$$\sigma = -\varepsilon_0 \frac{\partial V}{\partial r}\bigg|_{r=R}$$

$$= -\frac{q}{4\pi}\left[\frac{-\frac{1}{2}(2R - 2b\cos\theta)}{(R^2+b^2-2Rb\cos\theta)^{3/2}} - \frac{-\frac{1}{2}\alpha(2R-2c\cos\theta)}{(R^2+c^2-2Rc\cos\theta)^{3/2}}\right]$$

$$= -\frac{q}{4\pi}\left[\frac{b\cos\theta - R}{(R^2+b^2-2Rb\cos\theta)^{3/2}} + \frac{\alpha R - \alpha c\cos\theta}{(R^2+c^2-2Rc\cos\theta)^{3/2}}\right]$$

but $c = R^2/b$

$$\Rightarrow \quad R^2+c^2-2Rc\cos\theta$$

$$= R^2 + R^2\left(\frac{R^2}{b^2}\right) - \frac{2R^3}{b}\cos\theta$$

$$= \frac{R^2}{b^2}\left(b^2+R^2-2Rb\cos\theta\right)$$

$\alpha = \frac{R}{b}$

$$\Rightarrow \text{2nd term is} \quad \frac{\alpha(R - \cancel{\alpha}\overset{R^2/b}{c}\cos\theta)\,\cancel{\frac{R}{b}}\,\frac{b^2}{R^2}}{(b^2+R^2-2Rb\cos\theta)^{3/2}\left(\cancel{\frac{R^3}{b^3}}\right)}$$

$$\Rightarrow \quad \sigma = -\frac{q}{4\pi}\,\frac{b\cancel{\cos\theta} - R + b^2/R - \cancel{b\cos\theta}}{(b^2+R^2-2Rb\cos\theta)^{3/2}}$$

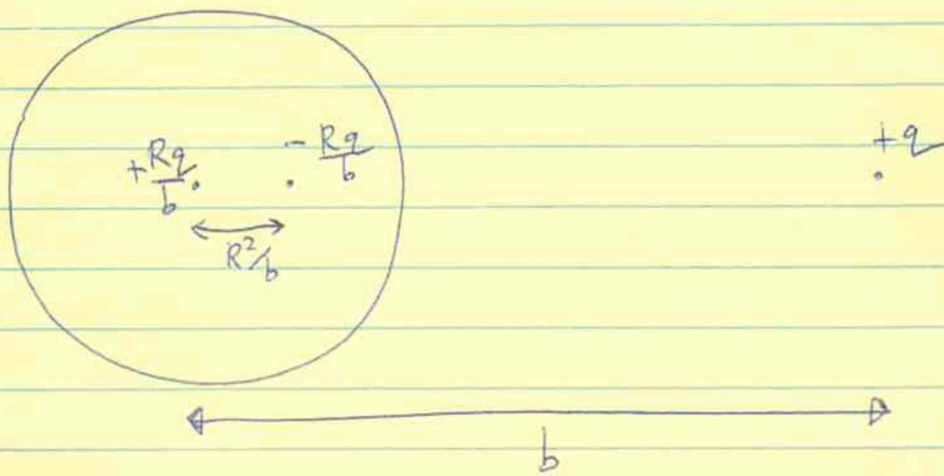
$$= -\frac{q}{4\pi R}\,\frac{b^2-R^2}{(b^2+R^2-2Rb\cos\theta)^{3/2}}$$

The force on the charge is

$$F = \frac{-q^2\, R/b}{4\pi\varepsilon_0 \left(b - R^2/b\right)^2} = \underline{\underline{-\frac{q^2}{4\pi\varepsilon_0}\,\frac{Rb}{(b^2-R^2)^2}}}$$

<u>What if the sphere is electrically neutral</u> (not connected to ground)?

Then we must add another image charge at the center of the sphere $+\frac{R}{b}q$. The surface of the sphere is still an equipotential because equipotentials of a point charge are spherical.



The potential at the surface is $V(R) = \frac{Rq}{b}\,\frac{1}{4\pi\varepsilon_0 R}$

$$= \underline{\frac{q}{4\pi\varepsilon_0 \cancel{q}\, b}}$$

The charge density is

$$\sigma = \frac{q}{4\pi R^2}\,\frac{R}{b}\left[1 - \frac{b(b^2-R^2)}{(b^2+R^2-2Rb\cos\theta)^{3/2}}\right]$$

For the grounded sphere, what is the energy stored?

$$W = -\int F\,dx$$

$$= -\int_{\infty}^{b} \frac{-q^2}{4\pi\varepsilon_0} \frac{Rx}{(x^2-R^2)^2}$$

substitute $y^2 = x^2 - R^2$

$y\,dy = x\,dx$

$$\Rightarrow \quad W = -\frac{q^2 R}{4\pi\varepsilon_0} \int_{\infty}^{\sqrt{b^2-R^2}} \frac{dy}{y^3}$$

$$= \frac{-q^2 R}{4\pi\varepsilon_0} \left[\frac{1}{b^2-R^2}\right] \frac{1}{2}$$

The image configuration has

$$U = \frac{1}{2} \sum q_i V_i$$

$$= \frac{-q^2\, R/b}{4\pi\varepsilon_0 (b - R^2/b)}$$

$$= -\frac{q^2 R}{4\pi\varepsilon_0 (b^2-R^2)}$$

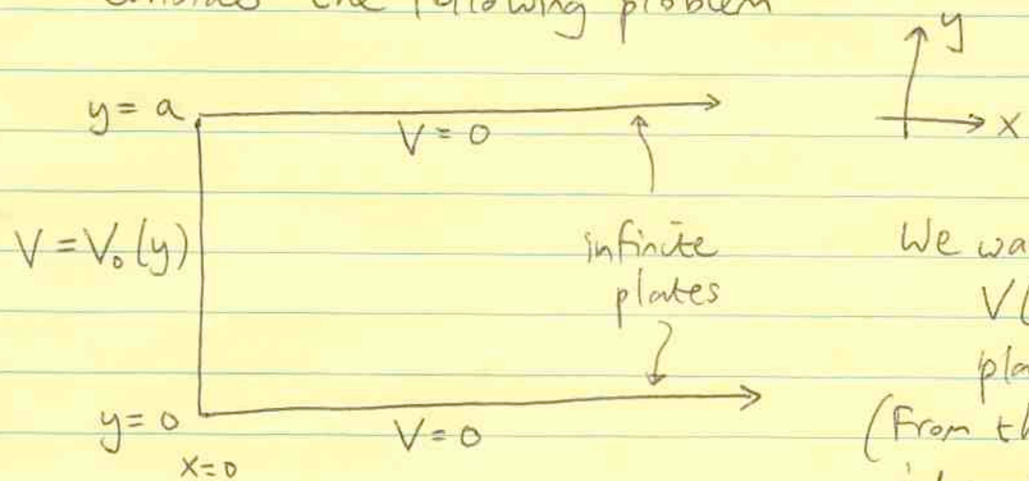
again, there is a factor of 2 difference.

PHYS 340 lecture 22
Oct 28, 2005

Separation of Variables in Cartesian coordinates

Separation of variables is a method to solve $\nabla^2 V = 0$ directly.

Consider the following problem



We want to find $V(x,y)$ between the plates.
(From the symmetry, V is independent of z).

V satisfies $\nabla^2 V = \dfrac{\partial^2 V}{\partial x^2} + \dfrac{\partial^2 V}{\partial y^2} = 0$

the boundary conditions are
$V = 0$ for $y = 0$, $y = a$, all x
$V = V_0(y)$ for $x = 0$
$V \to 0$ at $x \to \infty$

The procedure will be similar to the case where V was a function of one coordinate only – find the general solution, and apply the boundary conditions.

In seperation of variables, we look for a seperable solution:

$$V(x,y) = X(x)\, Y(y)$$

function of x only

function of y only.

then $\nabla^2 V = 0 \Rightarrow Y \dfrac{d^2X}{dx^2} + X \dfrac{d^2Y}{dy^2} = 0$

or $\boxed{\dfrac{1}{X}\dfrac{d^2X}{dx^2} + \dfrac{1}{Y}\dfrac{d^2Y}{dy^2} = 0}$

This equation is of the form

(function of x) + (function of y) = 0

and must be true for all x and y. The only way that this can be true is if each term is a constant.
(This is the key argument — make sure you understand it!)

We'll write the constant as k^2 — the "separation constant"

$$\frac{1}{X}\frac{d^2X}{dx^2} = k^2 \qquad \frac{1}{Y}\frac{d^2Y}{dy^2} = -k^2$$

the solutions are $X \propto \begin{Bmatrix} e^{kx} \\ e^{-kx} \end{Bmatrix}$ $\quad Y \propto \begin{Bmatrix} \sin ky \\ \cos ky \end{Bmatrix}$

[NB: I chose the sign of k^2 deliberately to make the x-functions exponential — see boundary condition number ① below.]

$\Rightarrow$ the general solution is any linear combination of

$$V(x,y) = \begin{cases} e^{kx} \sin ky \\ e^{kx} \cos ky \\ e^{-kx} \sin ky \\ e^{-kx} \cos ky \end{cases}$$

The idea is to make a linear combination of these solutions that satisfies the boundary conditions.
[Because the equation is linear, $V = V_1 + V_2$ is a solution if V_1 and V_2

are solutions, since $\nabla^2 V = \nabla^2(V_1 + V_2) = \nabla^2 V_1 + \nabla^2 V_2 = 0$

Now apply the boundary conditions

1. $V = 0$ for $x \to \infty$ $\Rightarrow$ we need the e^{-kx} solutions, because e^{+kx} diverges rapidly as $x \to \infty$.

2. $V = 0$ for $y = 0$ $\Rightarrow$ we need the $\sin(ky)$ solution which vanishes at $y = 0$ ($\cos ky$ is finite at $y = 0$).

3. $V = 0$ for $y = a$ $\Rightarrow$ $\sin(ka) = 0$

An infinite set of k's satisfy this boundary condition

$$k = \frac{n\pi}{a} \qquad n = 1, 2, 3 \ldots$$

$\Rightarrow$ The most general solution is

$$\boxed{V(x,y) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi y}{a}\right) e^{-n\pi x/a}}$$

The final boundary condition is

4. $V = V_0(y)$ at $x = 0$. This boundary condition determines the constants A_n.

$$V_0(y) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi y}{a}\right) \qquad \text{—} (*)$$

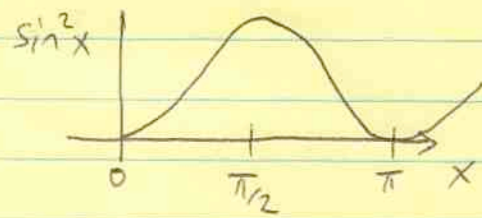
The A_n are the coefficients of a Fourier series expansion of $V_0(y)$.

This expansion works because the sines form a "complete set."

How to find A_n? Use <u>orthogonality</u> relations

$$\int_0^a \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi x}{a}\right) dx = \frac{a}{2}\,\delta_{mn}$$

The way to remember this is that the average value of $\sin^2 x$ is $\frac{1}{2}$, $\langle \sin^2 x \rangle = \dfrac{\int_0^{\pi/2} \sin^2 x\, dx}{\pi/2} = \dfrac{1}{2}$



$$\Rightarrow \quad \int_0^{\pi/2} \sin^2 x\, dx = \pi/4 .$$

Now multiply (✱) by $\sin\left(\frac{m\pi y}{a}\right)$ and integrate

$$\int_0^a V_0(y) \sin\left(\frac{m\pi y}{a}\right) dy = \sum_{n=1}^{\infty} A_n \underbrace{\int_0^a \sin\left(\frac{m\pi y}{a}\right) \sin\left(\frac{n\pi y}{a}\right) dy}_{\frac{a}{2}\delta_{mn}}$$

$$= \frac{a}{2} A_m$$

$$\Rightarrow \quad \boxed{A_m = \frac{2}{a} \int_0^a \sin\left(\frac{m\pi y}{a}\right) V_0(y)\, dy}$$

Let's do a specific example:

eg. $V_0(y) = \text{constant} = V_0$

$$A_m = \frac{2}{a}\int_0^a \sin\left(\frac{m\pi y}{a}\right) V_0 \, dy$$

$$= \frac{2V_0}{a}\left[-\cos\left(\frac{m\pi y}{a}\right)\frac{a}{m\pi}\right]_0^a$$

$$= \frac{2V_0}{m\pi}\left[1-\cos m\pi\right]$$

$$= \begin{cases} \dfrac{4V_0}{m\pi} & m \text{ odd} \\ 0 & m \text{ even} \end{cases}$$

$$\Rightarrow \boxed{V(x,y) = \sum_{n\,\text{odd}} \frac{4V_0}{n\pi} e^{-\frac{n\pi x}{a}} \sin\left(\frac{n\pi y}{a}\right)}$$

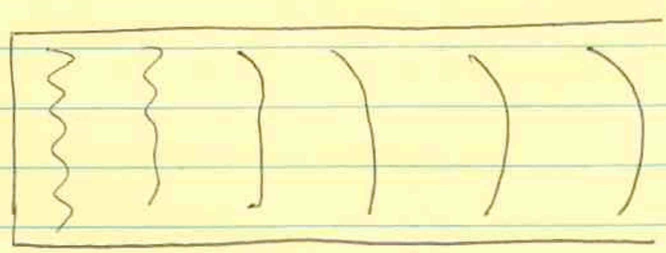
this is the solution for $V_0(y)$ = constant.

[As an aside, it turns out that this expression can be summed!

$$V(x,y) = \frac{2V_0}{\pi}\tan^{-1}\left(\frac{\sin \pi y/a}{\sinh \pi x/a}\right).$$
]

A few remarks:

① What does this solution look like? Large $n \Rightarrow$ short wavelength sine waves, which decay quickly in the x-direction. At large x, only $n=1$ survives

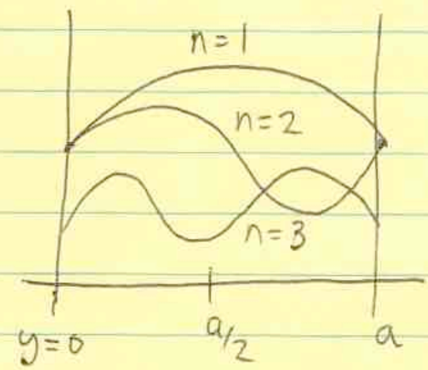


$\leftarrow V \simeq \frac{4V_0}{\pi}\sin\left(\frac{\pi y}{a}\right)e^{-\frac{\pi x}{a}}$

for $x \gg a$

The field decays much faster than for a point charge (exponential in x).

② why only odd n? Because $V_0(y) =$ constant is an even function about $y = a/2$



n=2 is an odd function and doesn't contribute to the sum.

Sometimes, this is a useful way to simplify the algebra.
eg. if V is an odd function about $x=0$, $\Rightarrow V \propto \sin kx$ not $\cos kx$.

If we'd done a different example

$$V_0(y) = \begin{cases} -V_0 & y < a/2 \\ +V_0 & y > a/2 \end{cases}$$

then only even n terms would count.

③ Because $\sinh(kx)$ and $\cosh(kx)$ are linear combinations of $e^{\pm kx}$, they are also solutions of $\frac{d^2X}{dx^2} = k^2 X$.

So an alternative set of solutions is

$$V(x,y) = \begin{Bmatrix} \sin ky \\ \cos ky \end{Bmatrix} \begin{Bmatrix} \sinh kx \\ \cosh kx \end{Bmatrix}$$

These functions are useful when there is a finite range in the x-direction. Again, "odd or even" considerations can help you choose $\sinh x$ or $\cosh x$.

(4) We've ignored $k^2=0$ solutions (linear functions of x or y). These are sometimes useful/necessary to include.

Separation of Variables in Spherical coordinates

In spherical coordinates, $\nabla^2 V = 0$ is

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial V}{\partial \theta}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial^2 V}{\partial \phi^2} = 0$$

Let's assume that V is independent of ϕ and look for a solution

$$V(r,\theta) = R(r)\,\Theta(\theta)$$

$$\Rightarrow \quad \frac{1}{R}\frac{\partial}{\partial r}\left(r^2 \frac{\partial R}{\partial r}\right) + \frac{1}{\Theta \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial \Theta}{\partial \theta}\right) = 0$$

This time, write the separation constant as $\ell(\ell+1)$.

$$\frac{1}{R}\frac{d}{dr}\left(r^2 \frac{dR}{dr}\right) = \ell(\ell+1) \qquad \text{— (1)}$$

$$\frac{1}{\Theta}\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta \frac{d\Theta}{d\theta}\right) = -\ell(\ell+1) \qquad \text{— (2)}$$

Look for a power law solution to (1)

$$R \propto r^n \quad \Rightarrow \quad \frac{1}{r^n}\frac{d}{dr}\left(n r^{n+1}\right) = \ell(\ell+1)$$

$$\frac{1}{\cancel{r^n}}\, n(n+1)\cancel{r^n} = \ell(\ell+1)$$

$$\Rightarrow \quad \underline{n = \ell} \quad \text{or} \quad \underline{n = -\ell-1}$$

$$\Rightarrow \quad \boxed{R(r) = A r^{\ell} + B r^{-\ell-1}}$$

Solutions to (2) are the Legendre polynomials in $\cos\theta$

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\,\frac{d\Theta}{d\theta}\right) = -\ell(\ell+1)\,\Theta$$

$$\Rightarrow \quad \boxed{\Theta(\theta) = P_\ell(\cos\theta)}$$

[if we write $\mu = \cos\theta$, then $\dfrac{d}{d\mu}\left((1-\mu^2)\dfrac{d\Theta}{d\mu}\right) = -\ell(\ell+1)\Theta$

$\Theta = P_\ell(\mu)$.]

The first few Legendre polynomials are

$$P_0(x) = 1$$
$$P_1(x) = x$$
$$P_2(x) = \frac{3x^2-1}{2}$$

The orthogonality relation is $\displaystyle\int_{-1}^{1} P_\ell(x)\,P_{\ell'}(x)\,dx$

$$= \int_0^\pi P_\ell(\cos\theta)\,P_{\ell'}(\cos\theta)\,\sin\theta\,d\theta$$

$$= \frac{2}{2\ell+1}\,\delta_{\ell\ell'}\,.$$

The general solution in spherical coordinates is therefore

$$\boxed{V(r,\theta) = \sum_{\ell=0}^{\infty}\left(A_\ell r^\ell + B_\ell r^{-\ell-1}\right) P_\ell(\cos\theta)}$$

You should see a hint of the multipole expansion in this general solution. If we look for solutions at large $r \to \infty$ we must throw away the r^l terms ($A_l = 0$)

$$\Rightarrow \quad V(r, \theta) = \sum_{l=0}^{\infty} B_l \frac{1}{r^{l+1}} P_l(\cos\theta)$$

l	r-dependence	θ-dependence	
0	~~none~~ $\frac{1}{r}$	none	monopole
1	$\frac{1}{r^2}$	$\cos\theta$	dipole
2	$\frac{1}{r^3}$	$\frac{3\cos^2\theta - 1}{2}$	quadrupole

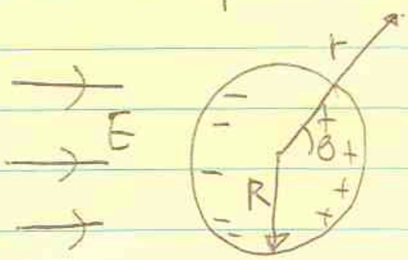
the radial and angular dependences that we found earlier emerge naturally from solving $\nabla^2 V = 0$.

(For an actual derivation of the multipole expansion, see eg. Jackson p137, beyond the scope of this course!)

Polarizability of a sphere

This is the classic example of applying the general solution to $\nabla^2 V = 0$ in spherical coordinates.

In a uniform electric field, a sphere of conducting material becomes polarized. The induced charge has a dipole potential.



To see this, note that we can write the uniform field as

$$V = -E r \cos\theta \qquad \left(\begin{array}{l} \text{Since } x = r\cos\theta \\ \phantom{\text{Since }} y = r\sin\theta \end{array} \right)$$

This is the boundary condition for $r \gg R$. The general solution to $\nabla^2 V = 0$ is

$$V(r,\theta) = \sum_{\ell=0}^{\infty} \left(A_\ell r^\ell + B_\ell r^{-\ell-1} \right) P_\ell(\cos\theta)$$

only $\ell = 1$ can match the boundary condition ($P_1(\cos\theta) = \cos\theta$)

As $r \to \infty$, $\quad V \to A_1 r \cos\theta \quad \Rightarrow \quad A_1 = -E$

$$\Rightarrow \quad V(r,\theta) = -E r\cos\theta + \frac{B_1}{r^2}\cos\theta .$$

But at $r = a$, we must have $V = 0$

$$\Rightarrow \quad -Ea + \frac{B_1}{a^2} = 0 \quad \Rightarrow \quad B_1 = Ea^3$$

$$\Rightarrow \quad \boxed{V(r,\theta) = -E r\cos\theta + \frac{Ea^3}{r^2}\cos\theta}$$

The second term is a dipole potential with $\underline{p} = 4\pi\varepsilon_0 a^3 \underline{E}$.

We write this as $\underline{p} = \alpha \underline{E}$ where α is the polarizability.

It is often given as $\dfrac{\alpha}{4\pi\varepsilon_0}$ which has units of volume. For a conducting sphere, $\dfrac{\alpha}{4\pi\varepsilon_0} = a^3$.

The surface charge density on the sphere is

$$\sigma = -\varepsilon_0 \frac{\partial V}{\partial r}\Big|_{r=a} = +\varepsilon_0 E \cos\theta + 2\varepsilon_0 E \cos\theta$$

$$= 3\varepsilon_0 E \cos\theta$$

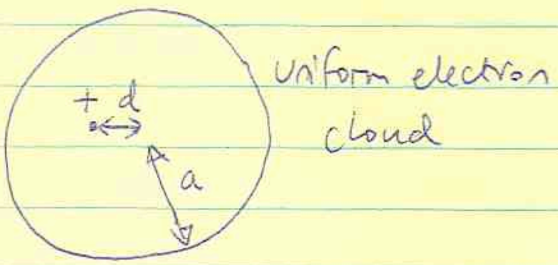
DIELECTRICS

We now turn to insulating materials, or dielectrics. Unlike conductors, in which electrons are free to move, the electrons in an insulator are bound to specific atoms or molecules. However, insulators become polarized by an applied electric field, either because atoms have intrinsic dipoles that rotate to align with the field, or because atoms gain an induced dipole.

① Induced dipoles

generally, we can write

$\underline{p} = \alpha \underline{E}$ e.g. simple model of an atom



$$E_e = \frac{\rho r}{3\varepsilon_0} \quad (r<a)$$

The displacement is such that $E_e(d) = E$ ← applied field

$$\Rightarrow \quad \frac{\rho d}{3\varepsilon_0} = \frac{Q d}{\varepsilon_0 4\pi a^3} = E$$

$$\Rightarrow \quad p = Qd = 4\pi\varepsilon_0 a^3 E$$

$$\Rightarrow \quad \boxed{\alpha = 4\pi\varepsilon_0 a^3} \quad \text{polarizability } \alpha$$

(Same as for the conducting sphere)

To calculate α for a real atom, we must use quantum mechanics and integrate over the electron wavefunction.

(term paper ??)

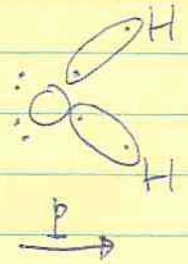
A typical order of magnitude is $\frac{\alpha}{4\pi\varepsilon_0} \sim a^3 \sim 10^{-30}\,m^3$

Some values:

(units of $10^{-30} m^3$)		
	H	0.667
	Li	24.3
(Z=55)	Cs	59.6 ← alkali metals loosely bound electrons
	He	0.205
(Z=54)	Xe	4.04 ← tightly bound electrons

② Polar molecules

eg. H_2O



HCl

$H^+ — Cl^-$

$\xleftarrow{\;p\;}$

units for dipole moment are debyes (D)

$$1D = 3.34 \times 10^{-30}\ Cm$$

Since a typical value might be (electron charge) x (atomic size)

$$\approx (1.6 \times 10^{-19} C)(10^{-10} m) = 1.6 \times 10^{-29}\ Cm,$$

this is an appropriate unit.

molecule	dipole moment (D)
HF	1.75
HCl	1.04
HBr	0.80
HI	0.83
H_2O	1.83
NH_3	1.48
haemoglobin	~100s

Polar molecules align with an external field (recall that the torque on a dipole is $\underline{p} \times \underline{E}$.)

The average alignment depends on temperature – specifically how the thermal energy per particle $k_B T$ compares with the potential energy $U = -\underline{p} \cdot \underline{E}$.

The probability of alignment is proportional to a Boltzmann factor

$$\exp\left(-\frac{U}{kT}\right) = \exp\left(+\frac{\underline{p} \cdot \underline{E}}{kT}\right)$$

$\Rightarrow$ mean dipole moment is

$$\langle \underline{p} \rangle = \frac{\int e^{-U/kT}\, \underline{p}\, d\Omega_p}{\int e^{-U/kT}\, d\Omega_p}$$

integrate over all orientations

p_0 = intrinsic dipole moment

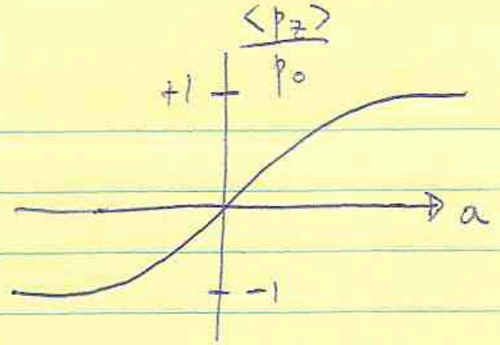
if θ is the angle from the z-axis,

$$\langle p_z \rangle = p_0 \frac{\int e^{-U/kT} \cos\theta \sin\theta\, d\theta}{\int e^{-U/kT} \sin\theta\, d\theta}$$

But $-\dfrac{U}{kT} = \dfrac{p_0 E \cos\theta}{kT} = ax$ where $a = \dfrac{p_0 E}{kT}$, $x = \cos\theta$

$$\Rightarrow \langle p_z \rangle = p_0 \frac{\int_{-1}^{1} e^{+ax}\, x\, dx}{\int_{-1}^{1} e^{ax}\, dx}$$

$$= p_0 \left[\coth a - \frac{1}{a}\right]$$ ← Langevin's formula



What are these energy scales?

room temperature

$$k_BT = 1.38 \times 10^{-23}\ \mathrm{J K^{-1}} \times 300\,\mathrm{K}$$
$$\simeq 5 \times 10^{-21}\ \mathrm{J}$$
$$= 0.03\ \mathrm{eV}$$

how big could $p_0 E$ be? Let's take E to be the breakdown electric field for air $= 10^6\ \mathrm{V/m}$.

$$\Rightarrow \quad p_0 E \simeq 10^{-30} \times 10^6\ \mathrm{V/m} \simeq 10^{-24}\ \mathrm{J}$$

$$\Rightarrow \quad a \text{ is } \simeq 3 \times 10^{-4} \ll 1.$$

So typically the dipole potential energy is much smaller than the thermal energy of the particles.

When a is small, $\coth a \simeq \frac{1}{a} + \frac{a}{3} + \frac{a^3}{45} + \cdots$

$$\Rightarrow \quad \langle p_z \rangle = \frac{p_0 a}{3}$$
$$= \frac{p_0^2 E}{3k_BT}$$
$$\Rightarrow \quad \boxed{\alpha = \frac{p_0^2}{3k_BT}} \quad \propto \frac{1}{T}$$

$\divideontimes$ If we combine ① and ② $\Rightarrow$ $\boxed{\alpha = \alpha_0 + \frac{p_0^2}{3k_BT}}$ Langevin-Debye formula

induced dipole (distortion of e^- wave function)

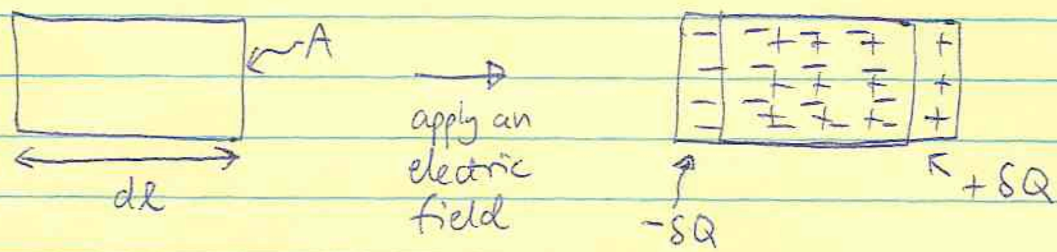
(alignment of polar molecules)

PHYS 340 lecture 25
Nov 4, 2005

Polarization and Bound charge

The polarization of the medium creates an electric field of its own, so that $\underline{E} = \underline{E}_{applied} + \underline{E}_P$. We now discuss how to write down $\underline{E}_P$.

Imagine a volume $dV = A\,dl$



When an electric field is applied, positive charges are displaced to the right, negative charges to the left.
At the surface, there is a net charge δQ.

Define the polarization $\underline{P}(\underline{r})$ as the mean dipole moment density.

Then $$\frac{\delta Q\,\delta l}{\delta l\,A} = P = \frac{\delta Q}{A}$$

the surface charge is $\boxed{\sigma_B = \underline{P}\cdot\underline{\hat{n}}}$

"B" means "bound charge"

(See Griffiths for a more mathematical derivation).

Integrate over the surface
$$Q_B = \int \sigma_B\, dA = \int \underline{P}\cdot d\underline{A}$$

This is the amount of charge that leaves the volume. Therefore, the volume acquires a net charge

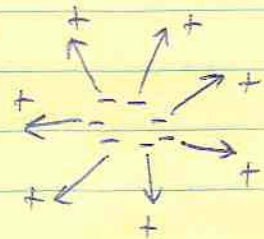
$$-Q_B = -\int \underline{P} \cdot d\underline{A}$$

or $$\int \rho_B \, dV = -\int \underline{\nabla} \cdot \underline{P} \, dV$$

since this holds for any volume $\Rightarrow$ $$\boxed{\rho_B = -\underline{\nabla} \cdot \underline{P}}$$

If $\underline{P}$ has a divergence, then there is bound charge within the dielectric.

eg.



The electric field due to the polarized charges

lecture 26
Nov 7, 2005

Gauss' law is $\underline{\nabla} \cdot \underline{E} = \rho / \varepsilon_0$. Now write ρ as the sum of two contributions $\rho = \rho_f + \rho_B$

"free" charges

$$\Rightarrow \quad \underline{\nabla} \cdot \underline{E} = \frac{\rho_f + \rho_B}{\varepsilon_0} = \frac{\rho_f}{\varepsilon_0} - \frac{\underline{\nabla} \cdot \underline{P}}{\varepsilon_0}$$

$$\Rightarrow \quad \boxed{\underline{\nabla} \cdot (\varepsilon_0 \underline{E} + \underline{P}) = \rho_f}$$

Here we define a new field $\underline{D}$ the "electric displacement"

$$\underline{D} = \varepsilon_0 \underline{E} + \underline{P} \quad \Rightarrow \quad \boxed{\underline{\nabla} \cdot \underline{D} = \rho_f}$$

The new field obeys Gauss' law, so we can use the symmetry of the problem to solve for $\underline{D}$ given the free charges ρ_f, without worrying about the bound charges.

However, given $\underline{D}$, we need an extra relation between $\underline{E}$ and $\underline{P}$ to determine them uniquely. This is known as a constitutive relation.

We will only consider <u>linear, isotropic, homogeneous dielectrics</u> (LIH dielectrics)

for which

$$\underline{P} = \chi_e \varepsilon_0 \underline{E} \qquad \chi_e \text{ is the susceptibility}$$

then $$\underline{D} = \varepsilon_0 \underline{E} + \underline{P} = \varepsilon_0 (1 + \chi_e) \underline{E}$$

$$= \varepsilon \underline{E} \qquad \varepsilon = \text{permittivity}$$

$$= \varepsilon_r \varepsilon_0 \underline{E} \qquad \varepsilon_r = \text{dielectric constant}$$

<u>Maxwell's equations for electrostatics in materials</u>

$$\underline{\nabla} \cdot \underline{D} = \rho_f \qquad \underline{\nabla} \times \underline{E} = 0$$

Some notes

(1) $\underline{\nabla} \times \underline{D}$ is not necessarily zero, since $\underline{\nabla} \times \underline{D} = \underline{\nabla} \times \underline{P}$, and it is possible to have $\underline{\nabla} \times \underline{P}$ non-zero. However in LIH dielectrics, $\underline{P} \propto \underline{E}$ and so $\underline{\nabla} \times \underline{P} = 0 \Rightarrow \underline{\nabla} \times \underline{D} = 0$. In this case, we can write $\underline{D}$ as the gradient of a scalar.

$$\underline{D} = \varepsilon \underline{E} \Rightarrow \underline{D} = -\varepsilon \underline{\nabla} V$$

$$\underline{\nabla} \cdot \underline{D} = \rho_f \Rightarrow \boxed{\nabla^2 V = -\rho_f / \varepsilon}$$

(2) How are ρ_f and ρ_B related for linear dielectrics?

$$\underline{P} = \chi \varepsilon_0 \underline{E} \Rightarrow \underline{\nabla} \cdot \underline{P} = \chi \varepsilon_0 \underline{\nabla} \cdot \underline{E}$$

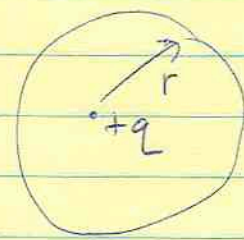
$$-\rho_B = \chi \rho_{total} = \chi(\rho_f + \rho_B)$$

$$\Rightarrow \boxed{\rho_B = \frac{-\rho_f \chi}{1+\chi}} = \left(\frac{1}{\varepsilon_r} - 1\right)\rho_f$$

So for a LIH dielectric, the bound charge has the same spatial profile as the free charge.

Examples

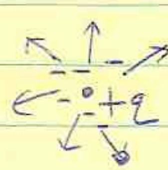
(1) Point charge in a dielectric medium



$$\underline{D} = \frac{q}{4\pi r^2}\hat{\underline{r}} \quad \text{from Gauss' law}$$

$$\Rightarrow \underline{E} = \frac{q}{4\pi \varepsilon r^2}\hat{\underline{r}}$$

The bound charge is a negative charge at the origin of strength $\left(\frac{1}{\varepsilon} - 1\right)q$



(2) Parallel plate capacitor

Gauss' law

$$\Rightarrow \underline{D} = \sigma(-\hat{\underline{z}})$$

$$\Rightarrow \boxed{E = \sigma/\varepsilon}$$

The potential difference between the plates is $V = Ed = \dfrac{\sigma d}{\varepsilon}$

$$\therefore \text{ Capacitance } C = \frac{\sigma A}{V} = \frac{\varepsilon A}{d} = \varepsilon_r \left(\frac{\varepsilon_0 A}{d} \right)$$

The capacitance is increased by the dielectric constant. Historically, this was how Faraday studied dielectrics, by measuring the increase in capacitance in the presence of insulating material.

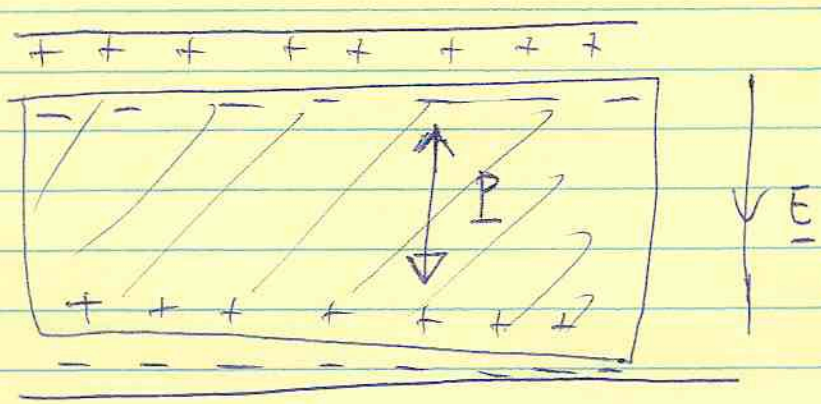
$$\boxed{C = \varepsilon_r C_{vacuum}}$$

What are some typical values?

	ε_r	E_{max} (10^6 V/m) dielectric strength
air	1.00059	3
polystyrene	2.5	20
water	80	—
strontium titanate	332	8

these properties important for capacitor design

The physics of the increase in capacitance is that there is a bound surface charge which screens the free charges on the plates.



Lecture 27
Nov 9, 2005

What is the energy stored?

Imagine charging up the capacitor. The voltage is $V = Q/C$ when the charge is Q, so

$$U_{cap} = \int_0^Q dQ\, V = \int_0^Q dQ\, \frac{Q}{C} = \frac{Q^2}{2C}$$

But the field energy is $Ad \times \frac{1}{2}\varepsilon_0 E^2$

$$= Ad \times \frac{1}{2}\varepsilon_0 \left(\frac{\sigma}{\varepsilon}\right)^2$$

$$= \frac{(\sigma A)^2}{2} \frac{d}{\varepsilon A} \frac{\varepsilon_0}{\varepsilon}$$

$$= \frac{1}{\varepsilon_r}\left(\frac{1}{2}\frac{Q^2}{C}\right)$$

The field energy is $\frac{1}{\varepsilon_r}$ of the energy stored. The missing piece is the work that must be done to establish the polarization.

In general, the energy stored is $\boxed{U = \int d^3\underline{r}\ \frac{1}{2}\underline{D}\cdot\underline{E}}$

for a linear dielectric (see Griffiths for derivation).

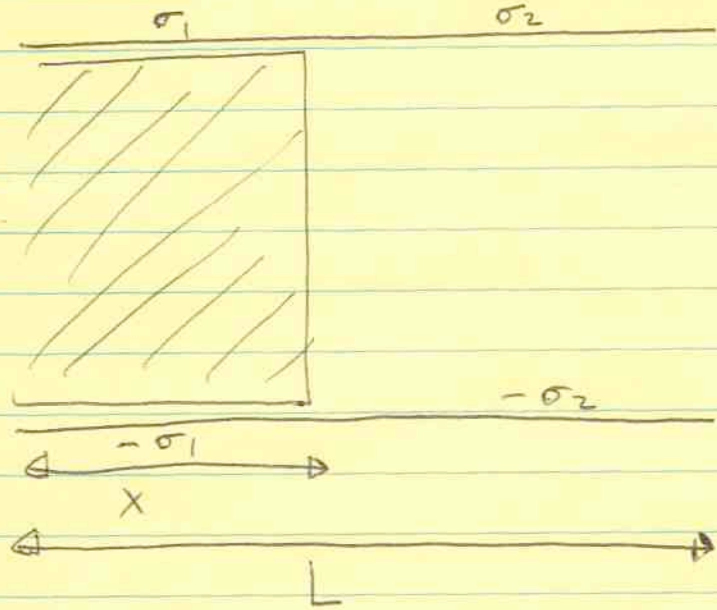
[For the example above $U_{cap} = (Ad) \times \frac{1}{2} DE$

$$= Ad \times \frac{1}{2}\frac{\sigma^2}{\varepsilon}$$

$$= \frac{d}{A\varepsilon} \times \frac{1}{2} \times (\sigma A)^2 = \frac{1}{2}\frac{Q^2}{C} \checkmark$$

Force on a dielectric

Consider a plane-parallel capacitor partially filled with dielectric.



To get the same voltage drop, the charge densities $\sigma_1 \neq \sigma_2$

$$\Delta V = Ed = \frac{\sigma_2}{\varepsilon_0} d = \frac{\sigma_1}{\varepsilon} d \Rightarrow \boxed{\sigma_1 = \varepsilon_r \sigma_2}$$

the charge density is higher in the dielectric region.

The capacitance is

$$C = \frac{Q}{\Delta V} = \frac{\varepsilon_0}{\sigma_2 d}\left(\sigma_1 x + \sigma_2 (L-x)\right) L$$

(assume square plates)

$$= \frac{\varepsilon_0 L}{\cancel{\sigma_2} d}\left(\varepsilon_r \cancel{\sigma_2} x + \cancel{\sigma_2}(L-x)\right)$$

$$\boxed{C = \frac{\varepsilon_0 L}{d}\left[x\chi + L\right]} \qquad (\chi = \varepsilon_r - 1)$$

There is a force on the dielectric because the dielectric would "like" to be all the way between the plates to minimize energy.

$$\text{force} = -\frac{dU}{dx} = -\frac{d}{dx}\left(\frac{1}{2}\frac{Q^2}{C}\right)$$

$$= \frac{1}{2}\frac{Q^2}{C^2}\frac{dC}{dx} = \underline{\underline{\frac{1}{2}V^2\frac{dC}{dx}}}$$

We have $\frac{dC}{dx} = \frac{\varepsilon_0 L x}{d}$

$\Rightarrow$ force is $\boxed{\frac{V^2 \varepsilon_0 L x}{2d}}$

If a battery is connected across the capacitor, the force is the same.

$$\frac{dU}{dx} = \frac{d}{dx}\left(\frac{1}{2}CV^2\right) - V\frac{dQ}{dx}$$

change in stored energy at constant V

work done by battery.

$Q = CV$ $\quad \frac{dQ}{dx} = V\frac{dC}{dx}$

$$\Rightarrow -\frac{dU}{dx} = \frac{1}{2}V^2\frac{dC}{dx} \checkmark$$

Lecture 28
Nov 11, 2005

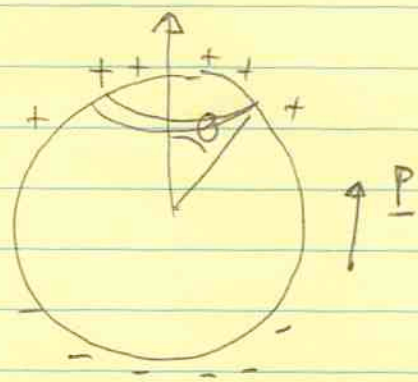
Clausius-Mossotti Formula

We discussed the fact that a dipole moment is induced in an atom by an applied magnetic field, $\underline{p} = \alpha \underline{E}^*$, where $\alpha = \alpha_0 + \frac{p_0^2}{3kT}$. However, when we calculated the effect of the polarization on the total electric field, we used the constitutive relation $\underline{P} = \chi \varepsilon_0 \underline{E}$, where $\underline{E}$ is the total electric field. What is the relation between α and χ?

The idea is to consider a spherical region inside a dielectric, small enough that the polarization is uniform, P (but large enough to contain many dipoles).

$\Rightarrow$ The surface charge is $\sigma_B = \underline{P} \cdot \underline{\hat{n}} = P \cos\theta$

The field due to the surface charge is



$$\underline{E} = -\int \frac{\sigma_B \cos\theta \, dA}{4\pi\varepsilon_0 a^2}$$

$$= -\int_0^{\pi} \frac{P \cos^2\theta \sin\theta \, d\theta \, 2\cancel{\pi}\cancel{a^2}}{2\,\cancel{4\pi}\varepsilon_0 \cancel{a^2}}$$

$$= \frac{\cancel{4}-P}{2\varepsilon_0} \int_{-1}^{1} dx \, x^2 \qquad (x = \cos\theta)$$

$$= -\frac{\underline{P}}{3\varepsilon_0}$$ ← this is the field at the center of the sphere.

$\Rightarrow$ The total field at the center is $\underline{E} = \underline{E}^* - \frac{\underline{P}}{3\varepsilon_0}$

But $\underline{P} = n\alpha \underline{E}^* = n\alpha \left[\underline{E} + \frac{\underline{P}}{3\varepsilon_0} \right]$

$$\Rightarrow \quad \underline{P}\left(1 - \frac{n\alpha}{3\varepsilon_0}\right) = n\alpha \underline{E}$$

but $\quad \underline{P} = \chi\varepsilon_0\underline{E}$

$$\Rightarrow \quad \varepsilon_0 \chi = \frac{n\alpha}{1 - \frac{n\alpha}{3\varepsilon_0}} \quad \text{or} \quad \boxed{\chi = \frac{\frac{n\alpha}{\varepsilon_0}}{1 - \frac{n\alpha}{3\varepsilon_0}}}$$

$$\Rightarrow \quad n\alpha\left(1 + \frac{\chi}{3\varepsilon_0}\right) = \chi\varepsilon_0$$

$$\Rightarrow \quad \alpha = \frac{3\varepsilon_0}{n}\left(\frac{\chi}{3+\chi}\right)$$

$$\boxed{\alpha = \frac{3\varepsilon_0}{n}\left(\frac{\varepsilon_r - 1}{\varepsilon_r + 2}\right)}$$

Clausius-Mossotti relation

links the microphysics (α) to the macrophysics (χ).

eg. water expect $\alpha \simeq \frac{p_0^2}{3kT}$ $\quad p_0 = 1.83\,D = 6.1 \times 10^{-30}\ \text{Cm}$

$$\Rightarrow \alpha = 3 \times 10^{-39} \text{ at } T = 300\text{K}$$

$\Rightarrow$ density of water is $\rho = 10^3\ \text{kg m}^{-3}$

$$\Rightarrow n \simeq \frac{\rho}{18 m_p} = 3.35 \times 10^{28}\ \text{m}^{-3}$$

$$\boxed{\frac{n\alpha}{\varepsilon_0} = 11.3}$$

$$\cancel{\Rightarrow \alpha = \frac{11/3}{1 - \frac{11.3}{3}}}$$

$\chi = 80$ so expect $\frac{n\alpha}{\varepsilon_0} = 3\left(\frac{80}{83}\right) = 2.89$

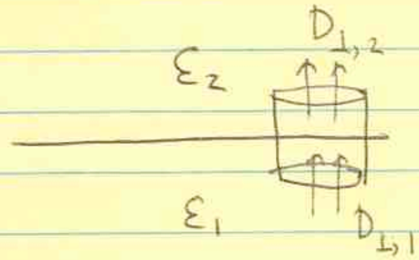
If we define an interparticle separation by $\frac{4\pi}{3} r_0^3 n = 1$

then $$\boxed{\frac{\alpha}{4\pi\varepsilon_0} = r_0^3 \left(\frac{\varepsilon_r - 1}{\varepsilon_r + 2}\right)}.$$

Boundary value problems

What are the boundary conditions at the interface between two dielectrics ?

1)

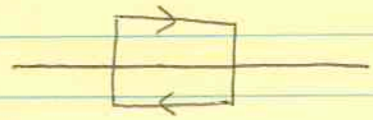


Gauss' Law

$$\Rightarrow \quad D_{\perp,2} - D_{\perp,1} = \sigma_f$$

$$[D_\perp] = \sigma_f$$

2)



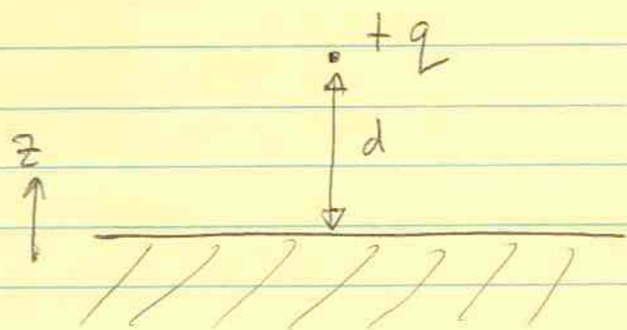
Since $\underline{\nabla} \times \underline{E} = 0$

$$[E_\parallel] = 0$$

$E_\parallel$ is continuous across the boundary.

3) The potential must be continuous across the boundary (discontinuous potential $\Rightarrow$ E or D diverges)

Point charge above a planar dielectric



We can solve this with the method of images, but we need to consider $z > 0$ and $z < 0$ seperately.

Above the plane ($z>0$)

$$V_{above} = \frac{1}{4\pi\varepsilon_0}\left[\frac{q}{\sqrt{x^2+y^2+(z-d)^2}} + \frac{q'}{\sqrt{x^2+y^2+(z+d)^2}}\right]$$

below the plane ($z<0$)

$$V_{below} = \frac{1}{4\pi\varepsilon_0}\frac{q''}{\sqrt{x^2+y^2+(z-d)^2}}$$

Boundary conditions:

$$V_{above} = V_{below} \text{ at } z=0 \Rightarrow \boxed{q+q'=q''}$$

$$[D_n]=0 \Rightarrow \left[\varepsilon\frac{\partial V}{\partial z}\right]=0$$

$$\frac{\partial V_{above}}{\partial z} = \frac{1}{4\pi\varepsilon_0}\left[\frac{-(z-d)q}{(x^2+y^2+(z-d)^2)^{3/2}} - \frac{(z+d)q'}{(x^2+y^2+(z+d)^2)^{3/2}}\right]$$

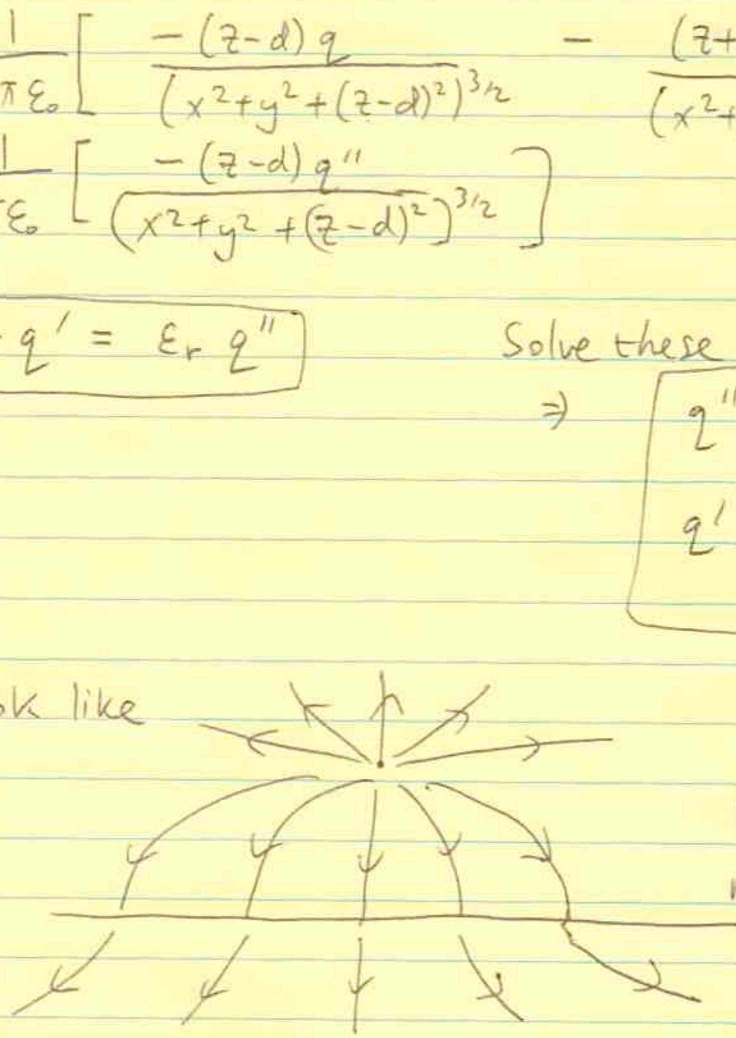
$$\frac{\partial V_{below}}{\partial z} = \frac{1}{4\pi\varepsilon_0}\left[\frac{-(z-d)q''}{(x^2+y^2+(z-d)^2)^{3/2}}\right]$$

$$\Rightarrow \boxed{q-q'=\varepsilon_r q''}$$

Solve these

$$\Rightarrow \boxed{\begin{aligned} q'' &= \frac{2}{1+\varepsilon_r}q \\ q' &= -\frac{\varepsilon_r-1}{\varepsilon_r+1}q \end{aligned}}$$

The field lines look like



$E_\perp$ drops on moving into the dielectric.

What is the bound surface charge?

$$\sigma_B = \underline{P} \cdot \underline{\hat{n}} = \chi \varepsilon_0 (E_{inside})_z$$

$$= -\frac{\chi d}{4\pi} \frac{q''}{(x^2+y^2+d^2)^{3/2}}$$

$$\boxed{\sigma_B = -\frac{qd}{2\pi} \frac{1}{(x^2+y^2+d^2)^{3/2}} \frac{\chi}{2+\chi}}$$

[A useful check is to take the $\chi \to \infty$ limit which gives the limit of a conductor

$$q' \to -q$$
$$q'' \to 0$$
$$\sigma_B \to -\frac{qd}{2\pi} \frac{1}{(x^2+y^2+d^2)^{3/2}}$$

This is because $\quad \underline{E} = \frac{\underline{D}}{\varepsilon} \to 0$ for $\varepsilon \to \infty$

or $\quad \underline{E} = \frac{\underline{P}}{\chi \varepsilon_0} \to 0$ for $\chi \to \infty$.]

Magnetic Materials

lecture 29
Nov 12th

Start with microphysics...

Dielectric materials develop a polarization when subjected to an applied electric field. ~~An elect~~ A magnetic field causes magnetization either by inducing magnetic dipoles or aligning intrinsic dipoles. However, the magnetization may be opposite to the applied field. In addition, quantum mechanics is key to understanding the magnetic response of materials.

Dipole moments can arise from the orbital motion of electrons, or from the intrinsic electron spin.

Consider an orbiting electron

$$L = m_e r^2 \omega = -m_e r^2 \frac{I}{e} 2\pi = -\left(\frac{2m_e}{e}\right) m$$

$$\Rightarrow \boxed{\underline{m} = -\frac{e}{2m_e} \underline{L}}$$

(we write $e = +1.6 \times 10^{-19}$ C giving us the minus sign)

angular momentum is quantized ~~in the at~~ in units of $\hbar$

$$\hbar = \frac{h}{2\pi} = 1.05 \times 10^{-34} \text{ Js}$$

$$\Rightarrow \quad L = n\hbar$$

$$|\underline{m}| = -n \frac{e\hbar}{2m_e} = -n\mu_B$$

where $\boxed{\mu_B = \frac{e\hbar}{2m_e} = 9.27 \times 10^{-24} \text{ Am}^2}$

is the Bohr magneton.

This is the size of the intrinsic electron dipole moment coming from its spin.

These are the main contributions to the magnetic properties of materials. The proton and neutron dipole moments are much smaller ($\approx e\hbar/2m_p$)

(1.41×10^{-26} Am^2 for the proton
-0.966×10^{-26} Am^2 for the neutron)

The three types of material are

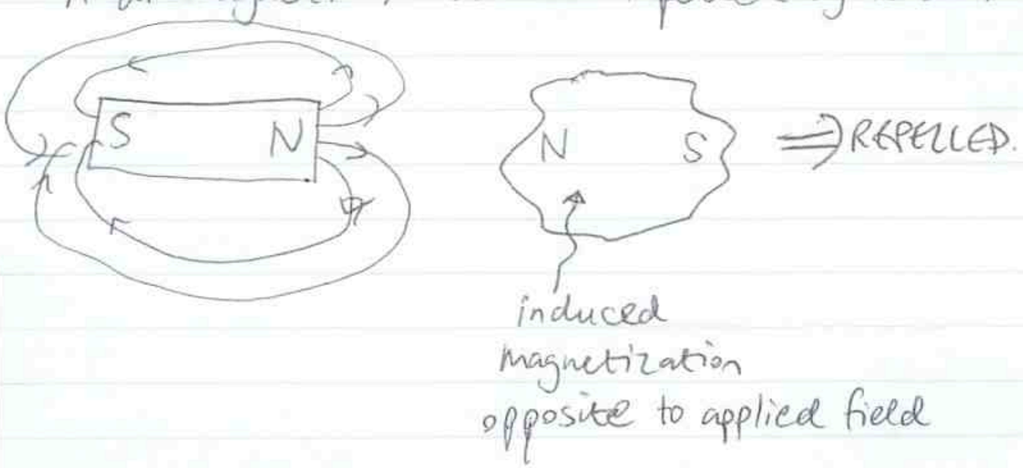
① diamagnetic — $\underline{M}$ opposite to $\underline{B}$, and weak

② paramagnetic — $\underline{M}$ parallel to $\underline{B}$, and weak

③ ferromagnetic — $\underline{M}$ " " " strong.

Diamagnetism

Most everyday materials are diamagnetic. Materials that have complete electron shells (no net spin or orbital angular momentum) have diamagnetic behaviour.
Applying a magnetic field $\underline{B}$ induces a magnetization opposite in direction but proportional to B, that disappears when B is removed.
Diamagnetism is independent of temperature.
A diamagnetic material is repelled by a bar magnet.

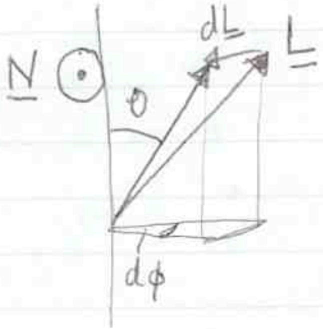


A classical model of diamagnetism:

~~A dipole created by~~

A dipole proportional to angular momentum PRECESSES in a magnetic field.

A reminder about precession:

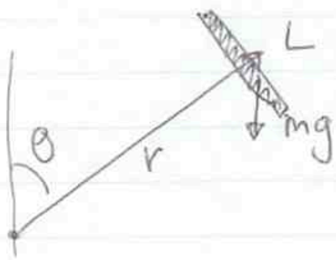


$$\frac{d\underline{L}}{dt} = \underline{N}$$

$$|d\underline{L}| = |\underline{N}|\, dt = L \sin\theta \, d\phi$$

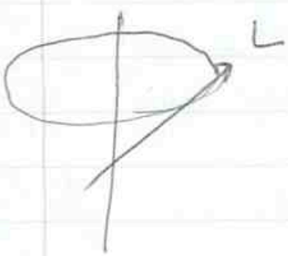
$$\Rightarrow \boxed{\frac{d\phi}{dt} = \frac{N}{L\sin\theta}}$$

eg. a spinning top



$$\text{torque} = mgr\sin\theta$$

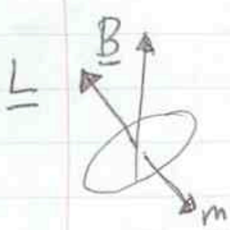
$$\Rightarrow \text{precession frequency} = \frac{mgr}{L} = \frac{mgr}{I\omega}$$



eg. $I = \frac{1}{2} M a^2$

$$\Rightarrow \omega_p = \frac{2gr}{a^2\omega}$$

We have a magnetic dipole in an applied magnetic field



$$\text{torque} = \underline{m} \times \underline{B} = -mB\sin\theta$$

$$\Rightarrow \omega_p = \frac{-mB}{L}$$

or $\boxed{\omega_p = \dfrac{eB}{2m_e}}$ This frequency is known as the Larmor frequency ω_L.

The precession gives rise to a magnetic moment opposite to $\underline{B}$.

$\underline{m} \times \underline{B}$ is out of page

precession is $\propto \hat{\underline{z}} \Rightarrow$ induced dipole is $\propto -\hat{\underline{z}}$ opposite to $\underline{B}$.

$\underline{m} \times \underline{B}$ is into page

induced dipole is again down.

(the precession is in the same direction in each case – extra angular momentum is the same)

So electrons with opposite angular momenta have an induced dipole in the same direction – opposite to B.

$$m_{induced} \simeq \frac{e}{2m_e} L \simeq \frac{e}{2m_e} m_e \langle r^2 \rangle \omega_L$$

$$\boxed{\underline{m} \simeq -\frac{e^2 \langle r^2 \rangle \underline{B}}{6m_e}}$$

inserted correct prefactor

Paramagnetism

Atoms or molecules with permanent dipole moments m_0. Due to an unpaired electron spin or orbital angular momentum. Typically $m_0 \sim \mu_B$.

As for dielectric materials with polarized molecules, there is a competition between the applied field trying to line up the dipoles and the thermal motions.

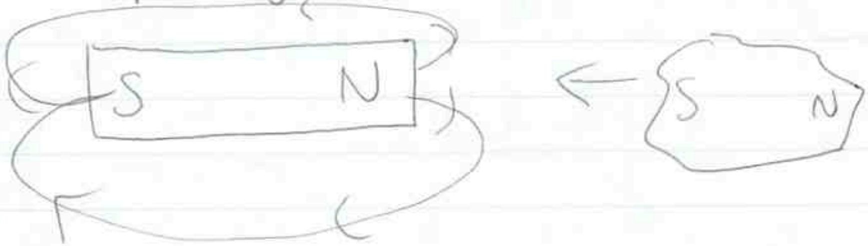
For $\frac{mB}{kT} \ll 1$ (usually the case)

$$\boxed{\langle m \rangle \approx \frac{m_0^2 B}{3kT}}$$

analogous to the dielectric result.

which varies as $\frac{1}{T}$

A paramagnetic material is ATTRACTED to a bar magnet.

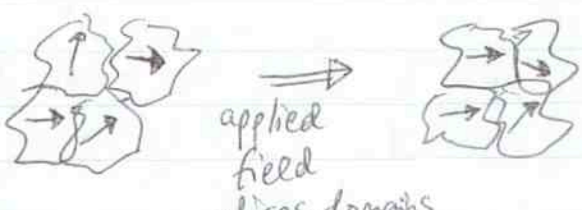


Ferromagnetism

This is the magnetic phenomenon most familiar from everyday experience.

Nearby dipoles interact to give magnetic domains. An external field is not necessary to maintain the magnetization. The response to an applied $\underline{B}$ field is <u>non-linear</u>.

We'll discuss ferromagnetic materials more later..



typical domain size is $10^{-10} - 10^{-12}\, m^3$

$10^{17} - 10^{20}$ atoms

Only 3 elements are ferromagnetic at RTP – Fe, Co, Ni

Lecture 30,
Nov 16, 2005

The field of a magnetized object

Last time, we talked about the microphysics of magnetic materials. An orbiting electron has a magnetic moment $\underline{m} = -\frac{e}{2m_e}\underline{L}$, typically of size $\mu_B = \frac{e\hbar}{2m_e}$ (Bohr magneton).

A dipole proportional to angular momentum precesses in an applied field at the Larmor frequency $\omega_L = eB/2m_e$.

For atoms with filled shells, the response is an induced dipole opposite to the induced field – DIAMAGNETISM.

$$\underline{m}_{dia} \simeq -\frac{e^2\langle r^2\rangle B}{6m_e} \quad \text{— ①}$$

Atoms with unpaired electrons have a permanent dipole moment. There is a tendency to align with the field – PARAMAGNETISM.

$$\underline{m}_{para} \simeq \frac{m_0^2\,\underline{B}}{3kT}. \quad \text{— ②}$$

A couple of notes:

(i) ② is true for $m_0 B \ll kT$. If $m_0 \simeq \mu_B$ then

$$m_0 B \simeq \mu_B B = 6\times10^{-5}\,\text{eV}\left(\frac{B}{1T}\right)$$

but $kT \simeq 2.5\times10^{-2}\,\text{eV}\left(\frac{T}{300K}\right)$

$$\Rightarrow \frac{m_0 B}{3kT} \lesssim 0.2\%$$

(ii) a nice way to rewrite ① is $m_{dia} \simeq \mu_B^2 B\left(\frac{e^2\langle r^2\rangle}{6m_e\mu_B^2}\right)$

The factor in brackets is $\frac{e^2\langle r^2\rangle}{6m_e}\frac{4m_e^2}{e^2\hbar^2} = \frac{2}{3}\frac{m_e\langle r^2\rangle}{\hbar^2}$

But for a hydrogen atom, the Bohr radius is $\boxed{r = \frac{1}{\alpha}\frac{\hbar}{m_e c}}$

$$\Rightarrow \quad \frac{2}{3}\frac{m_e\langle r^2\rangle}{\hbar^2} = \frac{2}{3}\frac{1}{\alpha^2}\frac{1}{m_e c^2}$$

$$= \frac{1}{3}\frac{1}{E_H}$$

$$\boxed{E_H = \alpha^2 m_e c^2/2 = 13.6\,\text{eV}.}$$

where $\alpha = \frac{e^2}{4\pi\varepsilon_0 \hbar c} = \frac{1}{137}$ is the fine structure constant.

$$\Rightarrow \quad \boxed{m_{dia} \simeq \mu_B \frac{\mu_B B}{3E_H}}$$

$\Rightarrow$ that paramagnetism (if present) is always larger than diamagnetism by a factor of $\frac{E_H}{kT} = \frac{13.6\text{eV}}{0.025\text{eV}} \simeq 500$

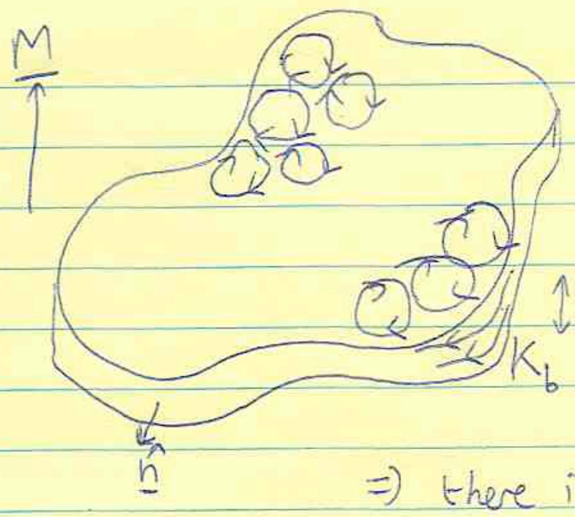
Magnetization and Bound Currents

Now think about the macroscopic effect of the magnetization.

We define $\underline{M}$ = dipole moment density

units $\frac{Am^2}{m^3}$ or $\frac{A}{m}$

Associated with $\underline{M}$ are bound currents. We'll use a simple heuristic argument to understand this.
Represent the individual dipoles as current loops.



Only currents at the surface contribute if $\underline{M}$ = constant.

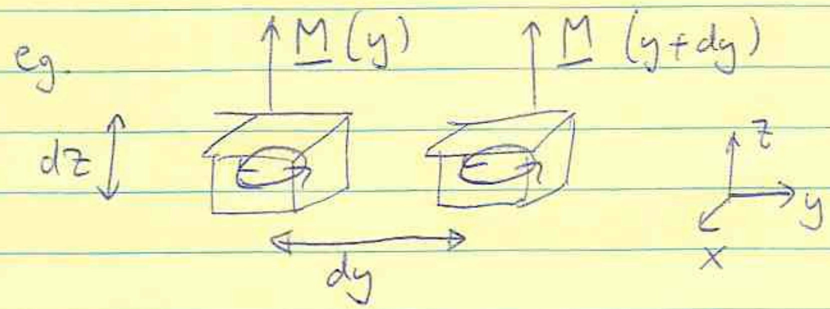
$$M = \frac{m}{ad} = \frac{Ia}{ad} = \frac{I}{d}$$

$\Rightarrow$ there is a surface current $\boxed{K_b = M}$

the direction is given by $\boxed{\underline{K}_b = \underline{M} \times \hat{\underline{n}}}$

If $\underline{M}$ is non-uniform, there are volume bound currents also.

eg.

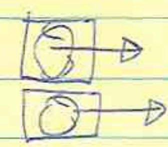


net current $I_x = \left[M_z(y+dy) - M_z(y) \right] dz$

$$= \frac{\partial M}{\partial y} \, dy \, dz = J_x \, dy \, dz$$

$$\Rightarrow J_x = \frac{\partial M_z}{\partial y}$$

Similarly, a non-uniform M in the y-direction would give $J_x = -\frac{\partial M_y}{\partial z}$



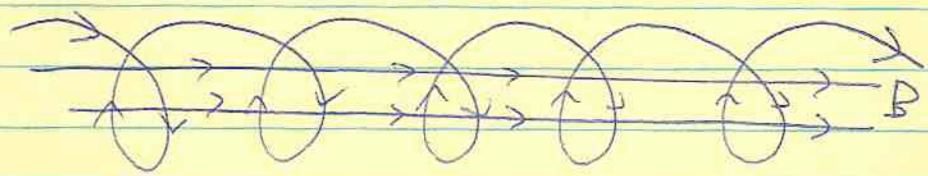
[note that $\underline{\nabla} \cdot \underline{J}_b = 0$]

$$\Rightarrow J_x = \frac{\partial M_z}{\partial y} - \frac{\partial M_y}{\partial z}$$

generalize $\Rightarrow$ $\boxed{\underline{J}_b = \underline{\nabla} \times \underline{M}}$ bound current density.

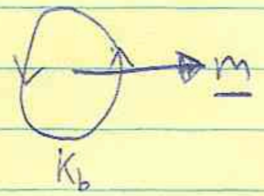
What is the effect of the bound currents?

Consider a solenoid filled with paramagnetic material.



The magnetization is in the direction of $\underline{B}$. Since $\underline{B}$ is uniform inside the solenoid, $\underline{M}$ is uniform $\Rightarrow$ no volume bound currents.

$J_b = 0$



The bound currents must flow in the same direction as the applied current

$\Rightarrow$ the effective current is larger

$\Rightarrow$ B is larger inside the solenoid

(opposite effect to a dielectric – $\underline{E}$ is decreased by putting a dielectric inside a capacitor).

How do we see this mathematically?

Write $\underline{J} = \underline{J}_f + \underline{J}_b$

free currents, bound currents

Ampère's law is $\underline{\nabla} \times \underline{B} = \mu_0 \underline{J} = \mu_0 (\underline{J}_f + \underline{J}_b)$

but $\underline{J}_b = \underline{\nabla} \times \underline{M}$

$$\Rightarrow \quad \underline{\nabla} \times \left(\frac{\underline{B}}{\mu_0} - \underline{M} \right) = \underline{J}_f$$

or $\boxed{\underline{\nabla} \times \underline{H} = \underline{J}_f}$ Ampere's law in a magnetic material

The field $\underline{H}$ is $\underline{H} = \frac{\underline{B}}{\mu_0} - \underline{M}$, or $\underline{B} = \mu_0(\underline{H} + \underline{M})$.

For a linear magnetic material, we can write

$\underline{M} = \chi_m \underline{H}$ where χ_m = magnetic susceptibility.

$$\underline{B} = \mu_0 (1 + \chi_m) \underline{H} = \mu \underline{H}$$

μ = permeability.

For the solenoid, the new Ampère's law $\Rightarrow$ $\boxed{H = nI}$

$\left(\oint \underline{H} \cdot d\underline{l} = I_{f,enc} \right)$

(previously we had $B = \mu_0 n I$)

$\Rightarrow$ $\boxed{B = \mu n I}$

larger than the vacuum case by an amount $\frac{\mu}{\mu_0} = 1 + \chi_m$.

For dia- and paramagnetic materials, χ_m is small.

eg.

material	χ_m	
bismuth	-1.6×10^{-4}	diamagnetic
H_2O	-9×10^{-6}	
He	-1.1×10^{-9}	
O_2	$+1.9 \times 10^{-6}$	paramagnetic
W (tungsten)	$+7.0 \times 10^{-5}$	
liquid O_2 (−200°C)	$+3.9 \times 10^{-3}$	

For ferromagnetic materials, B is a non-linear function of H, but the effective μ is $\simeq 1000 - 10^5$. More on this next time.

Lecture 31,
Nov 18, 2005

Last time we showed that the bound currents associated with magnetization $\underline{M}$ are

$$\boxed{\begin{aligned}\underline{K}_b &= \underline{M} \times \hat{\underline{n}} \\ \underline{J}_b &= \underline{\nabla} \times \underline{M}\end{aligned}}$$

Ampère's law $\quad \underline{\nabla} \times \underline{B} = \mu_0 (\underline{J}_f + \underline{J}_b)$

can be written $\quad \boxed{\underline{\nabla} \times \underline{H} = \underline{J}_f} \quad$ — (*)

where $\quad \underline{B} = \mu_0 (\underline{H} + \underline{M})$ defines a new field $\underline{H}$.

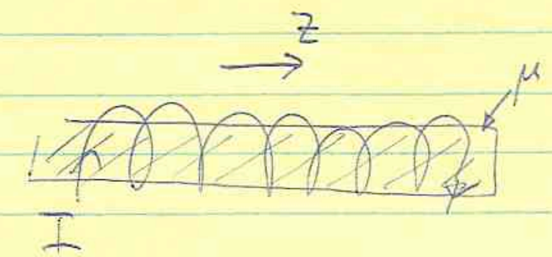
For a linear magnetic material, $\quad \underline{M} = \chi_m \underline{H}$
or $\quad \underline{B} = \mu \underline{H}$

where χ_m is the susceptibility
and $\mu = (1 + \chi_m)\mu_0$ is the permeability.

The procedure for finding $\underline{B}$ in~~side~~ a ~~magnetic~~ problem involving magnetic materials is to use Ampère's law to find $\underline{H}$ (if symmetry allows). Once $\underline{H}$ is determined, $\underline{B} = \mu \underline{H}$ and $\underline{M} = \chi_m \underline{H} \rightarrow$ bound currents.

Examples

1) $\boxed{\text{solenoid}}$



Ampère's law
$\Rightarrow \quad H = nI$

$$\Rightarrow \quad \boxed{B = \mu n I}$$

The magnetization is $\quad \underline{M} = \chi_m n I \,\hat{\underline{z}}$

There is a surface bound current

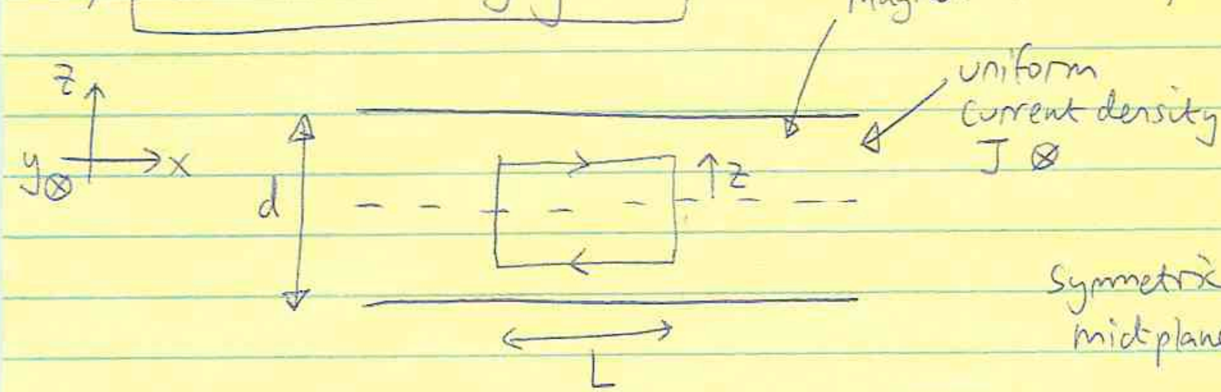
$$\underline{K}_b = \chi_m n I \ \hat{\underline{z}} \times \hat{\underline{r}} = \chi_m n I \ \hat{\underline{\phi}}$$

the applied current is $\underline{K}_f = nI\hat{\underline{\phi}}$

$\Rightarrow$ the total current is $nI(1+\chi_m) = \mu_r n I$ *

which gives $B = \mu n I$ ✓

2) $\boxed{\text{A current-carrying slab}}$



Symmetric about the midplane

Ampère's law $\Rightarrow$ $2LH = 2LzJ$

$$\boxed{\underline{H} = zJ\hat{\underline{x}}} \qquad |z| < d/2 \quad \text{inside}$$

$$2LH = 2L\frac{d}{2}J$$

$$\boxed{\underline{H} = \frac{Jd}{2}\hat{\underline{x}}} \qquad |z| > d/2 \quad \text{outside}$$

The B field outside is $\underline{B} = \mu_0 \dfrac{Jd}{2}\hat{\underline{x}}$

inside $\underline{B} = \mu J z \hat{\underline{x}}$

Using $\underline{H}$ means that we don't have to worry about the bound currents. But what do they look like?

$$\underline{M} = \chi_m \underline{H} = \chi_m z J \hat{\underline{x}}$$

well $\underline{K}_b = \underline{M} \times \hat{\underline{n}} = \pm \chi_m \frac{d}{2} J (\hat{\underline{x}} \times \hat{\underline{z}})$ Same for top and bottom surfaces

$$\boxed{\underline{K}_b = \mp \frac{\chi_m d J}{2} \hat{\underline{y}}}$$

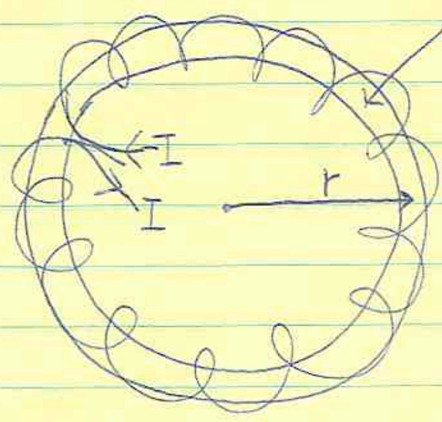
~~(+ on top, − on bottom)~~

$$\underline{J}_b = \underline{\nabla} \times \underline{M} = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ M_x & 0 & 0 \end{vmatrix} = \hat{\underline{y}} \, J \chi_m$$

$$\Rightarrow \quad \boxed{\underline{J}_b = \chi_m \underline{J}}$$

The net bound current that flows is zero, so the field outside is unchanged.
The field inside is larger by the ratio $\frac{\mu}{\mu_0}$ because the total current is $\underline{J} = \underline{J}_f + \underline{J}_b = \underline{J}_f (1 + \chi_m) = \mu_r \underline{J}_f$ ✓

3) $\boxed{\text{A toroidal magnet}}$



torus of magnetic material

By symmetry, $\underline{B} = B \hat{\underline{\phi}}$

integrate around a loop within the magnetic material

$$2\pi r H = NI$$

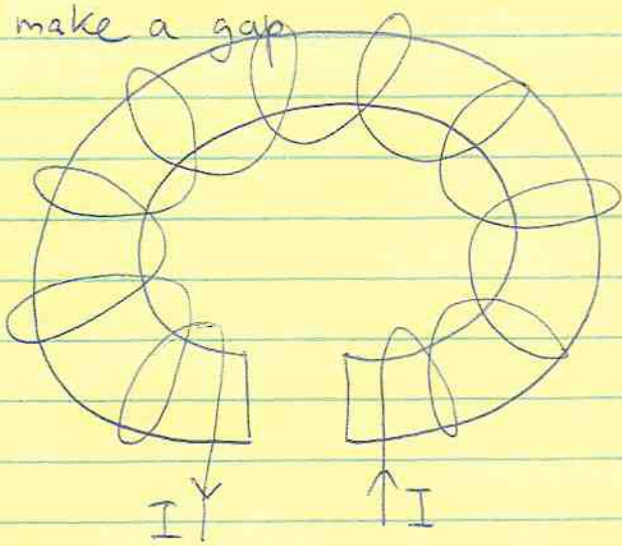
N: number of turns

$$\Rightarrow \quad H = \frac{NI}{2\pi r}$$

$$\boxed{B = \frac{\mu N I}{2\pi r}} \quad \text{inside}$$

$B = 0$ outside because any loop contains zero net current.

Now make a gap



What is the field in the gap?

$$\oint \underline{H} \cdot d\underline{l} = I_{f,enc}$$

$$\Rightarrow \boxed{L H_i + x H_g = NI} \quad (1)$$

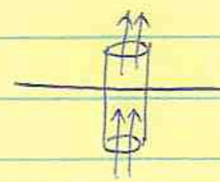
(i = iron g = gap)

where $L = 2\pi r - x$

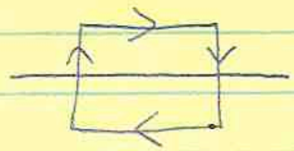
How do we relate H_i and H_g?

We need boundary conditions. Use the same argument as electrostatics to derive them.

$\underline{\nabla} \cdot \underline{B} = 0 \quad \Rightarrow \quad B_\perp$ is continuous



$\underline{\nabla} \times \underline{H} = \underline{J}_f$



$$\Rightarrow \quad [H_{\parallel}] = K_f \quad ?$$

in fact, need to be careful about directions

$$\boxed{[\underline{H}_{\parallel}] = \underline{K}_f \times \underline{\hat{n}}}$$

if there are no surface currents $\underline{K}_f = 0$ then $[H_{\parallel}] = 0$

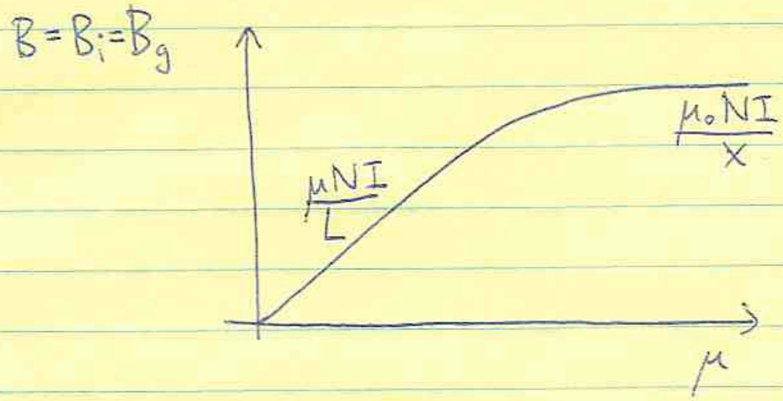
($H_{\parallel}$ is continuous)

For our problem, $[B_\perp] = 0 \Rightarrow B_i = B_g$

or $\boxed{\mu H_i = \mu_0 H_g}$ — ②

Solve ① and ② $\Rightarrow$ $\boxed{B_g = \dfrac{\mu_0 \mu N I}{L\mu_0 + x\mu}}$

if $\dfrac{x}{L} > \dfrac{\mu_0}{\mu}$ then $\underline{\underline{B_g = \dfrac{\mu_0 N I}{x}}}$ as if all the magnetic flux was in the gap!



PHYS 340 lecture 32
Nov 21, 2005

Ferromagnetic materials

These materials give rise to the familiar everyday magnetism.

Quantum-mechanical "exchange forces" between neighbouring dipoles cause them to line up.

[This is beyond the scope of this course, but the basic idea is that the electrostatic repulsion between neighbouring electrons means that the spatial wavefunction is antisymmetric

$$\psi = \psi_1(r_1)\,\psi_2(r_2) - \psi_2(r_1)\,\psi_1(r_2)$$

But Fermi statistics tell us that the overall wavefunction must be antisymmetric $\Rightarrow$ the spin part must be symmetric

$\Rightarrow$ triplet spin state

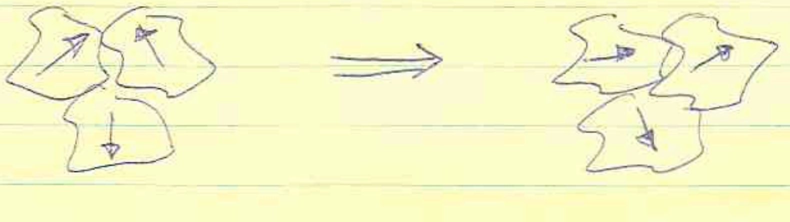
$\uparrow\uparrow$ or $\downarrow\downarrow$

or $(\uparrow\downarrow + \downarrow\uparrow)/2$.]

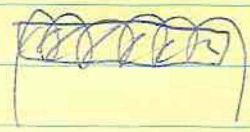
The result is magnetic domains whose size is set by the spin alignment (small scale) competing with the preference to reduce the overall dipole moment (large scale). The domain size is typically ~~microns~~ ($10^{17} - 10^{20}$ atoms).
$10^{-10} - 10^{-12}\,m^3$

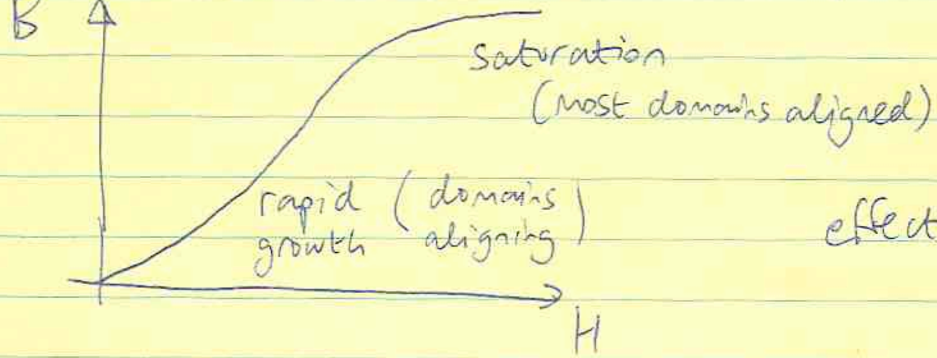
Only three elements are ferromagnetic at RTP — Fe, Co, Ni

Neighbouring domains align in an applied field: either by growth of favourably-aligned domains or realignment of existing domains.



The response is non-linear and dissipative / irreversible.

A typical magnetization curve: eg.  $H = nI$



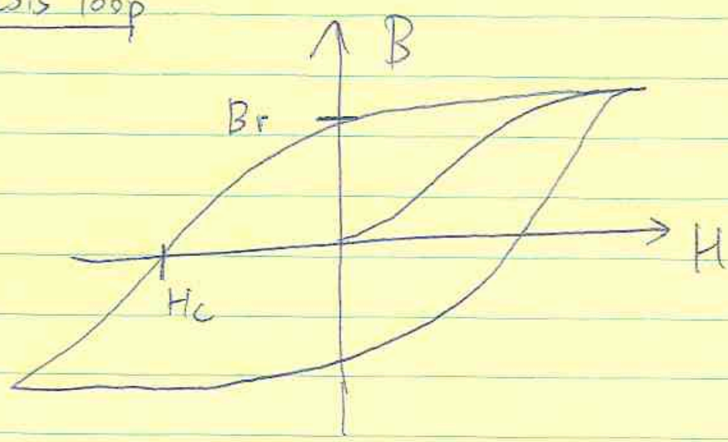
effective $\mu >> 1$

Each material has a "Curie-temperature" above which it is no longer ferromagnetic but paramagnetic.

eg.

Fe	$T_c = 1043$ K
Co	1388
Ni	627
Gd	292
Dy	88

When the applied field is removed and reversed, we get a <u>hysteresis loop</u>



B_r = <u>remnance</u> = residual field when applied field $\to 0$

H_c = <u>coercivity</u> = applied field needed to make $B = 0$.

A "hard" ferromagnetic material has $H_c \approx 10^4$ A/m
$B_r \approx 1$ T

A "soft" material has much smaller values of H_c and B_r — good for electromagnets because the field disappears when you turn off the current. ($H_c \sim 1-100$ A/m)

The area of the hysteresis loop $\int_{loop} \underline{B} \cdot d\underline{H}$ is the work done in moving the material through a complete cycle. This energy heats the magnet.

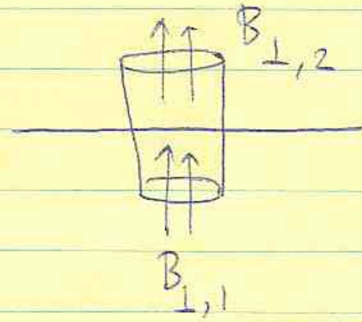
PHYS 340 lecture 33
Nov 23, 2005

Boundary conditions for magnetic fields

At the interface between two magnetic materials, what are the boundary conditions for $\underline{B}$ and $\underline{H}$? We get one from each of Maxwell's equations.

1) $\underline{\nabla} \cdot \underline{B} = 0$

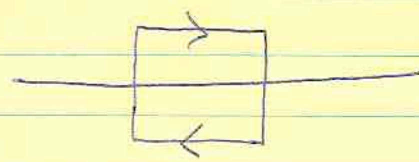
$\Rightarrow \boxed{B_\perp \text{ continuous}}$



parallel components give no contribution

2) $\underline{\nabla} \times \underline{H} = \underline{J}_f$

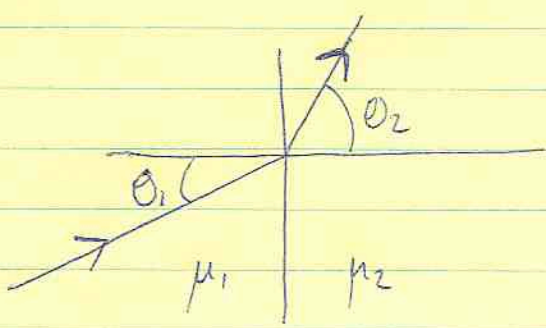
$\Rightarrow \boxed{[\underline{H}_{\parallel}] = \underline{K}_f \times \hat{\underline{n}}}$



only the parallel components contribute

if there are no free surface currents, then $H_\parallel$ is continuous.

These conditions mean that lines of $\underline{B}$ "refract" at the interface



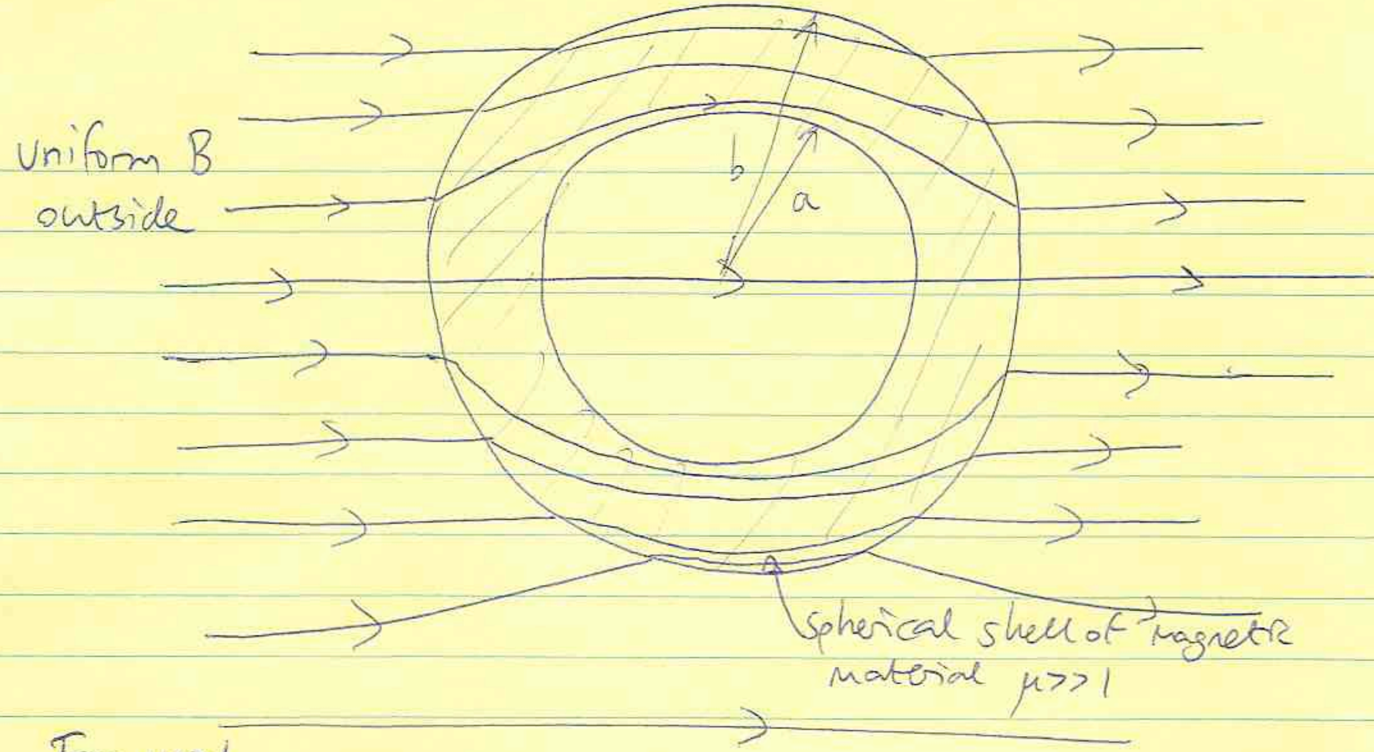
$\theta_2 > \theta_1$ if $\mu_2 > \mu_1$

$B_\perp$ is continuous

But $B_{\parallel,2} = \left(\frac{\mu_2}{\mu_1}\right) B_{\parallel,1}$

Example: magnetic shielding

Passive shielding using a spherical shell of magnetic material.
(see Jackson "Classical Electrodynamics" § 5.12)



For $\mu >> 1$,
the field inside the shell is $\dfrac{9}{2} \dfrac{B_0}{\mu} \dfrac{1}{1 - a^3/b^3} \ll B_0$

One way to solve this problem is to use the magnetic scalar potential. If there are no free currents, $\underline{\nabla} \times \underline{H} = 0 \Rightarrow$ we can write $\underline{H} = -\underline{\nabla} \phi$.

Since $\underline{B} = \mu_0 (\underline{H} + \underline{M})$

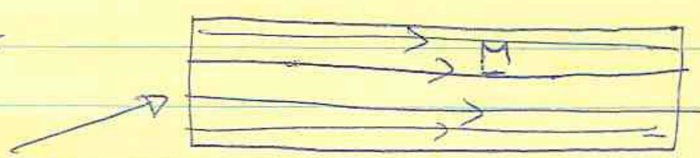
$$\underline{\nabla} \cdot \underline{B} = 0 = \mu_0 (\underline{\nabla} \cdot \underline{H} + \underline{\nabla} \cdot \underline{M})$$

$$\Rightarrow \quad \underline{\nabla} \cdot \underline{H} = \boxed{-\nabla^2 \phi = \underline{\nabla} \cdot \underline{M}}$$

So $-\underline{\nabla} \cdot \underline{M}$ acts as a source of $\underline{H}$ field lines.

This is beyond the scope of this course, but I wanted to mention it because it helps to see where the idea of north and south poles comes from.

eg. a bar magnet

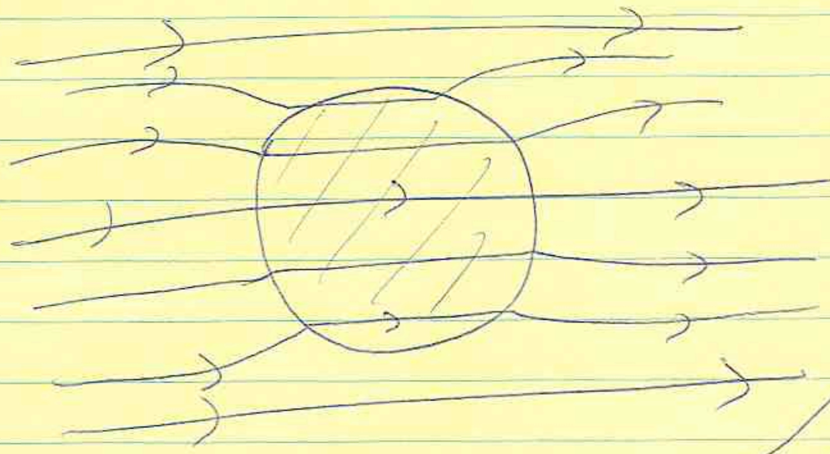


the $\underline{M}$ field line ends here $\Rightarrow \underline{\nabla} \cdot \underline{M} < 0$
$\Rightarrow$ Source of $\underline{H}$ field
$\Rightarrow$ North pole

at this end, $\underline{\nabla} \cdot \underline{M} > 0$
$\Rightarrow$ sink for $\underline{H}$ field lines $\Rightarrow$ South pole

(the following 4 pages not covered in class)

Example: sphere of magnetic material in a uniform field



[the field inside the sphere is uniform

the field outside is the original uniform field + a dipole field]

(just like a dielectric sphere in a uniform electric field)

to see this, write down solutions to $\nabla^2 \phi = 0$ where ϕ is the scalar magnetic potential. No time to do this here!
But if we assume the fields have this form, can derive the coefficients by matching boundary conditions.

inside $\underline{B} = \underline{B}_1$

outside $\underline{B} = \underline{B}_0 + \dfrac{\mu_0 m}{4\pi r^3}\left(2\cos\theta\,\hat{\underline{r}} + \sin\theta\,\hat{\underline{\theta}}\right)$

Now go to a point on the surface at angle θ

$B_\perp$ continuous

$\Rightarrow B_1 \cos\theta = \dfrac{\mu_0 m}{4\pi a^3}\, 2\cos\theta + B_0 \cos\theta$

$H_{\parallel}$ continuous

$\Rightarrow \dfrac{B_1}{\mu_r} \sin\theta = -\dfrac{\mu_0 m}{4\pi a^3} \sin\theta + B_0 \sin\theta$

$$\Rightarrow \quad \left.\begin{aligned} B_1 - B_0 &= +\frac{\mu_0 m}{2\pi a^3} \\ \frac{B_1}{\mu_r} - B_0 &= -\frac{\mu_0 m}{4\pi a^3} \end{aligned}\right\} \qquad B_1 + \frac{2B_1}{\mu} = 3B_0$$

$$\boxed{B_1 = \frac{3B_0\mu}{2+\mu}}$$

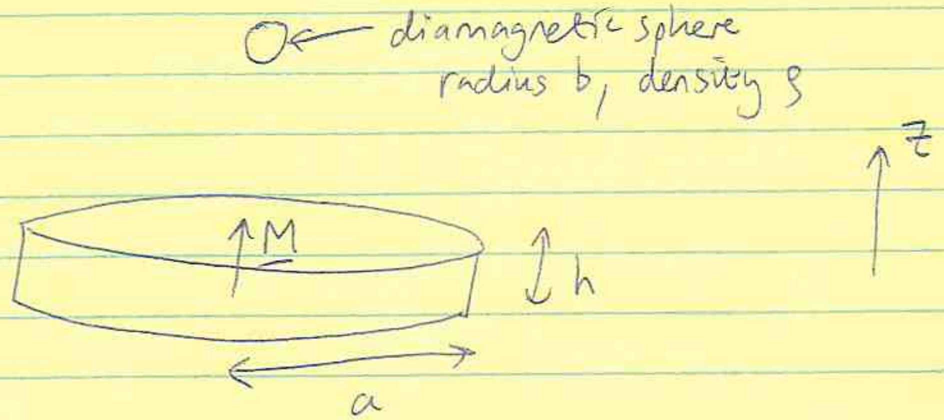
$$\frac{\mu_0 \underline{m}}{2\pi a^3} = B_0\left(\frac{3\mu_r}{2+\mu_r} - 1\right) = 2B_0\frac{(\mu_r - 1)}{(2+\mu_r)}$$

~~but th~~ $\Rightarrow$ $$\boxed{\underline{m} = 4\pi a^3 \frac{\underline{B_0}}{\mu_0}\frac{(\mu_r - 1)}{(2+\mu_r)}}$$

the magnetization is $\dfrac{m}{\frac{4\pi}{3}a^3} = \boxed{\dfrac{3B_0}{\mu_0}\left(\dfrac{\mu_r - 1}{\mu_r + 2}\right) = M}$

can rewrite B_1 as $\boxed{B_1 = B_0 + \frac{2}{3}M\mu_0}$

3) Diamagnetic levitation



the disk has a bound surface current $I = Mh$

the field along the axis is $B_z = \frac{\mu_0 I}{2} \frac{a^2}{(z^2+a^2)^{3/2}}$

$$= \frac{\mu_0 M h a^2}{2(z^2+a^2)^{3/2}}$$

the induced magnetization of the small sphere is

$$\left(\frac{\mu_r - 1}{\mu_r + 2}\right) \frac{B_z}{\mu_0}$$

$\Rightarrow$ the dipole moment is

$$\underline{m} = \frac{4\pi}{3} b^3 \underline{M}$$

$$= \frac{4\pi}{3} b^3 \left(\frac{\mu_r - 1}{\mu_r + 2}\right) \frac{M h a^2}{2(a^2+z^2)^{3/2}}$$

The force on the dipole is $\underline{\nabla}(\underline{m} \cdot \underline{B}) = \frac{d}{dz}\left(\frac{4\pi}{3} b^3 \left(\frac{\mu_r - 1}{\mu_r + 2}\right) \frac{B_z^2}{\mu_0}\right)$

$$= -\frac{4\pi b^3}{3} \rho g$$

$$\Rightarrow \boxed{-\rho g = \left(\frac{\mu_r - 1}{\mu_r + 2}\right) \frac{1}{\mu_0} 2 B_z \frac{dB_z}{dz}}$$

$$\frac{dB_z}{dz} = \frac{\mu_0 M h a^2}{2(z^2+a^2)^{5/2}}\left(-\frac{3}{2}2z\right)$$

$$= -\frac{3z}{(z^2+a^2)} B_z$$

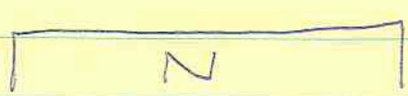
$$\Rightarrow \qquad \rho g = \cancel{-}\left(\frac{\mu_r - 1}{\mu_r + 2}\right)\frac{2B_z^2}{\mu_0}\frac{3z}{(z^2+a^2)}$$

$$\rho g = \left(\frac{\mu_r - 1}{\mu_r + 2}\right)\frac{6z}{(z^2+a^2)^4}\frac{\mu_0 M^2 h^2 a^4}{4}$$

$$\boxed{\rho g = \frac{3}{2}\frac{z}{(z^2+a^2)^4}\mu_0 M^2 h^2 a^4}$$

This gives the equilibrium height – in fact there are multiple solutions. Need to understand if the equilibrium is stable or unstable. No time to consider this question here. (see article by Simon et al 2001 Am J Phys 69 702)
A "science toy" that levitates a magnet is the Levitron (search and you'll find their webpage). This toy consists of a levitating permanent magnet

the dipole sees a net force upwards, but also wants to flip over because of the $\underline{m} \times \underline{B}$ torque



to stop it flipping over, the magnet is in the form of a spinning top
if it tries to flip over, conservation of angular momentum instead leads to precession.

PHYS 340 Lecture 34 Nov 25th

Electromagnetic Induction

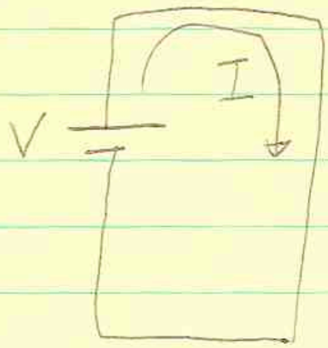
Now we move on to consider time varying fields. So far, electric and magnetic fields have been unrelated

$$\nabla \cdot \underline{E} = \rho/\varepsilon_0 \qquad \nabla \cdot \underline{B} = 0$$
$$\nabla \times \underline{E} = 0 \qquad \nabla \times \underline{B} = \mu_0 \underline{J}$$

What we'll find is that a time-changing magnetic field generates electric fields and vice-versa. This couples the two fields, making phenomena such as electromagnetic waves possible.

Electromotive force (EMF) from a battery

Consider a wire connected between the positive and negative terminals of a battery.



A current I flows around the loop, given by Ohm's law $\boxed{V = IR}$

R is the resistance of the wire in ohms.

Locally, what makes the current flow? There is an electric field inside the conductor that accelerates the particles, but they are constantly colliding with other particles, so that they drift along the wire with a terminal velocity

$$v_{drift} \approx \left(\frac{qE}{m}\right)\tau$$

time between collisions

The current density inside the wire is

$$\boxed{J = nq v_{drift}}$$

n = number density (m^{-3})

or $J = nq\left(\frac{qE\tau}{m}\right)$

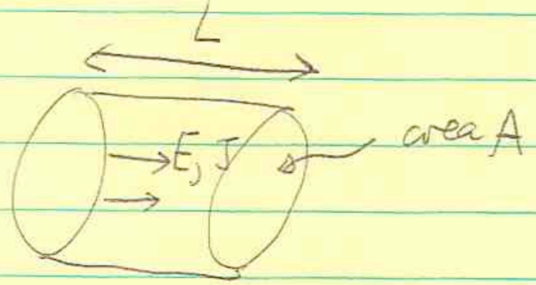
$$\boxed{J = \left(\frac{nq^2\tau}{m}\right) E}$$

This is a local version of Ohm's law $\boxed{\underline{J} = \sigma \underline{E}}$

σ = electrical conductivity

we also use $\rho = \frac{1}{\sigma}$ = resistivity

Now consider a section of straight wire



$\underline{J}$ and $\underline{E}$ point along the wire, and must be uniform because $\underline{\nabla} \times \underline{E} = 0$.

$$\therefore \quad I = JA = \sigma A E = \frac{\sigma A}{L} V$$

or $V = IR$

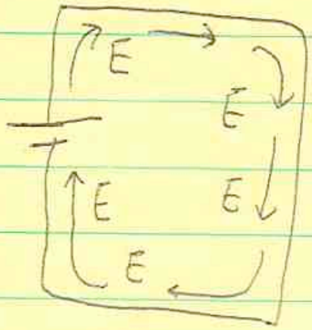
with $\boxed{R = \frac{L}{\sigma A} = \frac{\rho L}{A}}$

which gives the resistance in terms of microscopic conductivity + geometry (length and area here)

for more general geometries

$$R = \int \rho(\underline{r}) \frac{dx}{A}$$

Now think about a circuit



we know that $\oint \underline{E} \cdot d\underline{l} = 0$

(no loops in electrostatics)

So inside the battery, there must be an opposing field. The battery does work on charges coming into one terminal to move them against this internal field out to the opposite terminal.

The emf is the work done per unit charge,

$$EMF = \int_A^B \frac{\underline{F}}{q} \cdot d\underline{l}$$

$\underline{F}$ = internal force which moves the charges (for a battery, chemical energy)

or the voltage of the battery.

The work done by the battery is dissipated as the charges flow around the circuit. The rate of dissipation is

$$\text{dissipated power} = VI = I^2 R = \frac{V^2}{R}.$$

What about locally?

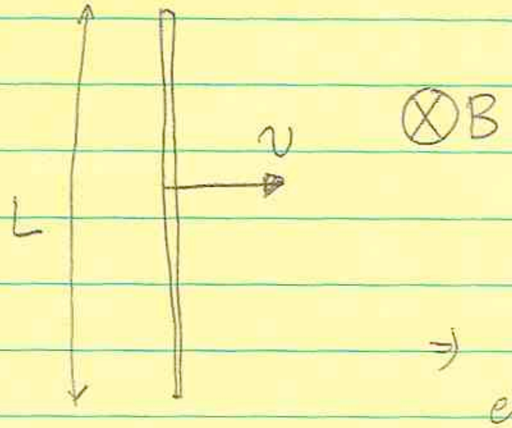
$$\frac{\text{dissipated power}}{\text{volume}} = VI = \frac{(EL)(JA)}{LA} = JE$$

$$= J^2/\sigma.$$

Motional EMF

If we take a wire in a magnetic field and move it, an EMF arises because of the Lorentz force on particles in the wire.

First think about an isolated wire

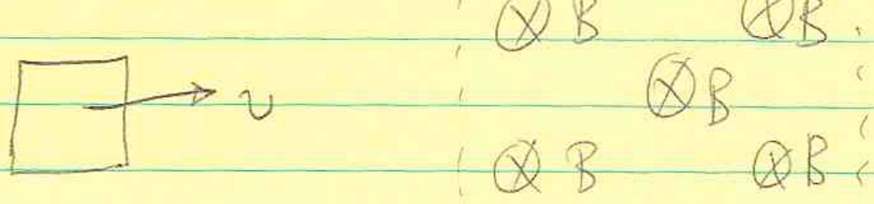


each charge carrier experiences a force qvB upwards

$\Rightarrow$ charge will flow until a compensating electric field $E = vB$ (pointing downwards)

The voltage drop along the wire is $V = \int \underline{E} \cdot \underline{dl} = vBL.$

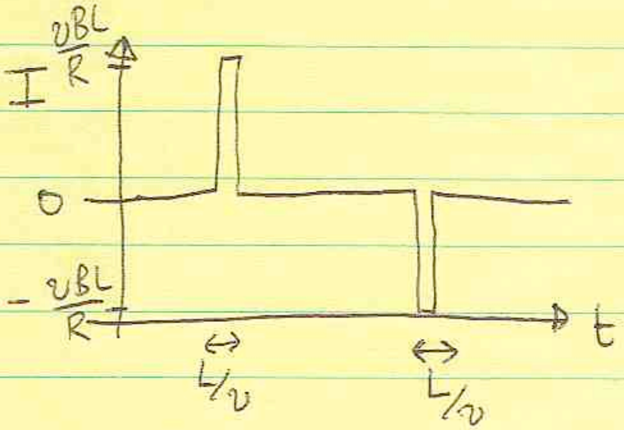
Now make a square loop and move it ~~to~~ into, through and out of a region with uniform B.



As the loop enters, the right hand side experiences a force per unit charge vB. This is just like the internal forces inside a battery. The emf is $\int vB\, dl = vBL$

The emf around the loop is $\oint (\underline{v} \times \underline{B}) . d\underline{l} = \mathcal{E}$

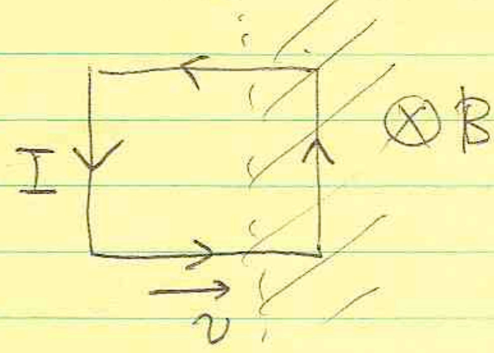
and the current is $\dfrac{\mathcal{E}}{R}$.



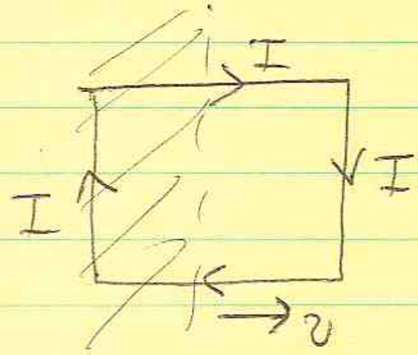
there is a current in the loop when it leaves and enters the magnetic field region.

~~There is a nice way to wr~~

Which direction does the current flow?



anticlockwise



clockwise

The ~~flu~~ current flows in such a way that ~~it opposes the~~ the magnetic field it produces OPPOSES the ~~a~~ change in the magnetic field inside the loop.

There is a nice way to write $\mathcal{E}$ in terms of $\dfrac{d\Phi}{dt}$ where

$\Phi = \int \underline{B} . d\underline{A}$ is the magnetic flux through the loop.

$$\Phi = BLx$$

$$\frac{d\Phi}{dt} = BL\frac{dx}{dt} = BLv = -\mathcal{E}$$

or $$\boxed{\mathcal{E} = -\frac{d\Phi}{dt}}$$

(the minus sign is because the current flows in such a way as to produce a field opposite to the original field).

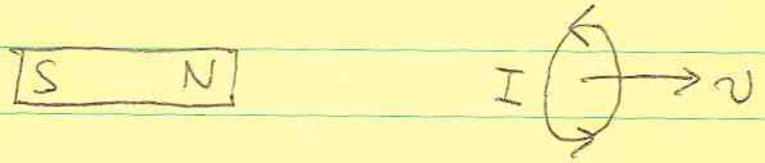
This result applies to a loop of arbitrary shape or ~~d~~velocity direction. (see Griffiths p 296)

~~Flux~~

Faraday's Experiments

Electromagnetic induction was discovered by Faraday in 1831. Experimentally, it is observed that:

1. a moving conductor loop in a non-uniform B has an induced current.



2. moving a bar magnet and holding the loop fixed also gives an induced current

(3) keeping both the magnet and loop fixed, but changing the strength of the magnet gives an induced current

All three effects are described by $\boxed{\mathcal{E} = -\frac{d\Phi}{dt}}$ **FARADAY'S LAW**

so whenever the magnetic flux through the loop changes, and for whatever reason, there is an induced emf.

In (1), the emf is a motional emf. But where is the emf coming from in (2) and (3) ?? We infer that a changing magnetic field induces an electric field, ie.

$$\mathcal{E} = \oint \underline{E} \cdot d\underline{l}$$

This integral is zero in electrostatics, but not when $\underline{B}$ is changing

$$\mathcal{E} = \oint \underline{E} \cdot d\underline{l} = -\frac{d\Phi}{dt} = -\frac{d}{dt}\int \underline{B} \cdot d\underline{A}$$

In differential form,

$$\oint \underline{E} \cdot d\underline{l} = \int \underline{\nabla} \times \underline{E} \cdot d\underline{A} = -\int \frac{\partial \underline{B}}{\partial t} \cdot d\underline{A}$$

$$\Rightarrow \quad \boxed{\underline{\nabla} \times \underline{E} = -\frac{\partial \underline{B}}{\partial t}}$$

~~This is the final~~

~~Wherever there~~

Comparing with Ampère's law $\underline{\nabla} \times \underline{B} = \mu_0 \underline{J}$,
we see that wherever there is a $\frac{\partial \underline{B}}{\partial t}$, a loop of electric field $\underline{E}$ is generated.

The direction of the induced current is given by LENZ'S LAW

"The direction of the induced emf is such as to oppose the change causing it."

PHYS 340 lecture 35 Nov 28th

Last time, we discussed electromagnetic induction.

FARADAY'S LAW $\mathcal{E} = -\frac{d\Phi}{dt}$

differential form $\underline{\nabla} \times \underline{E} = -\frac{\partial \underline{B}}{\partial t}$

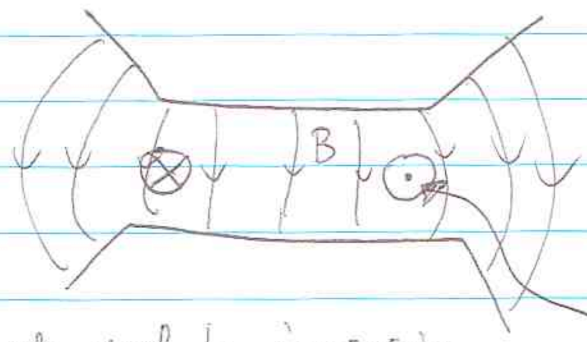
LENZ'S LAW

Griffiths very succinctly quotes this as "Nature abhors a change in flux."

(to abhor = "to regard with disgust and hatred")

Today, we'll look at some examples

1. Betatron



electron velocity

Electrons are accelerated by increasing B over time.

Electrons circulate at constant radius as they are being accelerated. ~~Since~~ They are accelerated by an electric field induced by the changing magnetic field.

$$\oint \underline{E} \cdot d\underline{l} = -\frac{d\Phi}{dt}$$

$$2\pi r E_\phi = -\frac{d\Phi}{dt} \Rightarrow E_\phi = \frac{-1}{2\pi r}\frac{d\Phi}{dt}$$

The gyroradius is $r = \frac{p}{eB}$

To keep this constant, we need $\frac{dp}{dt} = er\frac{dB}{dt}$

where B is the local B field at the radius of the orbiting electrons.

$$\therefore \quad \frac{dp}{dt} = -eE_\phi = \frac{e}{2\pi r}\frac{d\Phi}{dt} = er\frac{dB}{dt}$$

or $$\frac{dB}{dt} = \frac{1}{2}\frac{1}{\pi r^2}\frac{d\Phi}{dt}$$

$$\boxed{\frac{dB}{dt} = \frac{1}{2}\frac{d\bar{B}}{dt}}$$

the field is arranged so that this condition is satisfied

What is the final electron energy?

$$\int \cancel{dt}\, \frac{dp}{\cancel{dt}} = \int er\frac{dB}{\cancel{dt}}\,\cancel{dt}$$

$$\Rightarrow \quad p_f - p_i = er(B_f - B_i)$$

i = initial
f = final

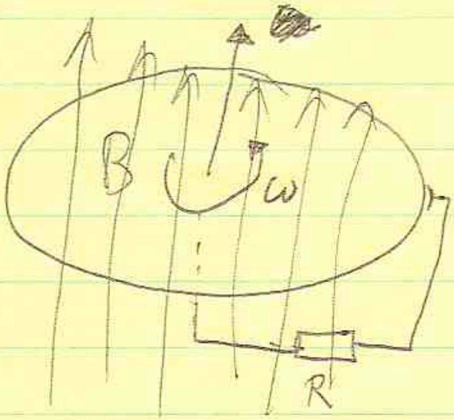
if $p_f \gg p_i$ and $B_f \gg B_i$,

$$p_f \simeq erB_f$$

eg. $B_f = 0.5\,T$
$r = 0.5\,m$

$E = p_f c = Berc = \underline{37\,MeV}$.

2. <u>a circular disk made of conducting material rotating in a uniform field</u>



the center and outer edge are connected by sliding contacts

a particle in the disk feels a force
$q v B = q \omega r B$ radially outwards.

$$\mathcal{E} = \int_0^a \cancel{q}\, \omega r B \, dr = \frac{1}{2} \omega a^2 B$$

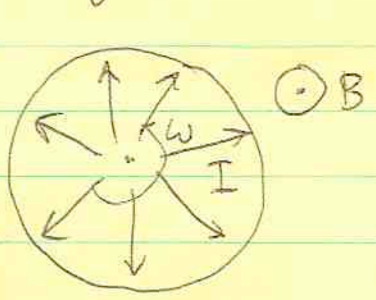
so a current $\boxed{I = \frac{\omega a^2 B}{2R}}$ flows around the circuit.

To apply $\mathcal{E} = -\frac{d\Phi}{dt}$ in this case is tricky, but we could

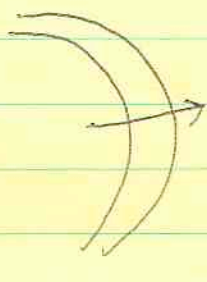
write $\mathcal{E} = -\frac{d}{dt}\left(\pi a^2 B\right) = \frac{\pi a^2 B}{P} = a^2 B \frac{\omega}{2\pi}$

which gives the correct answer.

A torque acts on the disk, <u>braking its rotation</u>



~~consider~~ at radius r, the current per unit length is $\frac{I}{2\pi r}$



$\underline{K} = \frac{I}{2\pi r} \hat{\underline{r}}$

the torque is
$dN = IBr$
$$\Rightarrow N = \int_0^a dN = \frac{1}{2} I B a^2$$

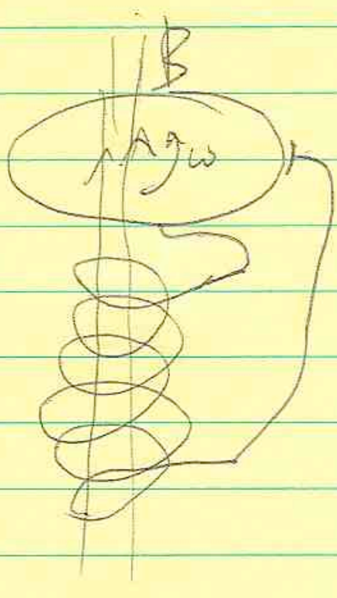
Whatever turns the disk must do work against this torque

$$\text{power} = N\omega = \frac{1}{2} B\omega a^2 I = \left(\frac{\omega a^2 B}{2}\right)^2 \frac{1}{R}$$
$$= \mathcal{E}^2/R$$

so the work done equals the ohmic dissipation in the wire.

In general,
A moving conductor in a non-uniform magnetic field experiences magnetic braking from induced <u>eddy currents</u>.

Finally, what if we put the current I through a solenoid pointing in the direction needed for the field $\approx \mu_0 n I$ to point in the same direction as the external field? Then we could turn the external field off!



then $I = \dfrac{\omega a^2 B}{2R}$

and $B = \dfrac{\mu_0 n I}{2}$ (roughly at the end of the solenoid)

$$\Rightarrow \frac{I\mu_0 n}{2} = \frac{2RI}{\omega a^2}$$

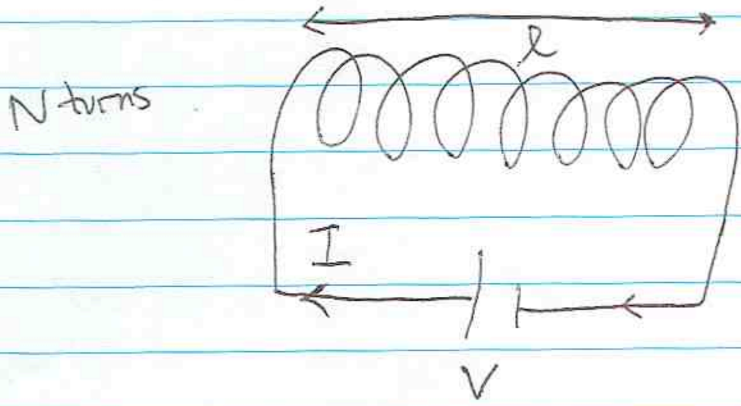
$$\Rightarrow \boxed{\omega = \frac{4R}{\mu_0 n a^2}}$$ frequency needed

This is a <u>self-excited dynamo</u>, ~~so~~ the basic idea is similar for the dynamo that generates the Earth's field.

Nov 30th
PHYS 340 lecture 36 ~~Nov 29th~~

~~(Today, Course evaluations)~~.
Also, we'll talk about <u>Inductance and Magnetic energy</u>

Think about connecting a battery to a solenoid



As the current starts to flow, the magnetic flux through each loop grows.
$\Rightarrow$ there must be an induced emf opposing the applied voltage.

We call this the "BACK EMF"

$$\mathcal{E} = -\frac{d\Phi}{dt} = -\frac{d}{dt}(BAN)$$
$$= -AN\frac{dB}{dt}$$
$$= -\frac{\mu_0 A N^2}{\ell}\frac{dI}{dt}$$

~~The total current which flows is given by~~
$$\cancel{V + \mathcal{E} = IR}$$

write this as $\mathcal{E} = -L\frac{dI}{dt}$

where $L = \frac{\mu_0 A N^2}{\ell}$ is the SELF INDUCTANCE of the solenoid.

The current which flows is given by
$$V + \mathcal{E} = IR$$

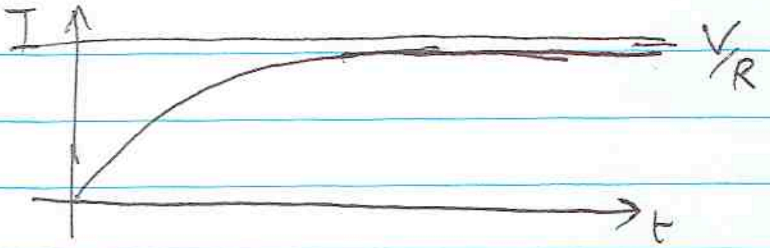
or $$V = L\frac{dI}{dt} + IR$$

$$\frac{dI}{dt} = -I\frac{R}{L} + \frac{V}{L}$$

$$\Rightarrow \quad I = Ae^{-\frac{Rt}{L}} + \frac{V}{R}$$

but at $t=0$ $I=0$ $\Rightarrow$ $A = -\frac{V}{R}$

$$\Rightarrow \quad \boxed{I = \frac{V}{R}\left[1 - e^{-Rt/L}\right]}$$



current builds up on a timescale L/R

What about the work done by the battery?

This is $$\int VI\,dt = \underbrace{\int_0^t I^2R\,dt}_{\text{ohmic dissipation}} + \underbrace{\int_0^I IL\frac{dI}{\cancel{dt}}\cancel{dt}}_{\text{work done against the back emf}}$$

The work done against the back emf is $\boxed{\frac{1}{2}LI^2}$

Where does this energy go? It must go into the magnetic field

$$\frac{1}{2}LI^2 = \frac{1}{2}\frac{\mu_0 A N^2}{\ell}I^2 = \frac{1}{2}\frac{A\ell}{\mu_0}\frac{\mu_0^2 N^2 I^2}{\ell^2}$$

$$= \left(\frac{1}{2}\frac{B^2}{\mu_0}\right)(A\ell)$$

volume

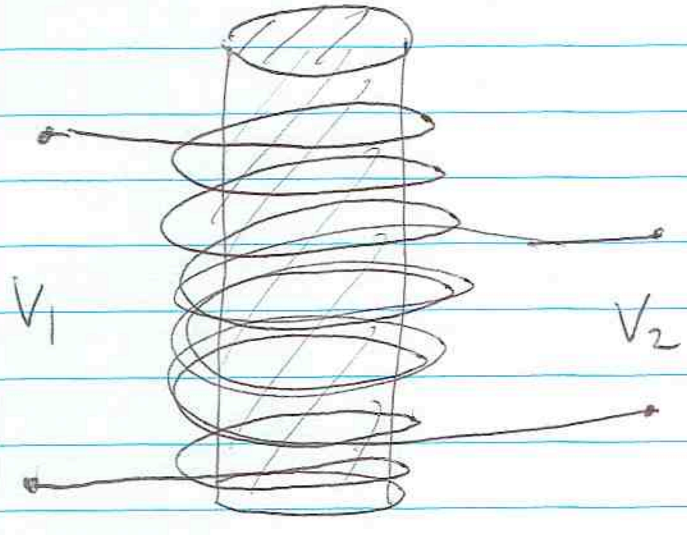
energy density

We infer that there is an energy density in the magnetic field of $\left(\frac{B^2}{2\mu_0}\right)$.

[Recall the energy density of the electric field ~~B~~ $\frac{1}{2}\varepsilon_0 E^2$.]

Even though magnetic forces do no work, wherever a ~~[illegible]~~ magnetic field is created, it involves a dB/dt, an induced emf and therefore work against the back emf, i.e. electrical forces.

One application of these ideas is a <u>transformer</u>.



Consider two coils wrapped around a cylinder of magnetic material.

The emf induced in coil 2 by a changing current in coil 1 is

$$\mathcal{E}_2 = -\frac{d}{dt}\left(B_1 A N_2\right)$$

$$= -\underbrace{\frac{\mu_0 N_1 N_2 A}{l_1}}_{\text{this is the mutual inductance } M} \frac{dI_1}{dt}$$

$$\mathcal{E}_2 = -M\frac{dI_1}{dt}$$

[can show that M is the same for $\mathcal{E}_1 = -M\frac{dI_2}{dt}$.

This is a general result for two circuits linked by magnetic flux. Often it is much easier to calculate M by considering $\mathcal{E}_2$ and dI_1/dt rather than $\mathcal{E}_1$ and dI_2/dt ~~for~~ as in this example. But once you have M, you can apply it to the reverse situation.]

In a transformer, we use an <u>alternating current</u>

$$V_1 = V\cos\omega t$$

Now assume negligible resistance for the coils

$$V_1 = L_1 \frac{dI_1}{dt} + \cancel{I_1 R_1} \to 0$$

$$\Rightarrow \quad \frac{dI_1}{dt} = \frac{V_1}{L_1} = \frac{V \cos \omega t}{L_1}$$

$$\Rightarrow \quad V_2 = \mathcal{E}_2 = -M \frac{dI_1}{dt} = -\frac{M}{L_1} V \cos \omega t$$

But $\dfrac{M}{L_1} = \dfrac{N_2}{N_1}$

$$\therefore \quad \boxed{V_2 = -\left(\frac{N_2}{N_1}\right) V_1}$$

$$(\text{output voltage}) = \left(\begin{matrix}\text{ratio of number} \\ \text{of turns}\end{matrix}\right) \times \left(\begin{matrix}\text{input} \\ \text{voltage}\end{matrix}\right)$$

Magnetic field energy

Remember that we can write $\underline{B} = \underline{\nabla} \times \underline{A}$ where $\underline{A}$ is the vector potential.

Faraday's law $\quad \dfrac{\partial \underline{B}}{\partial t} = -\underline{\nabla} \times \underline{E}$

can be rewritten $\quad \underline{\nabla} \times \dfrac{\partial \underline{A}}{\partial t} = -\underline{\nabla} \times \underline{E}$

$$\Rightarrow \quad \boxed{\underline{E} = \underbrace{-\underline{\nabla}\phi}_{\substack{\text{curl-free}\\ \text{part of } E\\ \text{from electric}\\ \text{potential } \phi}} \underbrace{- \frac{\partial \underline{A}}{\partial t}}_{\substack{\text{induced } E \text{ field}\\ \text{with non-zero curl.}}}}$$

The work done against the back emf is $\quad -\underline{J}.\underline{E} \quad$ (per second per unit volume)

$$= \underline{J}.\frac{\partial \underline{A}}{\partial t}$$

(we include the induced field only)

But because $\quad \underline{A}(r) = \displaystyle\int \frac{\underline{J}(\underline{r}')}{|\underline{r}-\underline{r}'|}\, d^3\underline{r}'$

$$\Rightarrow \quad \underline{J}.\frac{\partial \underline{A}}{\partial t} = \frac{\partial \underline{J}}{\partial t}.\underline{A}$$

$$\Rightarrow \quad \underline{J}.\frac{\partial \underline{A}}{\partial t} = \underline{\frac{1}{2}\frac{\partial}{\partial t}\left(\underline{J}.\underline{A}\right)}$$

Integrating over the whole of space and over time gives the total work done

$$\boxed{\frac{1}{2}\int \underline{J}.\underline{A}\, d^3\underline{r}} = W$$

Now let's manipulate this formula.

Ampère's law $\nabla \times \mathbf{H} = \mathbf{J}$

$$\Rightarrow \quad W = \frac{1}{2} \int d^3\mathbf{r} \, (\nabla \times \mathbf{H}) \cdot \mathbf{A}$$

But $\nabla \cdot (\mathbf{H} \times \mathbf{A}) = \mathbf{A} \cdot \nabla \times \mathbf{H} - \mathbf{H} \cdot \nabla \times \mathbf{A}$

$$\Rightarrow \quad W = \frac{1}{2} \int_V \mathbf{B} \cdot \mathbf{H} \, d^3\mathbf{r} \quad + \quad \underbrace{\int_S \cancel{A}(\mathbf{H} \times \mathbf{A}) \cdot d\mathbf{S}}$$

this surface term vanishes if the surface is well outside the current distribution

$$\Rightarrow \quad \boxed{W = \frac{1}{2} \int \mathbf{B} \cdot \mathbf{H} \, d^3\mathbf{r}}$$

where the volume is much larger than the current distribution

We interpret $\frac{1}{2} \mathbf{B} \cdot \mathbf{H}$ as the energy density in the magnetic field

PHYS 340 lecture 37 Dec 1st

First, summarize the important results from last time:

$$\mathcal{E} = -L\frac{dI}{dt}$$

"back emf" inductance

The work done against the back emf is

$$\frac{1}{2}LI^2 = \int \frac{1}{2}\underline{B}.\underline{H}\ d^3\underline{r}$$

where the energy density in the magnetic field is $\frac{1}{2}\underline{B}.\underline{H}$

(for electric fields, the energy density is $\frac{1}{2}\underline{D}.\underline{E}$).

<u>How to calculate inductance?</u>

1) in simple cases, use $\mathcal{E} = -\frac{d\Phi}{dt} = -L\frac{dI}{dt}$

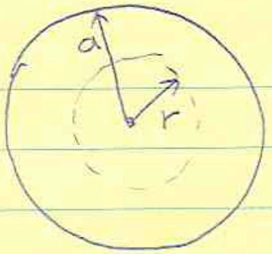
$$\Rightarrow \boxed{\Phi = LI}$$

eg. solenoid from last time $\mu_0 n I N A = LI$

$$\Rightarrow \boxed{L = \mu_0 n N A}.$$

2) often there isn't a well-defined single current loop with enclosed flux. Then it's better to use ENERGY.

eg. internal inductance of a current-carrying wire



current
I uniformly distributed

$$B_\phi = \frac{\mu_0 I r}{2\pi a^2}$$

The magnetic energy per unit length is $\displaystyle\int 2\pi r\, dr \, \frac{B_\phi^2}{2\mu_0}$

$$= \int 2\pi r\, dr \, \frac{1}{2\mu_0} \frac{\mu_0^2 I^2 r^2}{(2\pi a^2)^2}$$

$$= \frac{\mu_0 I^2}{4\pi a^4} \int_0^a r^3\, dr$$

$$= \frac{\mu_0 I^2}{16\pi}$$

compare this with $\frac{1}{2} L I^2 \Rightarrow \boxed{L = \frac{\mu_0}{8\pi}}$ internal inductance per unit length of a straight wire

<u>Completing Maxwell's equations: the displacement current</u>

So far we have the following equations governing the evolution of $\underline{E}$ and $\underline{B}$:

$$\underline{\nabla} \cdot \underline{E} = \rho/\varepsilon_0 \qquad \underline{\nabla} \cdot \underline{B} = 0$$

$$\underline{\nabla} \times \underline{E} = -\frac{\partial \underline{B}}{\partial t} \qquad \underline{\nabla} \times \underline{B} = \mu_0 \underline{J}$$

There is an inconsistency lurking here because charge conservation gives

$$\underline{\nabla} \cdot \underline{J} = -\frac{\partial \rho}{\partial t}$$

(to see this, integrate over a volume

$$\int_V \underline{\nabla} \cdot \underline{J}\, dV = \int_S \underline{J} \cdot d\underline{S} = -\frac{\partial}{\partial t}\int \rho\, dV = -\frac{\partial Q}{\partial t}$$

total charge leaving

rate of change of charge in the volume)

Take the divergence of Ampere's law

$$\underline{\nabla} \cdot (\underline{\nabla} \times \underline{B}) = \mu_0 \underline{\nabla} \cdot \underline{J}$$

$$0 = \cancel{\mu_0 \nabla} - \mu_0 \frac{\partial \rho}{\partial t}$$

only true when $\frac{\partial \rho}{\partial t} = 0$!

We need an additional term that will cancel the $\frac{\partial \rho}{\partial t}$! Since

$\frac{\partial \rho}{\partial t} = \frac{\partial}{\partial t}(\underline{\nabla} \cdot \varepsilon_0 \underline{E}) = \underline{\nabla} \cdot \left(\varepsilon_0 \frac{\partial \underline{E}}{\partial t} \right)$, a self-consistent version of Ampère's law is

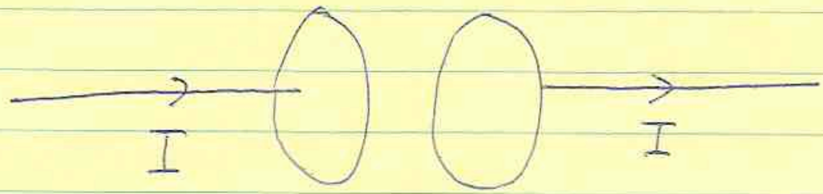
$$\boxed{\underline{\nabla} \times \underline{B} = \mu_0 \underline{J} + \underbrace{\mu_0 \varepsilon_0 \frac{\partial \underline{E}}{\partial t}}_{\text{this term is the displacement current}}}$$

Now the divergence of both sides vanishes.

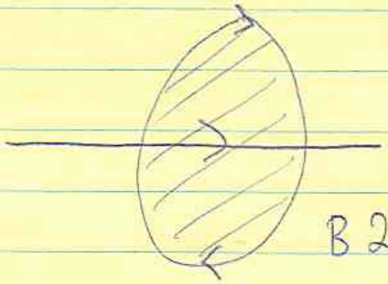
This was Maxwell's great contribution – to infer that

time changing electric fields produce magnetic fields

Classic example: charging a capacitor



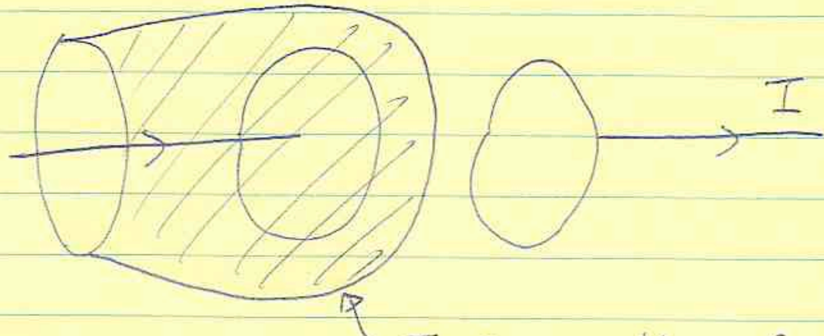
We can find the magnetic field using Ampère's law



in which we integrate over the loop and area shown

$$B\, 2\pi r = \mu_0 I$$

but we could choose a different area



$\underline{J} = 0$ over this surface!

We need the displacement current term

$$\int \mu_0 \varepsilon_0 \frac{\partial \underline{E}}{\partial t} \cdot d\underline{A} = \mu_0 \frac{d}{dt} Q_{enc} = \mu_0 I \checkmark$$

This gives the correct result.

Finally, $\boxed{\text{MAXWELL'S EQUATIONS}}$

$$\underline{\nabla} \cdot \underline{E} = \rho/\varepsilon_0 \qquad \underline{\nabla} \cdot \underline{B} = 0$$

$$\underline{\nabla} \times \underline{E} = -\frac{\partial \underline{B}}{\partial t} \qquad \underline{\nabla} \times \underline{B} = \mu_0 \underline{J} + \mu_0 \varepsilon_0 \frac{\partial \underline{E}}{\partial t}$$

with $\underline{\nabla} \cdot \underline{J} = -\dfrac{\partial \rho}{\partial t}$

in matter, we can write

$$\underline{\nabla} \cdot \underline{D} = \rho_f \qquad \underline{\nabla} \cdot \underline{B} = 0$$

$$\underline{\nabla} \times \underline{E} = -\frac{\partial \underline{B}}{\partial t} \qquad \underline{\nabla} \times \underline{H} = \underline{J}_f + \frac{\partial \underline{D}}{\partial t}$$

in which case, we need constitutive relations between $\underline{E}$ and $\underline{D}$, $\underline{B}$ and $\underline{H}$.

This is where we will stop. But let's have a quick look at the most famous / important / remarkable example – <u>electromagnetic waves</u>.

In vacuum, $\rho = 0$ and $\underline{J} = 0$.
Then,

$$\underline{\nabla} \times \underline{E} = -\frac{\partial \underline{B}}{\partial t}$$ ← take the curl of this equation.

LHS $\quad \underline{\nabla} \times (\underline{\nabla} \times \underline{E}) = -\nabla^2 \underline{E} + \underline{\nabla}\,(\cancelto{0}{\underline{\nabla} \cdot \underline{E}})$

RHS $\quad -\dfrac{\partial}{\partial t}(\underline{\nabla} \times \underline{B}) = -\varepsilon_0 \mu_0 \dfrac{\partial^2 \underline{E}}{\partial t^2}$

$$\Rightarrow \quad \boxed{\nabla^2 \underline{E} = \mu_0 \varepsilon_0 \frac{\partial^2 \underline{E}}{\partial t^2}}$$ this is a 3D wave equation

eg. in one dimension

$$\frac{\partial^2 f}{\partial t^2} = v^2 \frac{\partial^2 f}{\partial x^2}$$

general solution $f(x \pm vt)$

v = wave speed

Here, we have $v^2 = \frac{1}{\mu_0 \varepsilon_0} = \frac{1}{\cancel{4\pi 10^{-7}}} \cancel{4\pi 10^{-7}} c^2 = c^2$

the wave speed is the speed of light!

This is a truly remarkable result because remember that ε_0 is the constant in Coulomb's law which describes the force between two electric charges, μ_0 is the constant in the Biot-Savart law which describes the forces between currents $\rightarrow$ from these two constants we predict a wave whose speed equals the measured speed of light!

That's where we leave it for this class. More in PHYS 342/352...