PHYS 352
Electromagnetic Waves
Fall term 2018

Contents

Part 1: Fundamentals

These are notes for the first part of PHYS 352 Electromagnetic Waves. This course follows on from PHYS 350. At the end of that course, you will have seen the full set of Maxwell's equations, which in vacuum are

$$\begin{aligned} \nabla\cdot\mathbf{E} &= \frac{\rho}{\epsilon_0} & \nabla\times\mathbf{E} &= -\frac{\partial\mathbf{B}}{\partial t} \\ \nabla\cdot\mathbf{B} &= 0 & \nabla\times\mathbf{B} &= \mu_0\mathbf{J}+\mu_0\epsilon_0\frac{\partial\mathbf{E}}{\partial t} \end{aligned} \tag{1.1}$$

with

$$\nabla\cdot\mathbf{J} = -\frac{\partial\rho}{\partial t}. \tag{1.2}$$

In this course, we will investigate the implications and applications of these results. We will cover

- electromagnetic waves
- energy and momentum of electromagnetic fields
- electromagnetism and relativity
- electromagnetic waves in materials and plasmas
- waveguides and transmission lines
- electromagnetic radiation from accelerated charges
- numerical methods for solving problems in electromagnetism

By the end of the course, you will be able to calculate the properties of electromagnetic waves in a range of materials, calculate the radiation from arrangements of accelerating charges, and have a greater appreciation of the theory of electromagnetism and its relation to special relativity.

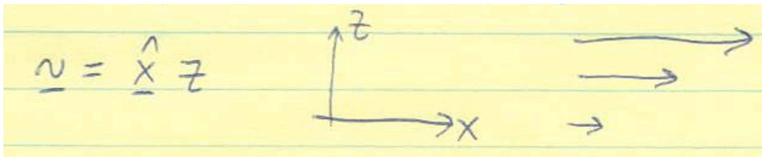
The spirit of the course is well-summed up by the "intermission" in Griffith's book. After working from statics to dynamics in the first seven chapters of the book, developing the full set of Maxwell's equations, Griffiths comments (I paraphrase) that the full power of electromagnetism now lies at your fingertips, and the fun is only just beginning. It is a disappointing ending to PHYS 350, but an exciting place to start PHYS 352!

Why study electromagnetism? One reason is that it is a fundamental part of physics (one of the four forces), but it is also ubiquitous in everyday life, technology, and in natural phenomena in geophysics, astrophysics or biophysics. The study of electromagnetism also introduces some advanced physics concepts, whether it be dealing with the abstract notions of fields or gauge invariance, or learning mathematical techniques such as the approaches for solving partial differential equations. In this course, we will cover all of these different aspects, going from the applications of electromagnetism to the basic structure of the theory of electromagnetism.

1.1 Important mathematical results

I will assume that you are familiar with the following results and concepts from vector calculus:

1. *Vector and scalar fields*, e.g. temperature $T(\mathbf{r})$, electrostatic potential $V(\mathbf{r})$, velocity of a fluid $\mathbf{v}(\mathbf{r})$, electric field $\mathbf{E}(\mathbf{r})$. Sketching a vector field, e.g. a shearing fluid flow



2. *Derivatives of fields*[1]

   - gradient operator (note that this is a vector)

$$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$$

   - gradient of a scalar field $\phi(\mathbf{r})$

$$\nabla\phi = \left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}\right)$$

   - gradient of a scalar field in the direction $\hat{\mathbf{n}}$

$$\hat{\mathbf{n}} \cdot \nabla\phi$$

   - divergence of a vector field (this is a scalar quantity)

$$\nabla \cdot \mathbf{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}$$

   - curl of a vector field (a vector)

$$\nabla \times \mathbf{E} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & E_y & E_z \end{vmatrix}$$

   - 2nd derivative (Laplacian) (a scalar)

$$\nabla \cdot \nabla\phi = \nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}$$

[1] We'll use Cartesian coordinates here. Depending on the symmetry of the problem, spherical or cylindrical coordinates may be necessary. For exam purposes, I will assume you know the Cartesian results but will give you the formulae for spherical or cylindrical coordinate systems.

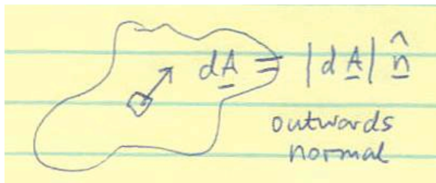
3. *Integrals of fields*

   - volume integral, e.g.

$$\int dV \phi(\mathbf{r}) = \int d^3\mathbf{r}\, \phi(\mathbf{r})$$

   - surface integral, e.g.

$$\int_S \mathbf{v} \cdot d\mathbf{A}$$



   - line integral, e.g.

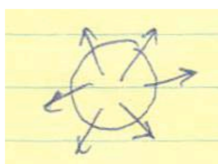
$$\int_{\text{path}} \mathbf{v} \cdot d\mathbf{l}$$

4. *Divergence theorem*

$$\int_S \mathbf{v} \cdot d\mathbf{A} = \int_V (\nabla \cdot \mathbf{v})\, dV$$

   where volume V is bounded by surface S.

Geometrical interpretation: if a vector field has $\nabla \cdot \mathbf{v} > 0$ at some point, there is a net flux across a closed surface ⇒ the vectors are "diverging"

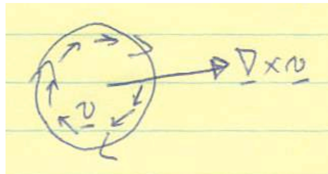


5. *Stoke's theorem*

$$\oint_{\text{loop}} \mathbf{v} \cdot d\mathbf{l} = \int_S (\nabla \times \mathbf{v}) \cdot d\mathbf{A}$$

   where S is any surface bounded by the loop, and the direction of $d\mathbf{A}$ is given by the right hand rule applied to the integration path.

Geometric interpretation: $|\nabla \times \mathbf{v}| > 0$ at some point indicates that the line integral around a small closed loop has a non-zero value $\Rightarrow$ the curl measures the "loopiness" of the field at each point



1.2 A review of the path to Maxwell's equations

To start with, let's review some basic ideas from PHYS 350. This will serve as an introduction to the notation we will use, to make sure we are all on the same page, but also we will focus on a physical understanding of each of Maxwell's equations. The overarching idea underlying electromagnetism at this level, as discussed in PHYS 350, is to move away from thinking about *forces* between charges and currents, and instead think about *fields*:

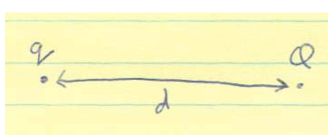
Charges produce electric fields which then act on other charges
Currents produce magnetic fields which then act on other currents

This contrasts with the approach usually taken in introductory courses which is to treat forces between charges and currents directly, e.g. through Coulomb's law. Instead, we think about the fields as physical objects that are sourced by charges and currents and in turn act on charges and currents through the Lorentz force. In this course, we will take this even further by considering electromagnetic waves that are wavelike disturbances in the fields and propagate even in vacuum when no charges or currents are present.

1.2.1 Electrostatics

Electrostatics begins with the observation of the force between two charges, expressed in Coulomb's law

$$F = \frac{qQ}{4\pi\epsilon_0 d^2},$$

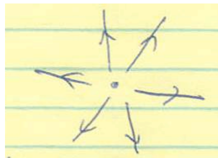


where ϵ_0 is the permittivity of free space.

The electric field is defined as $\mathbf{F} = q\mathbf{E}$ which gives for a point charge

$$\mathbf{E} = \frac{Q}{4\pi\epsilon_0 r^2}\hat{\mathbf{r}}.$$

Because the force between charges is always along the line between the charges, the electric field lines look like



We see that electric field lines "diverge" rather than "loop". Mathematically, this is described by

$$\text{Gauss}' \text{ law} \qquad \nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$$

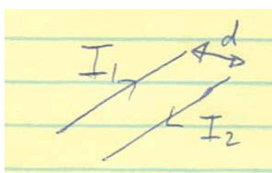
and the constraint

$$\nabla \times \mathbf{E} = 0$$

where ρ is the volume charge density (in Cm^{-3}). This can be summed up as "*electrostatic fields begin and end on charges*". The curl-free nature of $\mathbf{E}$ allows us to define the electrostatic potential through $\mathbf{E} = -\nabla\phi$, which is a useful route to solving for the electrostatic field in many cases.

1.2.2 Magnetostatics

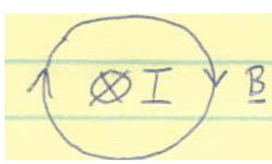
Here we begin with forces between currents, namely that parallel currents attract and oppositely-directed currents repel.



In the picture that each wire produces a magnetic field that acts on the charge carriers in the other wire with the Lorentz force

$$\mathbf{F} = q\left(\mathbf{v} \times \mathbf{B}\right),$$

we can use the observed forces to conclude that the magnetic field of a wire must consist of circular loops around each wire (the right hand rule gives the directions)



The idea that currents source magnetic field loops is expressed in Ampère's law

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$$

and the constraint

$$\nabla \cdot \mathbf{B} = 0$$

where $\mathbf{J}$ is the current density (current per unit area, Am^{-2}) and the constant μ_0 is the permeability of free space.

Some useful numbers:

charge on the electron	$e = -1.6 \times 10^{-19}$ C
	$\frac{1}{4\pi\epsilon_0} = 10^{-7} c^2$
permittivity of free space	$\epsilon_0 = 8.85 \times 10^{-12} \frac{\mathrm{C}^2}{\mathrm{Nm}^2}$
permeability of free space	$\frac{\mu_0}{4\pi} = 10^{-7} \frac{\mathrm{Tm}}{\mathrm{A}}$.

1.2.3 Magnetic induction

Now we move onto non-static fields. Time-dependent magnetic fields generate an electromotive force (emf) and currents. For example, consider a circular wire threaded by a time-dependent magnetic field. The emf is

$$\mathcal{E} = \oint \mathbf{E} \cdot d\mathbf{l} = -\int \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{A} = -\frac{d\Phi_m}{dt}$$

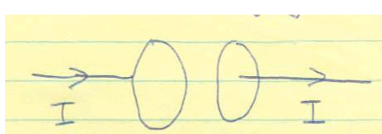
which is Faraday's law (the emf is given by the rate of change of magnetic flux) and Lenz's law (the minus sign). At a local point, we can use Stoke's theorem to write Faraday's law as

$$\nabla \times E = -\frac{\partial \mathbf{B}}{\partial t}.$$

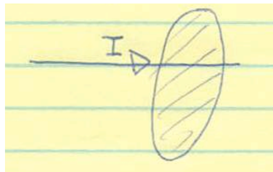
Electrostatic fields are curl free, sourced by charges, but Faraday's law tells us that time-dependent magnetic fields source electric field loops.

1.2.4 Displacement current

Time-dependent $\mathbf{E}$ fields act as a source for $\mathbf{B}$. The standard argument here is to consider charging a capacitor:



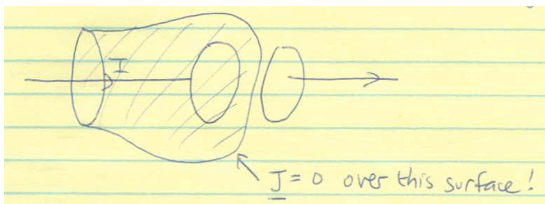
The charge stored is changing at a rate $dQ/dt = I$. Between the plates, $E = Q/\epsilon A$ grows with time, and Ampère's law tells us **B**. For example, if we draw a circular loop around the wire,



Ampere's law gives

$$B2\pi r = \mu_0 I \Rightarrow B = \frac{\mu_0 I}{2\pi r},$$

the standard result for the magnetic field of a wire. But we could have chosen a different area when evaluating Ampere's law:



which is bounded by the same loop, but now passes between the capacitor plates where there is no current! To make sure we get the same answer as before, there must be a displacement current term. By inspection of the final Maxwell equation

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

we see the form that the extra term takes. As well as $\mu_0 \mathbf{J}$, we include a term $\mu_0 \epsilon_0 \partial \mathbf{E}/\partial t$ on the right hand side of Ampere's law. With the displacement current term included for the capacitor, Ampere's law with the integration area that goes between the plates now gives

$$B2\pi r = \int \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \cdot d\mathbf{A} = \mu_0 \frac{dQ}{dt} = \mu_0 I,$$

the correct answer.

There is another argument for the displacement current term based on the symmetry of Maxwell's equations, and we'll come back to that in some depth when we discuss the relation between electromagnetism and relativity.

1.2.5 Maxwell's equations in vacuum and charge conservation

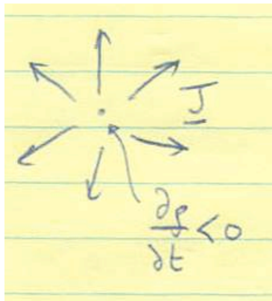
This completes Maxwell's equations in vacuum (we'll think about materials later).

$$\begin{aligned} \nabla \cdot \mathbf{E} &= \frac{\rho}{\epsilon_0} & \nabla \cdot \mathbf{B} &= 0 \\ \nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} & \nabla \times \mathbf{B} &= \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}. \end{aligned} \tag{1.3}$$

All that remains is to add a continuity equation relating $\mathbf{J}$ and ρ,

$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t} \tag{1.4}$$

which says that if the current vectors diverge there must be a local decrease in the charge density with time as the current carries charge away.

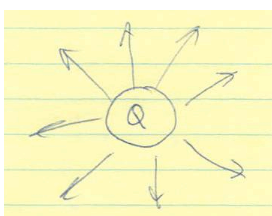


The charge conservation equation (1.4) can also be used to demonstrate that a displacement current term must exist in Ampere's law. Without the displacement current term, $\nabla \times \mathbf{B} = \mu_0 \mathbf{J} \Rightarrow \nabla \cdot \mathbf{J} = 0$ (take the divergence of both sides of Ampere's law). But $\nabla \cdot \mathbf{J} = 0$ holds only in the static case, and so we see that Ampere's law without the displacement current is not consistent with charge conservation for time-dependent situations. Adding the displacement current term makes this consistent

$$\nabla \cdot (\nabla \times \mathbf{B}) = \mu_0 \nabla \cdot \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial}{\partial t} \nabla \cdot \mathbf{E} = \mu_0 \left(\nabla \cdot \mathbf{J} + \frac{\partial \rho}{\partial t} \right) = 0$$

where in the second step we have used Gauss' law.

Feynman[2] has a nice physical version of this argument. Consider a source of radial current, e.g. a sphere of radioactive material that squirts out charged particles.



At radius r, the current density is $J(r)$. Charge conservation requires

$$\frac{\partial Q(r,t)}{\partial t} = -4\pi r^2 J(r),$$

where $Q(r,t)$ is the charge within a radius r. You might think that the current would produce a B field, but in this case if you try to apply Amperes law you will find from the symmetry of the problem that B has to vanish. Each radial stream of current has

[2] Vol. II page 18-3

a loop of field around it according to Ampere's law, which cancels the contribution from neighbouring current lines! So it is impossible to make this case consistent with $\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$. What happens is that the displacement current exactly cancels the $\mathbf{J}$ source term. At radius r, $E(r) = Q/4\pi\epsilon_0 r^2$ so that

$$\frac{\partial E}{\partial t} = \frac{1}{4\pi\epsilon_0 r^2}\frac{\partial Q}{\partial t} = -\frac{J}{\epsilon_0}$$

or

$$\mu_0\epsilon_0\frac{\partial E}{\partial t} = -\mu_0 J \Rightarrow \nabla \times \mathbf{B} = 0$$

and the solution is $B = 0$.

1.2.6 Maxwell's equations in materials

A reminder of how we treat electric and magnetic fields in materials is given in the Appendix. Here, we discuss first what the displacement current term looks like in materials, and then discuss how to derive the boundary conditions on the fields at interfaces between materials (we'll need this when we look at reflection of electromagnetic waves later).

Maxwell's equations in materials without the displacement current term are

$$\nabla \cdot \mathbf{B} = 0 \qquad \nabla \cdot \mathbf{D} = \rho_f$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \qquad \nabla \times \mathbf{H} = J_f$$

(take a look at the Appendix if you need a refresher on the definitions of $\mathbf{D}$ and $\mathbf{H}$). The new piece in time-dependent problems is that a changing polarization with time corresponds to a polarization current

$$J_P = \frac{\partial P}{\partial t}$$

that must be included in Ampere's law. Notice that the bound charge satisfies a continuity equation

$$\nabla \cdot \mathbf{J}_P = \frac{\partial}{\partial t}\nabla \cdot \mathbf{P} = -\frac{\partial \rho_B}{\partial t}.$$

Therefore Ampere's law has a displacement current term as we had in vacuum but also an addition term from polarization current,

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}_f + \mu_0 \mathbf{J}_B + \mu_0\frac{\partial \mathbf{P}}{\partial t} + \mu_0\epsilon_0\frac{\partial \mathbf{E}}{\partial t}.$$

Rewriting this as

$$\nabla \times \left(\frac{\mathbf{B}}{\mu_0} - \mathbf{M}\right) = \mathbf{J}_f + \frac{\partial}{\partial t}\left(\epsilon\mathbf{E} + \mathbf{P}\right)$$

we obtain

$$\nabla \times \mathbf{H} = \mathbf{J_f} + \frac{\partial \mathbf{D}}{\partial t}$$

which completes Maxwell's equations in materials.

Finally, a reminder about boundary conditions for **E**, **B**, **D**, and **H** at the interface between two materials. Recall that we derive boundary conditions by integrating Maxwell's equations across the surface from $-\epsilon$ to $+\epsilon$ and then let $\epsilon \to 0$. For example,

$$\nabla \cdot \mathbf{B} = 0 \Rightarrow \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} = 0.$$

Now imagine we have a surface whose normal vector is in the x-direction (so the surface is in the y-z plane). Integrate across:

$$\int_{-\epsilon}^{\epsilon} dx \left(\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} \right) = 0$$

$$\Rightarrow [B_x]_{-\epsilon}^{\epsilon} + \int_{-\epsilon}^{\epsilon} dx \left(\frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} \right) = 0.$$

The second term vanishes in the limit $\epsilon \to 0$ and so

$$[B_x]_{-\epsilon}^{\epsilon} = 0$$

showing that the perpendicular component of **B** is continuous across the surface.

You've probably seen this derived using a geometric argument, e.g. a Gaussian cylinder shrunk onto the boundary, but I've written it this way because for more complex differential equations, geometric arguments are not always possible. In that case, direct integration will let you derive a boundary condition. I'll leave the other boundary conditions as an exercise. For example, first try integrating $\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$ across the boundary. You should find that the parallel component of **E** is continuous (hint: look for the terms that are $\partial/\partial x$ of something, as they are the terms that will give a non-zero contribution in the limit $\epsilon \to 0$).

1.3 An immediate application: Electromagnetic waves

We can quickly get to electromagnetic waves in a few lines of vector algebra. This is actually a very important procedure that we will carry out later many times for different kinds of materials. In vacuum with no sources $\rho = 0$ and $\mathbf{J} = 0$, Maxwell's equations are[3]

$$\begin{aligned} \nabla \cdot \mathbf{E} &= 0 & \nabla \cdot \mathbf{B} &= 0 \\ \nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} & \nabla \times \mathbf{B} &= \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}. \end{aligned} \tag{1.5}$$

Now take the curl of Faraday's law. The left hand side is

$$\nabla \times (\nabla \times \mathbf{E}) = -\nabla^2 \mathbf{E} + \nabla (\nabla \cdot \mathbf{E})$$

[3]Just a note about the term "vacuum". I will use this term to mean we are outside any materials, so $\epsilon = \epsilon_0$ and $\mu = \mu_0$, but not in the sense of being "empty" space. So in our usage a vacuum can have some charge or current density to source the electromagnetic fields.

and the right hand side is

$$\nabla \times \left(-\frac{\partial \mathbf{B}}{\partial t}\right) = -\frac{\partial}{\partial t}(\nabla \times \mathbf{B}) = -\epsilon_0 \mu_0 \frac{\partial^2 \mathbf{E}}{\partial t^2}$$

which gives an equation governing **E**,

$$\nabla^2 \mathbf{E} = \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2}. \tag{1.6}$$

On the left hand side we used a vector identity to expand $\nabla \times \nabla \times \mathbf{E}$. Identities like this one are readily available, for example at the front of Griffith's book, but they are actually easy to derive for yourself, I've included an appendix to this chapter on that. I encourage you to take a look at it, it could be the most useful thing you learn in this course!

Returning to our result equation (2.27), you may recognize this as a 3D wave equation. In one dimension, a wave equation for quantity $f(x,t)$ is

$$\frac{\partial^2 f}{\partial t^2} = v^2 \frac{\partial^2 f}{\partial x^2}$$

with general solution $f(x,t) = f(x \pm vt)$, where v is the wave speed. Comparing with equation (2.27), we see that the wave speed in the electromagnetic case is

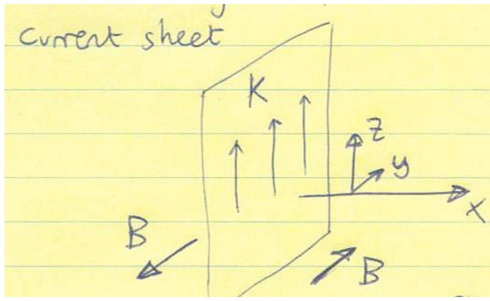
$$v^2 = \frac{1}{\epsilon_0 \mu_0} = \frac{1}{4\pi 10^{-7}} 4\pi 10^{-7} c^2 = c^2$$

the wave speed is the speed of light!

This result looks inevitable because these days we have units (SI) in which μ_0 is defined in terms of c^2, and we know that light is in fact an electromagnetic wave. But in a historical context, this is a truly remarkable result because remember that ϵ_0 is the constant in Coulomb's law, which describes the measured force between two electric charges, and μ_0 is the constant in the Biot-Savart law that describes the measured forces between currents. There is no obvious link to light, and yet from these two observations and the subsequently deduced Maxwell's equations, we predict a wave whose speed in fact equals the measured speed of light! In this way, our understanding of light as an electromagnetic wave came about.

1.4 General solutions to the wave equation

We'll look at properties of electromagnetic waves in the next section, but first consider the following more general example. We have an infinite plane current sheet with surface current K:



The time-independent solution is

$$B = \frac{\mu_0 K}{2}$$

independent of distance x and with a direction as shown in the diagram (right hand rule). But instead of a steady current, imagine the current is instead turned on at $t = 0$. What is the evolution in time?

The symmetry of this problem suggests that we try a solution $\mathbf{B} = B(x)\hat{\mathbf{y}}$ and $\mathbf{E} = E(x)\hat{\mathbf{z}}$. Maxwell's equations are then

$$\frac{\partial B}{\partial t} = \frac{\partial E}{\partial x} \tag{1.7}$$

$$\frac{\partial E}{\partial t} = c^2 \frac{\partial B}{\partial x}, \tag{1.8}$$

and we see that E and B satisfy a wave equation

$$\frac{\partial^2 E}{\partial t^2} = c^2 \frac{\partial^2 E}{\partial x^2} \tag{1.9}$$

and similarly for B. This is, of course, not surprising given our more general derivation of the wave equation earlier, but we choose this simple example because it gives a 1D wave equation that we can analyze.

The general solution of the 1D wave equation (1.9) is

$$f_1(x - ft) + f_2(x + ct)$$

where f_1 is a right-travelling component and f_2 is left travelling. To derive this general solution, we can define coordinates

$$\eta = x - ct \qquad\qquad \xi = x + ct$$

and then equation (1.9) becomes

$$\frac{\partial^2 E}{\partial \eta \partial \xi} \Rightarrow E = f_1(\eta) + f_2(\xi)\,. \tag{1.10}$$

Now consider these two different pieces, the left and right travelling components. If E is a function of $x \pm ct$ only, and independent of $x \mp ct$, then it must be the case that

$$\frac{\partial E}{\partial x} = \pm \frac{1}{c} \frac{\partial E}{\partial t}$$

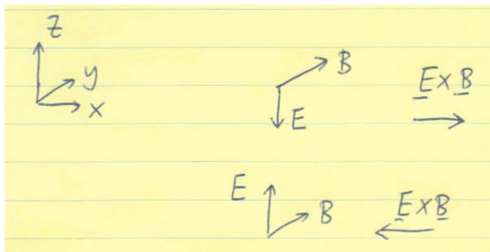
and therefore, using equation (1.7),

$$\frac{\partial B}{\partial t} = \pm \frac{\partial}{\partial t}\left(\frac{E}{c}\right)$$

or

$$B = \pm \frac{E}{c}.$$

The signs are such that the right travelling wave $f(x-ct)$ has $B = -E/c$ and the left going wave $f(x+ct)$ has $B = +E/c$. We can write this as the direction of propagation of the wave is $\propto \mathbf{E} \times \mathbf{B}$.



For a given case, therefore, we can write

$$E(x,t) = f_1(x-ct) + f_2(x+ct)$$

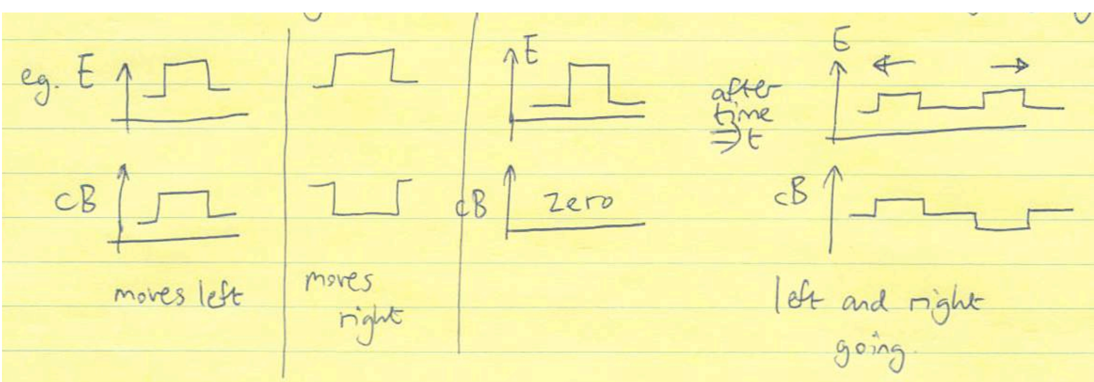
$$cB(x,t) = -f_1(x-ct) + f_2(x+ct)$$

for some choice of the functions f_1 and f_2. These functions are set by the boundary conditions, e.g. at $t = 0$

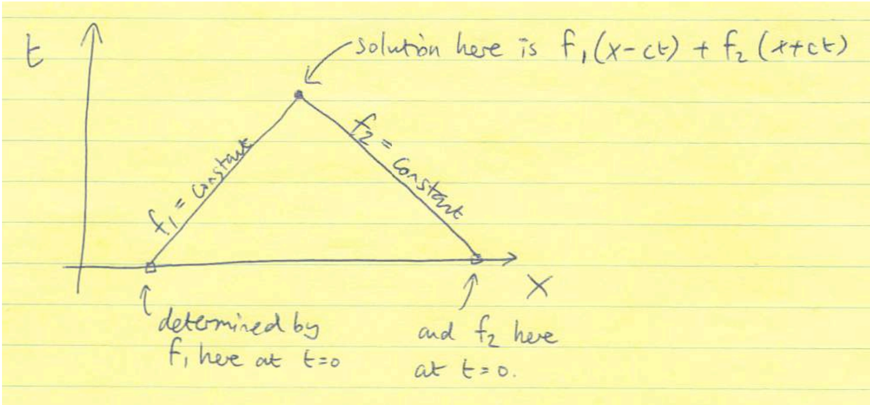
$$f_1(\eta) = \frac{E(x) - cB(x)}{2}$$

$$f_2(\xi) = \frac{E(x) + cB(x)}{2}.$$

This is just saying that if we choose the relative signs of E and B initially, we can send a wave either left or right:

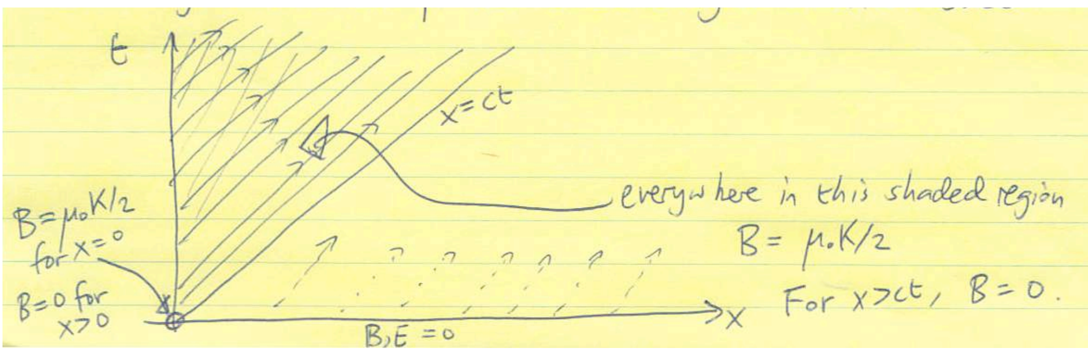


Graphically, the solution propagates from the initial conditions along "characteristic curves" with slope $\pm c$



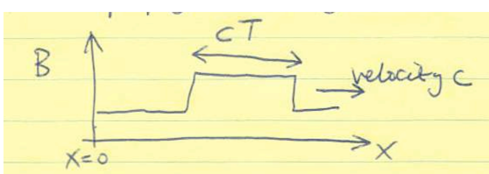
So that the solution at a given x and t is set by the f_1 component of the initial condition at $x - ct$, and the f_2 component of the initial condition at location $x + ct$ as shown in the diagram.

Let's go back now to our problem of switching on a current sheet.



We can see that the solution consists of two parts. One region, at $x > ct$ has $B = 0$ because the solution there is determined by characteristics originating at $x > 0$ and $t = 0$, where $B = 0$. The second region, $x < ct$ has characteristics that originate on the sheet at $x = 0$, where (just above the sheet) $B = \mu_0 K/2$, and so $B = \mu_0 K/2$ in that entire region, as in the static problem. What this is saying is that it takes a light travel time x/c before position x "knows" that the current has been turned on. At late times $> x/c$ the field corresponds to the static solution.

A similar argument gives the solution for the $x < 0$ domain. If, at a later time T, we turned the current off, we create an electromagnetic pulse that propagates away in each direction from the $x = 0$ plane:



We see that not only do the wave solutions propagate at the speed of light, but in general

Electromagnetic disturbances propagate at the speed of light.

1.5 Plane electromagnetic waves in vacuum

In the last section, we saw one method of solving the wave equations for **E** and **B**, the method of characteristics. Because these are linear equations, another approach is to use a Fourier decomposition, in other words think of the fields as a linear sum of *plane waves*

$$\mathbf{E} = \mathbf{E}_0 e^{i\mathbf{k}\cdot\mathbf{r}} e^{-i\omega t}$$

$$\mathbf{B} = \mathbf{B}_0 e^{i\mathbf{k}\cdot\mathbf{r}} e^{-i\omega t}.$$

Recall that when we write the solution as $\propto e^{ikx}$, we really mean the real part $\mathrm{Re}\left(e^{ikx}\right) = \cos kx$. The real part always gives the physical quantities, the complex notation provides a much more convenient way to keep track of the phases.

Substituting the plane wave solution into the wave equation gives

$$-k^2 \mathbf{E}_0 = \frac{-\omega^2}{c^2} \mathbf{E}_0$$

or the the *dispersion relation*

$$\omega = \pm ck,$$

the relation between the wave frequency ω and wave vector **k**. The sign $\pm$ gives the propagation direction of the wave.

Maxwell's equations give us other interesting properties of these waves. Since $\nabla \cdot \mathbf{E} = 0$ and $\nabla \cdot \mathbf{B} = 0$, we see that

$$\mathbf{k} \cdot \mathbf{E}_0 = 0 \qquad\qquad \mathbf{k} \cdot \mathbf{B}_0 = 0$$

so that the wave is *transverse*, meaning that the electric and magnetic fields are perpendicular to the propagation direction. The other two Maxwell equations involving the curl of **E** and **B** give

$$\nabla \times \mathbf{B} = \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} \Rightarrow \mathbf{k} \times \mathbf{B}_0 = -\frac{\omega}{c^2}\mathbf{E}_0$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \Rightarrow \mathbf{k} \times \mathbf{E}_0 = +\omega \mathbf{B}_0$$

which imply

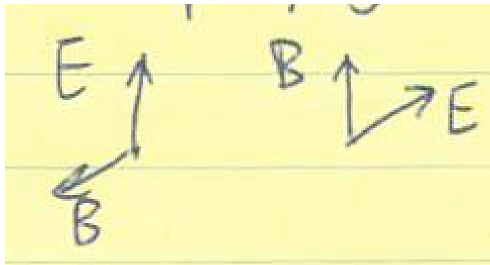
$$\mathbf{E}_0 \cdot \mathbf{B}_0 = 0 \qquad\qquad |\mathbf{E}_0| = c\,|\mathbf{B}_0|\,.$$

The first result you have probably seen before, that the E and B directions are mutually orthogonal in an electromagnetic wave; the second giving the ratio of E and B we have seen before in the previous section.

Let's go through some important results regarding EM waves:

- These plane wave solutions are important because the wave equation is linear so any solution can be written as a linear combination (Fourier expansion) of plane waves.

- Because **E** is perpendicular to the direction of propagation, there are two linearly-independent polarizations



Both cases propagate in the same direction ($\mathbf{E} \times \mathbf{B}$ points in the same direction) but they have orthogonal E's and orthogonal B's.

- The dispersion relation is the familiar relation between the frequency and wavelength of an electromagnetic wave $f = c/\lambda$. It is good to have some sense of the order of magnitude of wavelengths and/or frequencies (I find wavelengths easier to remember) of the different parts of the EM spectrum. I've included a table taken from Pollock & Stump's book on the next page. As we will see later, the physical size of the wavelength is important for example for antenna design or for the scattering efficiency of electromagnetic waves from particles, so it is good to have a sense of the wavelengths in different regions of the EM spectrum.

TABLE 11.3 The electromagnetic spectrum

Frequency (Hz)	Description	Wavelength
10^2	super low frequency (SLF) radio waves	3000 km
	submarine communication	
10^3	ultra low frequency (ULF) radio waves	300 km
10^4	very low frequency (VLF) radio waves	30 km
10^5	low frequency (LF) radio waves	3 km
	marine radio	
10^6	medium frequency (MF) radio waves	300 m
	AM radio is 0.53×10^6 to 1.60×10^6 Hz.	
10^7	high frequency (HF)	30 m
	short-wave radio	
10^8	(VHF) aircraft radio and navigation	3 m
	FM radio is 0.87×10^8 to 1.08×10^8 Hz.	
	TV channels 2–13	
10^9	(UHF) cellular telephones, radar,	30 cm
	microwave ovens, TV channels 14–83	
10^{10}	(SHF) microwaves, radar, mobile radio	3 cm
10^{11}	extremely high frequency (EHF)	3 mm
	Cosmic microwave background	
	maximum is at 3×10^{11} Hz.	
10^{12}	far infrared	0.3 mm
10^{13}	far infrared	30 μm
10^{14}	near infrared	3 μm
	Visible light is 3.9×10^{14} to 7.6×10^{14} Hz.	
10^{15}	near ultraviolet	0.3 μm
10^{16}	vacuum ultraviolet	30 nm
10^{17}	soft X rays	3 nm
10^{18}	soft X rays	0.3 nm
10^{19}	hard X rays	30 pm
10^{20}	gamma rays	3 pm
10^{21}	gamma rays	0.3 pm
10^{22}	cosmic gamma rays	30 fm

1.6 Conservation of energy and momentum

Another interesting application of Maxwell's equations is to help to understand energy conservation in electromagnetism.

1.6.1 Conservation of energy and the Poynting flux

Consider a system of charges and currents. The work done per second by the fields on charge q is

$$\mathbf{F} \cdot \mathbf{v} = q\mathbf{v} \cdot (\mathbf{E} + \mathbf{v} \times \mathbf{B}) = q\mathbf{v} \cdot \mathbf{E}$$

where $\mathbf{F}$ is the Lorentz force, and notice that only electric fields do any work because the magnetic force is always perpendicular to the motion. The work done per second per unit volume is

$$nq\mathbf{v} \cdot \mathbf{E} = \mathbf{J} \cdot \mathbf{E}$$

where n is the number density of charges. This is the work done by the fields, which goes into the kinetic energy of the charges. By conservation of energy, the rate of change of the energy density in the electric and magnetic fields is therefore

$$-\mathbf{J} \cdot \mathbf{E}.$$

Maxwell's equations can be used to evaluate this quantity in terms of the fields

$$-\mathbf{J} \cdot \mathbf{E} = -\frac{1}{\mu_0} (\nabla \times \mathbf{B}) \cdot \mathbf{E} + \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \cdot \mathbf{E}.$$

The last term is

$$\epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \cdot \mathbf{E} = \frac{\partial}{\partial t} \left(\frac{1}{2} \epsilon_0 E^2 \right)$$

which looks promising because recall that $(1/2)\epsilon_0 E^2$ is the energy density in the electric field. To simplify the first term, we can use the identity[4]

$$\nabla \cdot (\mathbf{E} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{E}) - \mathbf{E} \cdot (\nabla \times \mathbf{B})$$

which gives

$$-\mathbf{J} \cdot \mathbf{E} = \nabla \cdot \left(\frac{\mathbf{E} \times \mathbf{B}}{\mu_0} \right) - \frac{1}{\mu_0} \mathbf{B} \cdot \nabla \times \mathbf{E} + \frac{\partial}{\partial t} \left(\frac{1}{2} \epsilon_0 E^2 \right).$$

But $\nabla \times \mathbf{E} = -\partial \mathbf{B} / \partial t$ (Faraday's law), and so

$$-\mathbf{J} \cdot \mathbf{E} = \nabla \cdot \left(\frac{\mathbf{E} \times \mathbf{B}}{\mu_0} \right) + \frac{\partial}{\partial t} \left(\frac{B^2}{2\mu_0} + \frac{\epsilon_0 E^2}{2} \right), \tag{1.11}$$

where again we recognize the energy density in the magnetic field $B^2/2\mu_0$ appearing in the $\partial / \partial t$ term.

Equation (4.131) describes the conservation of energy, and is in so-called flux conservative form. The quantity

$$\frac{B^2}{2\mu_0} + \frac{1}{2} \epsilon_0 E^2$$

[4] You could think of this as an integration by parts.

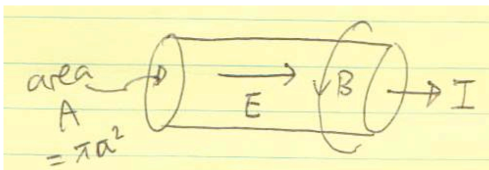
is the energy per unit volume in the fields, and

$$\frac{1}{\mu_0}\mathbf{E} \times \mathbf{B} \equiv \mathbf{S} \tag{1.12}$$

is the energy flux carried by the fields (energy per unit area per second) which is known as the Poynting flux $\mathbf{S}$. (Recall that we already met the vector $\mathbf{E} \times \mathbf{B}$ when talking about the direction of propagation of EM waves). In words, equation (4.131) states that the local rate of change of energy density is given by the divergence of the Poynting flux plus any exchange of energy with the charged particles.

1.6.2 Example: steady current in a wire

A simple example is a steady current in a wire $\mathbf{J} = \sigma \mathbf{E}$ where σ is the electrical conductivity. Steady means that there is no time-dependence, $\partial/\partial t = 0$.



At the surface of the wire $r = a$, the magnetic field is $\mathbf{B} = \hat{\text{Œ}}\mu_0 I/2\pi a$ from Ampère's law, and the electric field points along the wire $\mathbf{E} = E\hat{\mathbf{z}}$. The Poynting flux is therefore directed radially inwards, and is

$$\mathbf{S} = -\hat{\mathbf{r}}\,\frac{IE}{2\pi a}. \tag{1.13}$$

This is the energy per unit area flowing into the wire. Multiplying by the circumference $2\pi a$, we get the energy per unit length per second

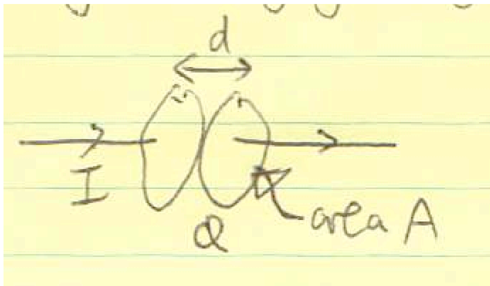
$$2\pi a S = EI = \frac{V}{L}I = I^2\frac{R}{L} \tag{1.14}$$

where V is the voltage difference across length L of the wire, and R is the resistance of length L. So we see that the Poynting flux into the wire is equal to the ohmic dissipation inside the wire.

Not only is the total Poynting flux into the wire at its surface equal to the ohmic dissipation inside, but the Poynting flux changes with radius inside the wire, such that the difference between the Poynting flux at $r + dr$ and that at r matches the ohmic dissipation between r and $r + dr$. I will leave this for you to work out (see the list of problems at the end of the chapter). Here, we just note that the ohmic dissipation per unit volume is I^2R/AL where A is the cross-section of the wire, or $J^2RA/L = J^2/\sigma = \mathbf{J}\cdot\mathbf{E}$.

1.6.3 Example: Charging a cylindrical capacitor

Another typical example to look at is a cylindrical capacitor with charge $Q(t)$ that is charging with a current $I = dQ/dt$.



The electric field between the plates is $E = Q/\epsilon_0 A$, and the total electrostatic energy is

$$U_E = \frac{1}{2}\epsilon_0 E^2 A d = \frac{Q^2}{2\epsilon_0 A} d,$$

changing at a rate

$$\frac{dU_E}{dt} = \frac{d}{\epsilon_0 A} Q \frac{dQ}{dt} = \frac{d\, QI}{\epsilon_0 A}.$$

The point here is that this energy has to come from somewhere, and from our energy conservation law, it must come from a net Poynting flux into the volume between the plates.

To evaluate the Poynting flux, we need the **B** field, which is given between the plates by the displacement current

$$2\pi r B_\phi = \pi r^2 \mu_0 \epsilon_0 \frac{dE}{dt}$$

so that at $r = a$, the Poynting flux is

$$S = \frac{a\epsilon_0}{2}\frac{dE}{dt} E$$

radially inwards. The total flux of energy is

$$2\pi a d\, S = d\pi a^2 \epsilon_0 E \frac{dE}{dt} = d\pi a^2 \epsilon_0 E \,\frac{I}{\epsilon_0 A} = \frac{QId}{\epsilon_0 A}$$

matching the rate of change of electrostatic energy between the plates.

This example is actually a bit subtle because we have made an assumption that the charge is added slowly enough that the quasi-static approximation holds — at each time t, we calculate the electric field as $E = Q/\epsilon_0 A$ just as we would for a capacitor in electrostatics. We ignore the "back emf" or the inductance in this approximation, and therefore the magnetic energy. This is a good approximation as long as the charging timescale is long compared to the light crossing time across the capacitor.

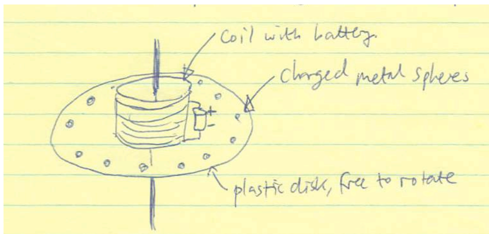
1.6.4 Conservation of momentum

We'll derive this in detail later when we think about relativity, but let's just mention here that as well as an energy flux there is also a momentum flux

$$\text{momentum flux} = \frac{\mathbf{S}}{c} = \frac{1}{\mu_0 c}\mathbf{E} \times \mathbf{B}.$$

To see why it must be S/c, recall that photons are massless particles and therefore have $E = (p^2c^2 + m^2c^4)^{1/2} = pc$, so that the energy and momentum fluxes are simply related by a factor of c.

A famous example of the momentum flux is Feyman's disk paradox.



act in only one way (and that is the *right way*, naturally). So in physics a paradox is only a confusion in our own understanding. Here is our paradox.

Imagine that we construct a device like that shown in Fig. 17–5. There is a thin, circular plastic disc supported on a concentric shaft with excellent bearings, so that it is quite free to rotate. On the disc is a coil of wire in the form of a short solenoid concentric with the axis of rotation. This solenoid carries a steady current I provided by a small battery, also mounted on the disc. Near the edge of the disc and spaced uniformly around its circumference are a number of small metal spheres insulated from each other and from the solenoid by the plastic material of the disc. Each of these small conducting spheres is charged with the same electrostatic charge Q. Everything is quite stationary, and the disc is at rest. Suppose now that by some accident—or by prearrangement—the current in the solenoid is interrupted, without, however, any intervention from the outside. So long as the current continued, there was a magnetic flux through the solenoid more or less parallel to the axis of the disc. When the current is interrupted, this flux must go to zero. There will, therefore, be an electric field induced which will circulate around in circles centered at the axis. The charged spheres on the perimeter of the disc will all experience an electric field tangential to the perimeter of the disc. This electric force is in the same sense for all the charges and so will result in a net torque on the disc. From these arguments we would expect that as the current in the solenoid disappears, the disc would begin to rotate. If we knew the moment of inertia of the disc, the current in the solenoid, and the charges on the small spheres, we could compute the resulting angular velocity.

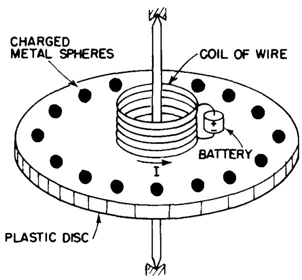


Fig. 17–5. Will the disc rotate if the current I is stopped?

But we could also make a different argument. Using the principle of the conservation of angular momentum, we could say that the angular momentum of the disc with all its equipment is initially zero, and so the angular momentum of the assembly should remain zero. There should be no rotation when the current is stopped. Which argument is correct? Will the disc rotate or will it not? We will leave this question for you to think about.

We should warn you that the correct answer does not depend on any nonessential feature, such as the asymmetric position of a battery, for example. In fact, you can imagine an ideal situation such as the following: The solenoid is made of superconducting wire through which there is a current. After the disc has been carefully placed at rest, the temperature of the solenoid is allowed to rise slowly When the temperature of the wire reaches the transition temperature between superconductivity and normal conductivity, the current in the solenoid will be brought to zero by the resistance of the wire. The flux will, as before, fall to zero, and there will be an electric field around the axis. We should also warn you that the solution is not easy, nor is it a trick. When you figure it out, you will have discovered an important principle of electromagnetism.

17–5 Alternating-current generator

In the remainder of this chapter we apply the principles of Section 17–1 to analyze a number of the phenomena discussed in Chapter 16. We first look in more detail at the alternating-current generator. Such a generator consists basically of a coil of wire rotating in a uniform magnetic field. The same result can also be achieved by a fixed coil in a magnetic field whose direction rotates in the manner described in the last chapter. We will consider only the former case. Suppose we have a circular coil of wire which can be turned on an axis along one of its diameters. Let this coil be located in a uniform magnetic field perpendicular to the axis of rotation, as in Fig. 17–6 We also imagine that the two ends of the coil are brought to external connections through some kind of sliding contacts.

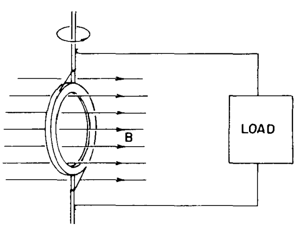


Fig. 17–6. A coil of wire rotating in a uniform magnetic field—the basic idea of the ac generator.

Due to the rotation of the coil, the magnetic flux through it will be changing. The circuit of the coil will therefore have an emf in it. Let S be the area of the coil and θ the angle between the magnetic field and the normal to the plane of the coil.*

* Now that we are using the letter A for the vector potential, we prefer to let S stand for a Surface area.

17–6

Intially, a current flows in the small central coil, with everything stationary. Then the current stops for some reason. In that case, the B field from the central coil goes to zero, and so there is an emf $\mathcal{E} = -d\Phi/dt$ that acts on the charged spheres and causes the disk to rotate.

The "paradox" or puzzle is to ask where does the angular momentum in the disk's rotation come from? Angular momentum should be conserved and yet the system was stationary initially. The answer is that the fields themselves have an angular momentum content in the initial state. As the fields decay, the angular momentum is transferred into the rotational motion of the disk.

1.6.5 Conservation of energy in a material

I'll leave this as an exercise, but if you use the Maxwell's equations for a material to derive an energy equation, you should find that the energy density is

$$\frac{1}{2}\mathbf{E}\cdot\mathbf{D}+\frac{1}{2}\mathbf{B}\cdot\mathbf{H}$$

and the Poynting vector is

$$\mathbf{S}=\mathbf{E}\times\mathbf{H}.$$

1.6.6 Application to electromagnetic waves

- The energy flux in the wave is given by the Poynting vector

  $$|\mathbf{S}|=\frac{|\mathbf{E}\times\mathbf{B}|}{\mu_0}=\frac{E_0B_0}{\mu_0}\cos^2\left(\mathbf{k}\cdot\mathbf{r}-\omega t\right)$$

  The time-averaged intensity is (using the fact that $\langle\cos^2\rangle=1/2$)

  $$\langle S\rangle=\frac{E_0B_0}{2\mu_0}=\frac{cE_0^2}{2\mu_0c^2}=\frac{1}{2}c\epsilon_0E_0^2=\frac{1}{2}c\frac{B_0^2}{\mu_0}=\frac{c}{2}\left(\frac{B^2}{2\mu_0}+\frac{\epsilon_0E_0^2}{2}\right)$$

  this has the expected form (energy flux) = (velocity) x (energy density). Note that the electric and magnetic energy densities contribute equally.

- As we discussed earlier, the momentum flux is $\langle S\rangle/c$ which gives rise to *radiation pressure*. The pressure on an absorbing surface is

  $$\frac{\langle S\rangle}{c}=\frac{1}{2}\epsilon_0E_0^2.$$

1.7 Scalar and vector potentials for time-dependent fields

In electrostatics and magnetostatics, the constraints $\nabla\times\mathbf{E}=0$ and $\nabla\cdot\mathbf{B}=0$ mean that we can write the fields in terms of potentials,

$$\mathbf{E}=-\nabla\phi\qquad\qquad\mathbf{B}=\nabla\times\mathbf{A}\qquad(1.15)$$

so that the fields are completely specified by a scalar potential ϕ in the electrostatic case and by a vector field $\mathbf{A}$ in the magnetic case. It is worth thinking about the number of degrees of freedom needed to specify these fields at each point in space. In the electrostatic case, one number at each point in space (the scalar ϕ) is all that is needed to specify the electric field that nominally has three independent components E_x, E_y and E_z at each point in space. The reason is that the constraint that **E** be curl-free substantially reduces the allowed types of vector field and the independent degrees of freedom at each point.

In the magnetic case, it is a little more complicated because the vector potential is a vector field and so has three components at each point. But the constrained nature of B ($\nabla \cdot \mathbf{B} = 0$) means that only two of the three components are independent, since the curl of **A** corresponds to the physical field. I am free to add any curl-free vector field to **A** without changing the physical field **B**:

$$\mathbf{A} \to \mathbf{A}' = \mathbf{A} + \nabla\lambda$$

$$\nabla \times \mathbf{A}' = \nabla \times \mathbf{A}.$$

I can do this by choosing $\nabla \cdot \mathbf{A}$ to have a specific value which is referred to as "choosing a gauge". For static problems as you may have seen in the past, a useful gauge choice is to set $\nabla \cdot \mathbf{A} = 0$, the *Coulomb gauge*. For time-dependent problems, as we will see, a different gauge choice is usually made.

In the time-dependent case, $\nabla \cdot \mathbf{B} = 0$ still holds and so we can still define a vector potential **A** such that

$$\mathbf{B} = \nabla \times \mathbf{A}. \tag{1.16}$$

Substituting this into Ampere's law gives

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} = -\frac{\partial}{\partial t}\nabla \times \mathbf{A}$$

$$\Rightarrow \nabla \times \left(\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t}\right) = 0,$$

which replaces the constraint $\nabla \times \mathbf{E}$ in electrostatics. This means that we can define a scalar potential ϕ such that

$$\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} = -\nabla\phi$$

or

$$\mathbf{E} = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t}. \tag{1.17}$$

Equations (1.16) and (1.17) give the electric and magnetic fields in the time-dependent case in terms of potentials **A** and ϕ which now depend on both position and time.

As before, there is a gauge choice to be made. The gauge transformation is now

$$\mathbf{A} \to \mathbf{A}' = \mathbf{A} + \nabla\lambda$$

$$\phi \to \phi' = \phi - \frac{\partial \lambda}{\partial t}$$

which you can verify does not change the physical fields **E** and **B**. We could still apply the Coulomb gauge here, but a more convenient choice for time-dependent problems (that simplifies calculations) is to use the *Lorentz gauge*

$$\nabla \cdot \mathbf{A} = -\frac{1}{c^2}\frac{\partial \phi}{\partial t}. \tag{1.18}$$

We see immediately why this is a good choice if we rewrite Maxwell's equations in terms of the potentials **A** and ϕ instead of the fields **E** and **B**.

Start with Faraday's law

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t}.$$

The LHS is

$$\nabla \times \nabla \times \mathbf{A} = \nabla\left(\nabla \cdot \mathbf{A}\right) - \nabla^2 \mathbf{A} = -\frac{1}{c^2}\nabla\frac{\partial \phi}{\partial t} - \nabla^2 \mathbf{A},$$

where we use the gauge choice to replace $\nabla \cdot \mathbf{A}$. The RHS is

$$\mu_0 \mathbf{J} + \frac{1}{c^2}\left[-\frac{\partial}{\partial t}\nabla\phi - \frac{\partial^2 \mathbf{A}}{\partial t^2}\right],$$

which has a term that cancels one of the terms of the LHS, leaving the result

$$\nabla^2 \mathbf{A} - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu_0 \mathbf{J}. \tag{1.19}$$

A similar equation can be derived for ϕ, this time starting from

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} = -\nabla^2\phi - \frac{\partial}{\partial t}\nabla \cdot \mathbf{A} = -\nabla^2\phi + \frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2}$$

so that

$$\nabla^2\phi - \frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} = -\frac{\rho}{\epsilon_0}. \tag{1.20}$$

By choosing the Lorentz gauge, we have obtained separate wave equations[5] for ϕ and **A**, in which ϕ is sourced by ρ and **A** is sourced by **J**. These wave equations will lead us later into general solutions to time-dependent problems in terms of *retarded potentials*.

1.8 Radiation from an accelerated charge

We've already seen how the charge density ρ or current density **J** acts as a source in the wave equations for the potentials ϕ and **A**. Similarly, we saw that changes in the surface current in section 1.4 led to launching of a propagating electromagnetic disturbance. As we will see, the key factor is acceleration of charges:

Accelerating charges radiate

To conclude this first part of the course, let's go through a simple argument that shows why this so. The same argument also gives a simple derivation of Larmor's formula for the power radiated by an accelerating charge. This argument is originally due to J. J. Thompson and is presented in Longair's book *High Energy Astrophysics*.

[5] Notice again the high degree of symmetry between these two equations for ϕ and **A**. We will come back to this in the relativity section where we'll write a single relativistically invariant wave equation for a 4-potential sourced by a 4-current.

Consider instantaneously accelerating a charge for time Δt, changing its velocity by an amount Δv. In a frame moving with the charge initially, it begins to move, to a position $x = (\Delta v)t$ at time t later. For radial distances from the charge $r < ct$, the field lines "know" that the charge has moved and point back to the charge at its present location. But for $r > ct$, the field lines point back to the original charge position (the origin).

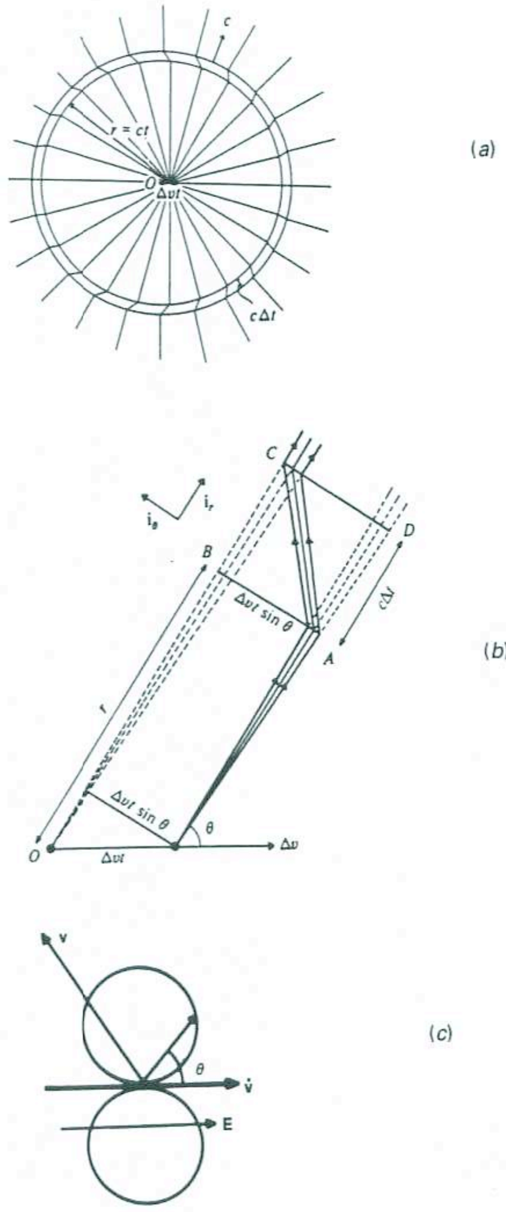
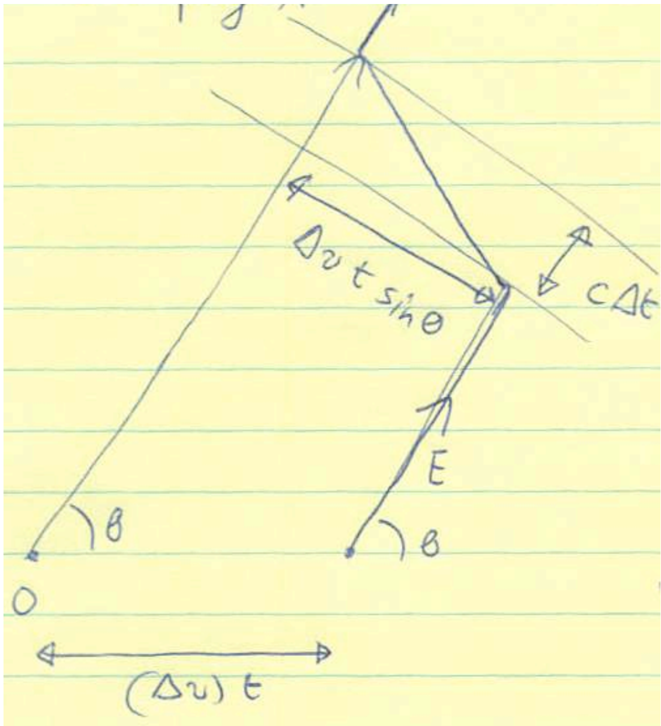


Figure 3.1. (*a*) Illustrating J. J. Thomson's method of demonstrating the radiation of an accelerated charged particle. The diagram shows schematically the configuration of electric field lines at time t due to a charge accelerated to a velocity Δv in time Δt at $t = 0$. (From M. S. Longair (1984). *Theoretical concepts in physics*, page 192, Cambridge: Cambridge University Press.)



The field lines transition between these two behaviours in a small region of size $c\Delta t$ in radius and $(\Delta v)t \sin\theta$ in horizontal extent (see the diagram above). Since $\nabla \cdot \mathbf{E} = 0$ in vacuum, the electric flux $\int \mathbf{E}.d\mathbf{A}$ going vertically into this region must be the same as the horizontal electric flux within the region,

$$E_\theta c\Delta t = E_r(\Delta v)t \sin\theta$$

or

$$E_\theta = E_r \frac{t\Delta v}{c\Delta t} \sin\theta.$$

But for a point charge, $E_r = q/4\pi\epsilon_0 r^2 = q/4\pi\epsilon_0(ct)^2$, and $\Delta v/\Delta t$ is the acceleration a, giving

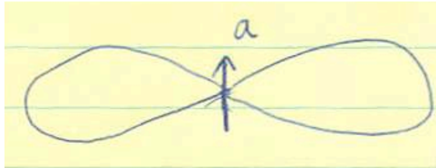
$$E_\theta = \frac{qa \sin\theta}{4\pi\epsilon_0 c^2 r}.$$

A similar argument can be made for $\mathbf{B} = B_\phi \hat{\text{Œ}}$. The result is that an EM wave propagates radially outwards with $E = cB$ and speed c.

The Poynting flux in the transition region where $E_\theta > 0$ is

$$|\mathbf{S}| = \frac{EB}{\mu_0} = \epsilon_0 E^2 c = \frac{(qa)^2 \sin^2\theta}{16\pi^2\epsilon_0 c^3 r^2}. \quad (1.21)$$

Note that $S \propto \sin^2\theta$, so that the radiation pattern is a dipole, with radiation emitted mostly perpendicular to the acceleration.



The total power is

$$\int 2\pi r^2 \sin\theta d\theta \frac{(qa)^2 \sin^2\theta}{16\pi^2\epsilon_0 c^3 r^2}$$

or

$$\text{Power} = \frac{(qa)^2}{6\pi\epsilon_0 c^3}. \tag{1.22}$$

Here, we use the result

$$\int_0^\pi \sin^3\theta d\theta = \int_{-1}^1 (1-\mu^2) d\mu = 2 - \frac{2}{3} = \frac{4}{3}.$$

Equation (4.146) is one of our most important results, giving the power radiated by an accelerated charge. We'll derive it in a more rigorous way later, starting with the wave equation for the vector potential, and use it in many different situations throughout the course.

Note that $E_\theta \propto 1/r$ rather than $1/r^2$ for the Coulomb field. This $1/r$ field is called the radiation field or acceleration field and leads to $S \propto 1/r^2$ or a total power that stays constant as the EM pulse propagates to infinity.

SUMMARY

Here are the main ideas and results that we covered in this part of the course:
Maxwell's equations You should be able to write these down for vacuum and materials. The continuity equation for charges.

Electromagnetic waves. Derivation of the wave equation for electromagnetic waves in vacuum, $\nabla^2\mathbf{E} - \mu_0\epsilon_0\partial^2\mathbf{E}/\partial t^2 = 0$ and the identification of the wave speed $c^2 = 1/\mu_0\epsilon_0$.

Conservation of energy. Energy density in electric and magnetic fields

$$U = \frac{B^2}{2\mu_0} + \frac{1}{2}\epsilon_0 E^2$$

Poynting flux

$$\mathbf{S} = \frac{1}{\mu_0}\mathbf{E} \times \mathbf{B}$$

Energy conservation

$$-\mathbf{J}\cdot\mathbf{E} = \frac{\partial U}{\partial t} + \nabla\cdot\mathbf{S}$$

Examples: Poynting flux into a wire, the energy flow in a charging capacitor.

Conductors. Current in a conductor $\mathbf{J} = \sigma\mathbf{E}$. Ohmic dissipation per unit volume J^2/σ.

General solutions of the wave equation. The idea that electromagnetic disturbances propagate at the speed of light. The general solution of the wave equation $\mathbf{E}(x \pm ct)$ with $B = \pm E/c$.

Plane EM waves in vacuum.

$$\mathbf{E} = \mathbf{E}_0 e^{i\mathbf{k}\cdot\mathbf{r}} e^{-i\omega t}$$

Dispersion relation $\omega = \pm ck$. The wave is transverse $\mathbf{k}\cdot\mathbf{E}_0 = 0$, $\mathbf{k}\cdot\mathbf{B}_0 = 0$. Time averaged intensity is $\langle S\rangle = c\epsilon_0 E^2/2$. Electric and magnetic energy densities contribute equally. Momentum flux $\langle S\rangle/c$. Two linearly-independent polarizations.

Scalar and vector potentials. Gauge transformations in the time-dependent case. The Coulomb gauge $\nabla\cdot\mathbf{A} = 0$ and Lorentz gauge $\nabla\cdot\mathbf{A} = -(1/c^2)(\partial\phi/\partial t)$. Wave equations for the potentials

$$\nabla^2\mathbf{A} - \frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2} = -\mu_0\mathbf{J}$$

$$\nabla^2\phi - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = -\frac{\rho}{\epsilon_0}$$

Materials. Free and bound currents and charge densities and how they relate to the polarization $\mathbf{P}$ or $\mathbf{M}$. LIH dielectrics and the relations between $\mathbf{D}$, $\mathbf{P}$, $\mathbf{E}$, ϵ, χ_e etc. and the same for magnetic fields. The energy density for a LIH dielectric

$$\frac{1}{2}\mathbf{E}\cdot\mathbf{D} + \frac{1}{2}\mathbf{B}\cdot\mathbf{H}$$

Boundary conditions. The general technique of integrating the differential equation across the boundary to derive boundary conditions at an interface. Boundary conditions for time-dependent problems. Continuity of $E_\parallel$ and $B_\perp$, how the change in $D_\perp$ and $H_\parallel$ depend on the free surface charge density and current density respectively.

You should be able to:

- Use Maxwell's equations to derive the wave equations for the fields $\mathbf{E}$ and $\mathbf{B}$, or the potentials $\mathbf{A}$ and ϕ, in vacuum. A key vector identity is $\nabla\times\nabla\times\mathbf{A} = -\nabla^2\mathbf{A}$ for a divergence-free field. You should probably just know this.
- Know how to obtain the fields $\mathbf{E}$ and $\mathbf{B}$ from the potentials $\mathbf{A}$ and ϕ in a time-dependent context.
- Be able to describe the concepts of gauge choice (including the differences between Lorentz and Coulomb gauge)
- Write down the energy flux and energy density of an EM wave in terms of the electric and magnetic field strengths.
- Evaluate the Poynting flux and use it to determine the energy flux, momentum flux, or momentum density (linear or angular momentum) in the fields and talk about the energy flow.

Appendix A: Index notation and proving vector identities

Proving vector identities is very straightforward. You just need four things:

1. Einstein summation convention $\mathbf{A} \cdot \mathbf{B} = A_i B_i$
2. The Kronecker delta: $\delta_{ij} = 1$ if $i = j$, or 0 otherwise. E.g., this allows the dot product to be written $\mathbf{A} \cdot \mathbf{B} = \delta_{ij} A_i B_j$.
3. The Levi-Civita tensor: $\epsilon_{ijk} = 1$ if ijk is an even permutation (123,231,312), $\epsilon_{ijk} = -1$ if ijk is an odd permutation (321,213,132), and 0 otherwise (if any indices are repeated). A way to represent cross-products, e.g. $(\mathbf{A} \times \mathbf{B})_i = \epsilon_{ijk} A_j B_k$.
4. The identity $\epsilon_{ijk}\epsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$.

Examples:

1. Proof of the "BAC-CAB" rule for double cross products.

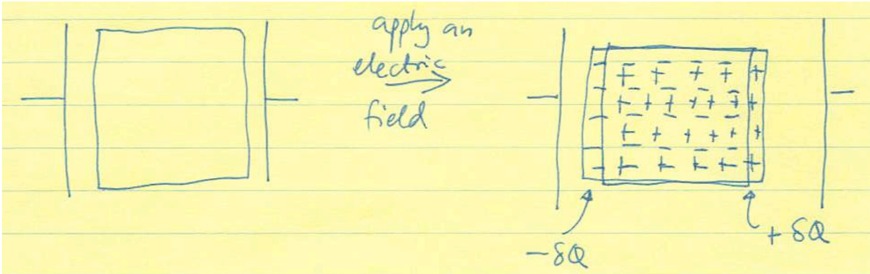
$$\begin{aligned} [\mathbf{A} \times (\mathbf{B} \times \mathbf{C})]_i &= \epsilon_{ijk} A_j \epsilon_{klm} B_l C_m \\ &= \epsilon_{ijk}\epsilon_{klm} A_j B_l C_m \\ &= \left(\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}\right) A_j B_l C_m \\ &= A_j B_i C_j - A_j B_j C_i \\ &= \left[\mathbf{B}\left(\mathbf{A} \cdot \mathbf{C}\right) - \mathbf{C}\left(\mathbf{A} \cdot \mathbf{B}\right)\right]_i \end{aligned}$$

2. An example with a derivative

$$\begin{aligned} [\mathbf{u} \times (\nabla \times \mathbf{u})]_i &= \epsilon_{ijk} u_j \epsilon_{klm} \partial_l (u_m) \\ &= u_j \partial_i u_j - u_j \partial_j u_i \\ &= \left[\nabla \frac{1}{2} u^2 - \mathbf{u} \cdot \nabla \mathbf{u}\right]_i \end{aligned} \tag{1.23}$$

Appendix B: A reminder about magnetic and electric fields in materials

We give a reminder here about the definitions of the fields $\mathbf{D}$ and $\mathbf{H}$ etc. in materials. First, consider electric fields. In response to an applied electric field, a dielectric becomes polarized. The polarization field $\mathbf{P}(\mathbf{r})$ is the local dipole moment density (dipole moment per unit volume). For example, consider a dielectric inserted into a plane-parallel capacitor



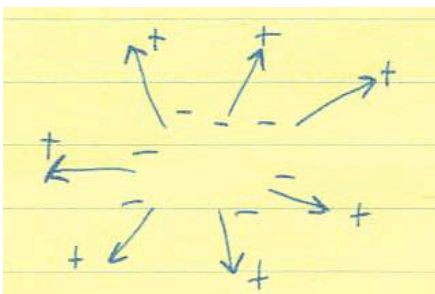
In this case,

$$P = \frac{\delta Q \delta l}{A \delta l} = \frac{\delta Q}{A},$$

where A is the area of the capacitor plates and δl is the small thickness of the layer of bound charges on either side of the dielectric. The general result is that there is a bound surface charge on a polarized dielectric given by

$$\sigma_B = \mathbf{P} \cdot \hat{\mathbf{n}}.$$

More generally still, inside a dielectric, if P has a divergence, then there will be a local bound charge:



The bound charge density is

$$\rho_B = -\nabla \cdot \mathbf{P}.$$

Gauss' law is

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} = \frac{\rho_f + \rho_B}{\epsilon_0}$$

where subscripts f and B refer to free and bound charges respectively. Writing in terms of P,

$$\epsilon_0 \nabla \cdot \mathbf{E} = \rho_f - \nabla \cdot \mathbf{P}$$

$$\Rightarrow \nabla \cdot (\epsilon_0 \mathbf{E} + \mathbf{P}) = \rho_f$$

which we write as

$$\nabla \cdot \mathbf{D} = \rho_f$$

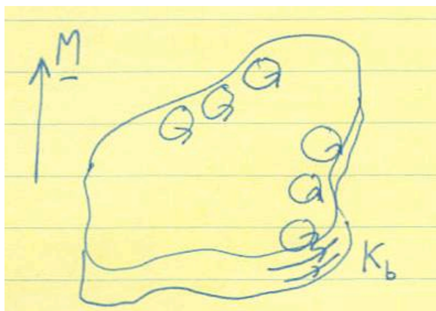
defining the electric displacement field $\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P}$. This is very powerful because we can solve Gauss' law knowing only the free charge density, we don't need to know what is happening in the dielectric beyond a "constitutive relation" between $\mathbf{P}$ and $\mathbf{E}$.

The simplest case is the linear, isotropic, homogeneous (LIH) dielectric for which $\mathbf{P} = \chi_e \epsilon_0 \mathbf{E}$ where χ_e is the susceptibility that measures the polarization of the material in response to an applied electric field. Then

$$\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P} = \epsilon_0 \left(1 + \chi_e\right) \mathbf{E} = \epsilon \mathbf{E} = \epsilon_r \epsilon_0 \mathbf{E}$$

where ϵ is the permittivity and ϵ_r the relative permittivity or dielectric constant. More complicated materials could have a non-linear relation between **P** and **E**, or an anisotropic response in which **P** points in a different direction to **E**, and χ_e is a tensor rather than a scalar, but we won't consider such cases here.

We follow the same approach for magnetic materials, defining **M**, the magnetic dipole moment density. First consider an example where **M** is constant within the material,



If we write the local magnetic dipole moment as $m = Ia$ (I is the current and a the area of each current loop), the dipole moment density is $M = Ia/ad = I/d$ where d is the thickness of the material. Inside the material the bound current loops cancel one another, but at the surface there is a bound surface current $K = I/d = M$, or for the general case $\mathbf{K}_b = \mathbf{M} \times \hat{\mathbf{n}}$. If **M** is non-uniform within the material, there are bound volume currents also,

$$\mathbf{J}_b = \nabla \times \mathbf{M}.$$

Again, we define a new field **H** so that we only have to worry about the free currents rather than the bound ones. We write for magnetostatics

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} = \mu_0 (J_f + J_B) = \mu_0 \left(J_f + \nabla \times \mathbf{M}\right)$$

$$\Rightarrow \nabla \times \left(\frac{\mathbf{B}}{\mu_0} - \mathbf{M}\right) = \mathbf{J}_f$$

or

$$\nabla \times \mathbf{H} = \mathbf{J_f}$$

which defines $\mathbf{H} = \mathbf{B}/\mu_0 - \mathbf{M}$.

For a linear magnetic material, $\mathbf{M} = \chi_m \mathbf{H}$, where χ_m is the magnetic susceptibility. Then

$$\mathbf{B} = \mu_0 (\mathbf{H} + \mathbf{M}) = \mu_0 (1 + \chi_m) \mathbf{H} = \mu \mathbf{H},$$

where μ is the permeability of the material.

Appendix C: Phase and group velocities

The phase velocity of a wave is $v_p = \omega/k$. To see why this is the case, write the wave with frequency $\omega(k)$ as

$$Ae^{i(kx-\omega t)} = Ae^{ik\left(x-v_p t\right)}$$

so that in a frame moving with velocity v_p the wave will be stationary.

When the phase velocity is a function of frequency, the different frequency components of a wave-packet will move at different speeds. The group velocity $v_g = d\omega/dk$ gives the speed at which a wave packet propagates. To see this, we write a general expression for a wavepacket

$$A(x,t) = \int dkA(k)e^{i(kx-\omega(k)t)} = \int dkA(k)e^{i\phi(k)}$$

and the argument is that the maximal amplitude will come from the part of the integrand where the phase varies slowly with k, otherwise the integrand for different values of k will cancel out. Setting $\partial\phi/\partial k = 0$ gives

$$x = \frac{\partial\omega}{\partial k}t = v_g t$$

implying that the location of maximal amplitude is moving with a velocity v_g.

Part 2: Electromagnetic Waves in Materials

These are notes for the second part of PHYS 352 Electromagnetic Waves. In Part 1, we looked at wave solutions to Maxwell's equations and the properties of electromagnetic waves in vacuum. Next, we will consider wave propagation in materials. In a material, the electric field of the wave can induce charge and current densities that feedback on the wave through the source terms in Maxwell's equations. For example, this gives rise to the slower speed of light in glass as compared to air that leads to familiar optics effects such as refraction.

We start by looking at materials where the electrons are free to move (conductors and plasmas) (§2.1), and then consider the bound electrons in dielectrics (§2.2). We'll then calculate what happens at the interface between two materials and recover some standard results from optics (§2.3). Finally, we'll use Larmor's formula to derive the scattering cross-section for photons from single electrons (§2.4).

2.1 Electromagnetic waves in plasmas and conductors

We begin by considering materials where the electrons are free to move, such as a conductor or a plasma[6]. We have already seen the relation $\mathbf{J} = \sigma \mathbf{E}$ for a conductor, where σ is the electrical conductivity. This describes how a current arises in a conductor as the conduction electrons respond to an applied electric field. But the response of a material to an applied time-dependent electric field may be more complicated, and in particular can be out of phase with the applied electric field. An example is a dilute (low density) plasma in which collisions between particles are not important, in which case the current is 90 degrees out of phase with the electric field, and the effective conductivity is imaginary.

In general, the relation between the electric field and the current can be both complex (in phase and out of phase components) and frequency-dependent, so we can write

$$\mathbf{J} = \sigma(\omega)\, \mathbf{E}, \tag{2.24}$$

where σ is a complex number that depends on frequency. The plan in this section is to first derive the dispersion relation in the presence of this current, derive the form of σ for conductors and plasmas, and then consider the resulting wave properties in plasmas and conductors.

2.1.1 Dispersion relation in a material with free electrons

To see how the presence of a current density $\mathbf{J}$ changes the dispersion relation, we derive the wave equation following the same procedure as for EM waves in a vacuum. Start with Ampere's law,

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} = \mu_0 \sigma \mathbf{E} + \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t}. \tag{2.25}$$

[6] A plasma is a high temperature gas in which the atoms are fully or partially ionized, giving a gas of positive ions and free electrons.

Taking the curl and using the identity $\nabla \times (\nabla \times \mathbf{B}) = -\nabla^2\mathbf{B} + \nabla(\nabla \cdot \mathbf{B}) = -\nabla^2\mathbf{B}$ gives

$$-\nabla^2\mathbf{B} = \mu_0\sigma\nabla \times \mathbf{E} + \frac{1}{c^2}\frac{\partial}{\partial t}\nabla \times \mathbf{E}, \tag{2.26}$$

or

$$-\nabla^2\mathbf{B} = -\mu_0\sigma\frac{\partial \mathbf{B}}{\partial t} - \frac{1}{c^2}\frac{\partial^2\mathbf{B}}{\partial t^2}, \tag{2.27}$$

where we use Faraday's law $\nabla \times \mathbf{E} = -\partial\mathbf{B}/\partial t$.

To obtain a dispersion relation, we consider plane waves, i.e. $\mathbf{B} = \mathbf{B}_0 e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}$. Then equation (2.27) gives

$$k^2 = i\omega\mu_0\sigma + \frac{\omega^2}{c^2}. \tag{2.28}$$

The dispersion relation has an extra term compared to the dispersion relation in vacuum ($\omega^2 = c^2k^2$), that as we will see gives rise to some interesting effects. First, let's go over how to derive $\sigma(\omega)$.

2.1.2 The electron equation of motion and the relation between J and E

For a conductor, we have already seen the relation $\mathbf{J} = \sigma\mathbf{E}$, where the real constant σ is the electrical conductivity. Where does this come from? The microscopic picture is that the electrons are moving at their *terminal velocity*. They are continuously accelerated by the applied electric field, but quickly collide (for example, with the atoms in the metal) resetting their velocity to zero. The net effect is that they develop a drift velocity v given by

$$v \approx -\frac{eE}{m}\tau \tag{2.29}$$

where the acceleration is $-eE/m$ and τ is the time between collisions. The current is then $J = -nev$ where n is the number density of electrons, or

$$J = -nev = \frac{ne^2\tau}{m}E = \sigma E \tag{2.30}$$

which gives σ in terms of the electron number density and collision time.

In the context of EM waves, the **J–E** relation for a conductor is the one we should use when the electrons have many collisions during one wave period. But in a dilute plasma, the collision time is much longer than a wave period, so that the electric field from the wave accelerates the electrons freely, without any collisions occurring. In that case, the equation of motion of an electron in the plasma is

$$\frac{dv}{dt} = -\frac{eE}{m} \tag{2.31}$$

or, because all quantities are $\propto e^{-i\omega t}$ so that $dv/dt = -i\omega v$,

$$-i\omega v = -\frac{eE}{m} \tag{2.32}$$

giving a current density

$$\mathbf{J} = -ne\mathbf{v} = i\frac{ne^2}{m\omega}\mathbf{E}. \tag{2.33}$$

Comparing with equation (2.30), we see that the units are right (we've replaced τ with $1/\omega$), but now the conductivity is imaginary, meaning that the current is 90 degrees out of phase with the electric field, and frequency-dependent.

These are in fact two limiting cases, and it is possible to write down a more general expression which takes into account both the acceleration of the electrons and the drag force from collisions. For an example, see the 2012 Midterm question 2.

2.1.3 EM waves in a plasma

Now let's look at the properties of the EM waves. We consider the plasma case first, as it is a bit more straightforward. Inserting the conductivity $\sigma = ine^2/m\omega$ into equation (2.28) gives

$$k^2 = -\frac{\mu_0 ne^2}{m} + \frac{\omega^2}{c^2} \tag{2.34}$$

or

$$\frac{k^2c^2}{\omega^2} = 1 - \frac{\omega_p^2}{\omega^2} \tag{2.35}$$

where we have defined the *plasma frequency* ω_p given by

$$\omega_p^2 = \frac{ne^2}{\epsilon_0 m}.$$

For high frequencies $\omega \gg \omega_p$, the dispersion relation is the same as the vacuum case $\omega^2 = k^2c^2$. A way to think about this is that the wave frequency is so large that the electrons do not have time to respond to the wave, so the wave propagates as if in vacuum. At low frequencies, we see a new effect: when $\omega < \omega_p$, then $k^2 < 0$ and the wave no longer oscillates but *evanesces*. The wave does not propagate because the electrons are able to move and short out the electric field.

Numerically, the plasma frequency is

$$f_p = \frac{\omega_p}{2\pi} = 9\ \mathrm{kHz}\left(\frac{n}{\mathrm{cm}^{-3}}\right)^{1/2}. \tag{2.36}$$

In the Earth's ionosphere, the electron number density is $n \sim 10^4$–$10^5\ \mathrm{cm}^{-3}$, giving $f_p \sim 1$–3 MHz. Waves below this frequency cannot propagate in the ionosphere and are reflected back to Earth. For low frequency waves with $\omega \ll \omega_p$, the decay length in the evanescent region is $2\pi/ik = c/f_p \approx 100$ m for the ionosphere. Reflection of waves by the ionosphere is crucial for radio wave propagation in the Earth's atmosphere, allowing communication over large distances.

Faraday's law gives $\omega\mathbf{B}_0 = \mathbf{k} \times \mathbf{E}_0$ so we see that if the wave is evanescent (k is imaginary), then E and B are 90 degrees out of phase with each other. The Poynting vector is then

$$|\mathbf{S}| = \frac{|\mathbf{B}_0|\,|\mathbf{E}_0|}{\mu_0}\sin\left(-\omega t\right)\cos\left(-\omega t\right)e^{-2k_I x} \tag{2.37}$$

which has a vanishing time-average. This implies that the evanescent wave carries no energy into the plasma. An incident wave is reflected from a plasma when $\omega < \omega_p$.

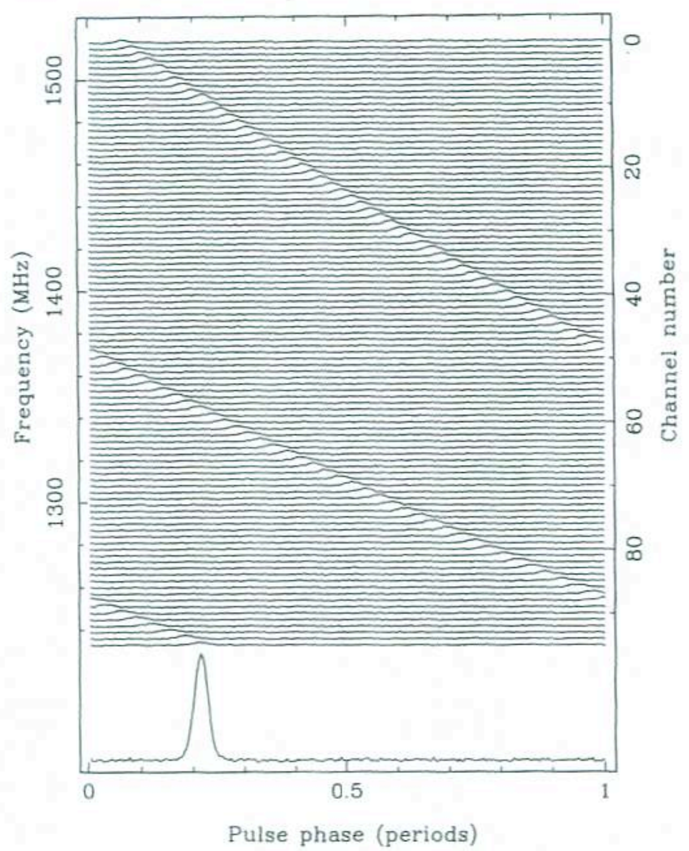
From the dispersion relation, we see that the phase and group velocities of propagating waves depend on ω: the waves are *dispersive*. As you can confirm from the dispersion relation, the velocities are

$$v_p = \frac{\omega}{k} = \frac{c}{n} > c \qquad v_g = \frac{\partial \omega}{\partial k} = cn < c \tag{2.38}$$

where the refractive index of the plasma n is given by

$$n^2 = 1 - \left(\frac{\omega_p}{\omega}\right)^2. \tag{2.39}$$

A famous example of this from astrophysics is that the radio pulses from pulsars arrive later at lower radio frequencies, with a characteristic quadratic dependence of arrival time on frequency (see the figure for an example). The delay is due to the fact that the pulses travel through ionized gas in our galaxy before reaching Earth.



radio pulsar arrival time depends on frequency.

Fig. 3.1. Frequency dispersion in pulse arrival time for PSR B1641−45, recorded in 96 adjacent frequency channels, each 3 MHz wide, centred on 1380 MHz.

2.1.4 EM waves in a conductor

In a conductor, we use a constant conductivity σ in equation (2.28), giving the dispersion relation for a conductor

$$\frac{k^2c^2}{\omega^2} = 1 + \frac{i\sigma}{\epsilon_0\omega}. \tag{2.40}$$

This is a bit more difficult to deal with because the wavevector now has both real and imaginary parts (in the plasma case, k was either pure real or pure imaginary). To find the real and imaginary parts, we write explicitly $k = k_R + ik_I$ and substitute that into the dispersion relation. Equating real and imaginary parts on both sides of the equation, we find

$$k_R^2 - k_I^2 = \frac{\omega^2}{c^2} \tag{2.41}$$

and

$$2k_Rk_I = \frac{\sigma\omega}{c^2\epsilon_0} = \mu_0\sigma\omega \tag{2.42}$$

which can be solved to find k_R and k_I.

A useful limit is the "good conductor" limit $\sigma \gg \epsilon_0\omega$. This limit corresponds to the term $J = \sigma E$ being much greater than the displacement current term $\epsilon_0\partial E/\partial t$ in Ampere's law. In that case, the solution to equations (2.41) and (2.42) is

$$k_R = k_I = \frac{1}{c}\left(\frac{\sigma\omega}{2\epsilon_0}\right)^{1/2}, \tag{2.43}$$

(It is instructive to derive this result directly from Maxwell's equations by dropping the displacement current term from Ampere's law before deriving the dispersion relation for the waves). The complex k vector can be written

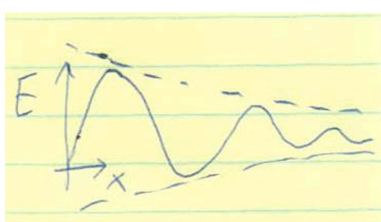
$$k = \left(\frac{1+i}{\sqrt{2}}\right)(\mu_0\sigma\omega)^{1/2} = e^{i\pi/4}(\mu_0\sigma\omega)^{1/2}. \tag{2.44}$$

The phase velocity of the wave is $v_p = c(2\omega\epsilon_0/\sigma)^{1/2} \ll c$ and group velocity is $v_g = 2v_p$.

Most importantly, we see that the wave amplitude decays $\mathbf{E} \propto e^{-k_I x} \propto e^{-x/\delta}$ where

$$\delta = \frac{1}{k_I} = \left(\frac{2}{\mu_0\sigma\omega}\right)^{1/2} \tag{2.45}$$

is the *skin depth* of the conductor at that frequency.



Numerically, for a typical conductor, $\sigma \sim 10^8$ in SI units, giving $\sigma/\epsilon_0\omega \approx 10^{18}$ Hz$/f \approx (\lambda/3 \times 10^{-10}$ m$)$. The skin depth is then $\delta \approx 0.05$ m$(f/\text{Hz})^{-1/2}$. For example, optical light with $\lambda \approx 0.5\ \mu$m has $f \approx 6 \times 10^{14}$ Hz and $\sigma/\epsilon_0\omega \gtrsim 1000$, $\delta \approx 2 \times 10^{-9}$ m. Metal shielding is often used to screen experimental apparatus from stray radiation fields.

An instructive exercise is to calculate the Poynting flux of the evanescent wave in the conductor. If x is the distance along the propagation direction, you should find

$$\langle S \rangle = \frac{E_0^2}{2} \left(\frac{\sigma}{2\mu_0\omega} \right)^{1/2} e^{-2k_I x} \tag{2.46}$$

giving

$$\frac{d\langle S \rangle}{dx} = -\frac{E_0^2 \sigma}{2} e^{-2k_I x} = -\langle \frac{J^2}{\sigma} \rangle, \tag{2.47}$$

a beautiful result that tells you what happens to the energy in the decaying wave.

2.2 Bound electrons

We now consider materials with bound electrons, i.e. insulators. We know how to write down Maxwell's equations for linear, isotropic, homogeneous (LIH) dielectrics, so we use that as a starting point.

2.2.1 EM waves in LIH Dielectrics

We know from our discussion of materials with free electrons that the response of the material depends on the frequency of the wave. At low frequencies, the response should be the same as the static response, i.e. we can write the polarizations as $\mathbf{P} = \chi_e \epsilon_0 \mathbf{E}$ or $\mathbf{M} = \chi_m \mathbf{H}$ and therefore $\mathbf{D} = \epsilon \mathbf{E}$ and $\mathbf{H} = \mathbf{B}/\mu$ in Maxwell's equations. The resulting equations are the same as for the vacuum case but with $\epsilon_0 \to \epsilon$ and $\mu_0 \to \mu$.

Without doing any calculations we can see then that the wave speed in the LIH case will be given by

$$c'^2 = \frac{1}{\mu\epsilon} = \frac{c^2}{\epsilon_r \mu_r}. \tag{2.48}$$

EM waves propagate more slowly in the material than in vacuum by a factor of

$$n = \frac{c}{c'} = \sqrt{\epsilon_r \mu_r} \tag{2.49}$$

which is the *refractive index* of the material. The dispersion relation for the waves is $\omega = c'k = ck/n$. In practise, μ_r is very close to 1, and so $n \approx \sqrt{\epsilon_r}$. For example, glass has $n \approx 1.5$.

All of this should break down at high enough wave frequencies that the response time of the material becomes important. We consider a simple model for this next.

2.2.2 The Lorentz dielectric

The Lorentz dielectric is simple model of a dielectric in which the electron's motion is treated as a forced, damped, simple harmonic oscillator. The bound nature of the electron is included by putting the electron in a harmonic potential (hence its simple harmonic motion). The model also includes the drag that leads to the conductivity of a conductor, and includes the acceleration term that gives the out-of-phase response of a plasma. This makes it very interesting to study as an "all in one" model of the response of a material to an EM wave.

The equation of motion of the electron in this case is

$$-\omega^2\mathbf{x} + \omega_0^2\mathbf{x} - i\omega\gamma\mathbf{x} = \frac{-e\mathbf{E}}{m} \tag{2.50}$$

where $\mathbf{x}$ is the displacement of an electron. The oscillator is described by its characteristic frequency ω_0 (which tells you about the confining potential) and the damping rate γ (for example the collision rate of the electron with atoms in the material). The electron's motion is forced by the applied electric field which has amplitude $\mathbf{E}_0$ and frequency ω. Assuming that the electron displacement takes the form $\mathbf{x} = \mathbf{x_0}e^{-i\omega t}$, equation (2.50) gives

$$\mathbf{x_0} = \frac{e\mathbf{E_0}/m}{\omega^2 - \omega_0^2 + i\omega\gamma} \tag{2.51}$$

for the amplitude of the electron motion.

There are two ways to incorporate this into Maxwell's equation and derive the dispersion relation (both give the same answer). The first is to write down the polarization of the material per unit volume

$$\mathbf{P} = -ne\mathbf{x} \tag{2.52}$$

and insert a term $\partial\mathbf{P}/\partial t = -i\omega\mathbf{P}$ into Ampère's law. Alternatively, we can write the current density

$$\mathbf{J} = -ne\mathbf{v} = i\omega ne\mathbf{x}, \tag{2.53}$$

which allows us to use the dispersion relation we derived above for a current $\mathbf{J} = \sigma\mathbf{E}$. We'll follow this second approach here because it allows us to check different limits of the conductivity and see the connection to the plasma and conductor from the previous section.

Using equation (2.51), the current density associated with the electron motion is

$$\mathbf{J} = \frac{i\omega ne^2\mathbf{E}/m}{\omega^2 - \omega_0^2 + i\omega\gamma}. \tag{2.54}$$

In terms of the plasma frequency $\omega_p^2 = ne^2/m\epsilon_0$, we can write

$$\mathbf{J} = \sigma(\omega)\,\mathbf{E} = \frac{i\omega\omega_p^2\epsilon_0}{\omega^2 - \omega_0^2 + i\omega\gamma}\mathbf{E}. \tag{2.55}$$

It is worth checking that this formula has the appropriate limits. For unbound electrons $\omega_0 = 0$ and a large collision rate $\omega \ll \gamma$, we get $\sigma = \omega_p^2 \epsilon_0/\gamma = ne^2/m\gamma$ which is the conductivity of a conductor if $\gamma = 1/\tau$. For unbound electrons ($\omega_0 = 0$) with no damping ($\gamma = 0$), we obtain $\mathbf{J} = i(\omega_p^2\epsilon_0/\omega)\mathbf{E} = i(ne^2/m\omega)\mathbf{E}$ which is the relation for a plasma.

Having obtained an expression for $\sigma(\omega)$, equation (2.28) gives us the dispersion relation. We can write this as a permittivity $\epsilon/\epsilon_0 = n^2 = c^2k^2/\omega^2$, or

$$\epsilon(\omega) = \epsilon_0 \left[1 - \frac{\omega_p^2}{\omega^2 - \omega_0^2 + i\omega\gamma}\right], \tag{2.56}$$

or as a susceptibility $\chi(t)$ given by $\epsilon = \epsilon_0(1+\chi)$ as

$$\chi(\omega) = \frac{\omega_p^2}{\omega_0^2 - \omega^2 + i\omega\gamma}. \tag{2.57}$$

A new limit that we haven't encountered previously is to take $\omega \to 0$. Then

$$\epsilon \to \epsilon_0 + \frac{\omega_p^2\epsilon_0}{\omega_0^2} = \epsilon_0 + \frac{ne^2}{m\omega_0^2} \tag{2.58}$$

which we interpret as the static dielectric constant $\epsilon(0)$. We now have a way to think about the dielectric constant that we've used so many times in electrostatic problems in terms of the microphysics of the material: its plasma frequency and its bound potentials as reflected in ω_0^2.

What is fascinating about this highly simplified model is that the resulting $\epsilon(\omega)$ has the same qualitative properties as many observed materials. Writing $\epsilon = \epsilon' + i\epsilon''$, the real and imaginary parts of $\epsilon(\omega)$ are

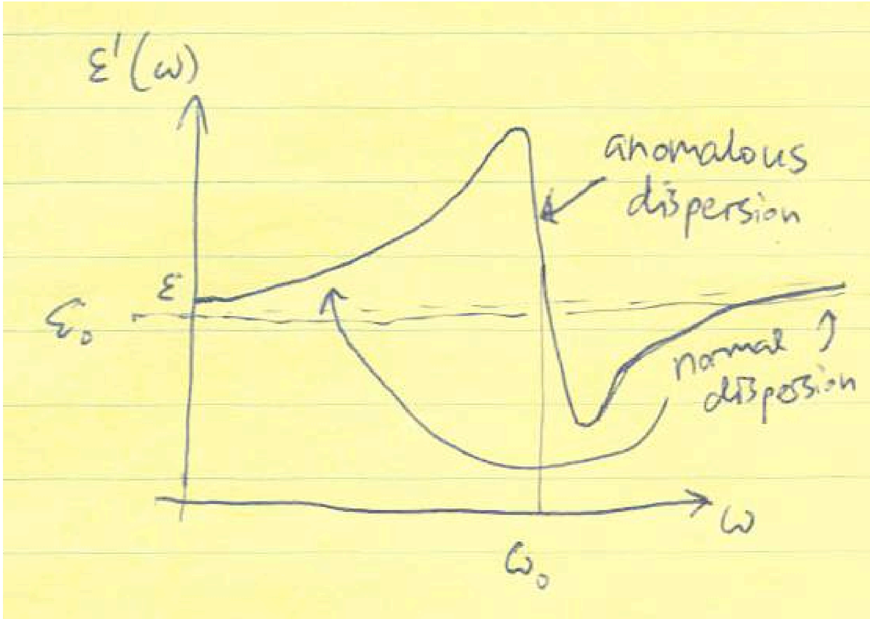
$$\epsilon'(\omega) = \epsilon_0 + \frac{\epsilon_0\omega_p^2(\omega_0^2 - \omega^2)}{(\omega^2 - \omega_0^2)^2 + \gamma^2\omega^2} \qquad \text{real part} \tag{2.59}$$

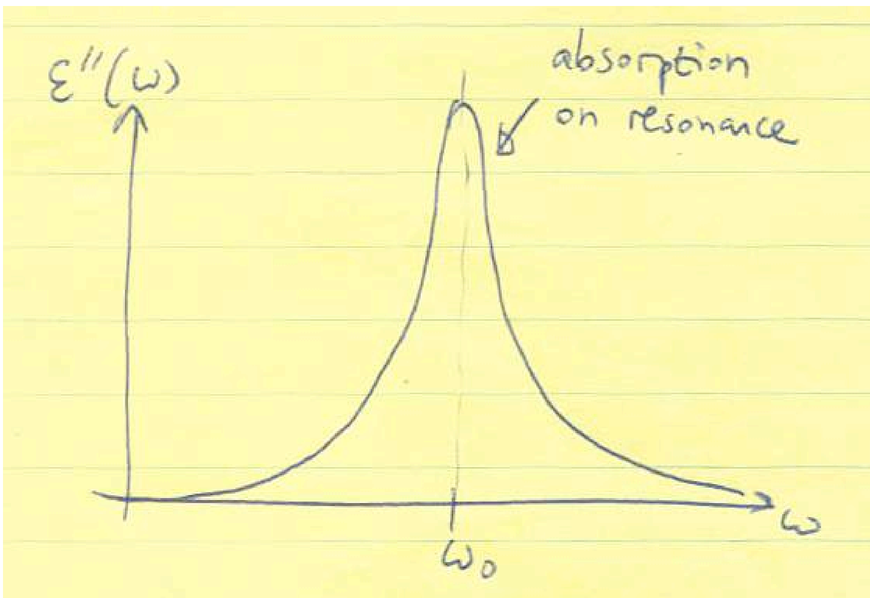
$$\epsilon''(\omega) = \frac{\epsilon_0\omega_p^2\omega\gamma}{(\omega^2 - \omega_0^2)^2 + \gamma^2\omega^2} \qquad \text{imaginary part.} \tag{2.60}$$

Sketches of each of these as a function of frequency are shown below. In terms of the refractive index n where $n^2 = \epsilon$, the phase velocity of the wave is $v_p = c/n_r(\omega)$, where n_r is real part of n. The group velocity is

$$v_g = \frac{d\omega}{dk} = \frac{c}{n_r + \omega dn_r/d\omega}. \tag{2.61}$$

In a region of frequency where $dn_r/d\omega < 0$, the group velocity can become larger than c ("superluminal") or even negative. These ranges of frequency are known as "anomalous dispersion", see the label on the sketch of ϵ' below. The opposite behaviour is in materials where $dn_r/d\omega$ is very large (an extreme case of "normal dispersion"): then the group velocity can be $v_g \ll c$, known as "slow light". These are all great project topics! I've also included a figure from Jackson that shows the real and imaginary parts of ϵ for water. Note the qualitative similarities to the Lorentz model: absorption is associated with anomalous dispersion.


ε'(ω)
anomalous dispersion
ε
ε₀
normal dispersion
ω
ω₀


ε''(ω)
absorption on resonance
ω
ω₀

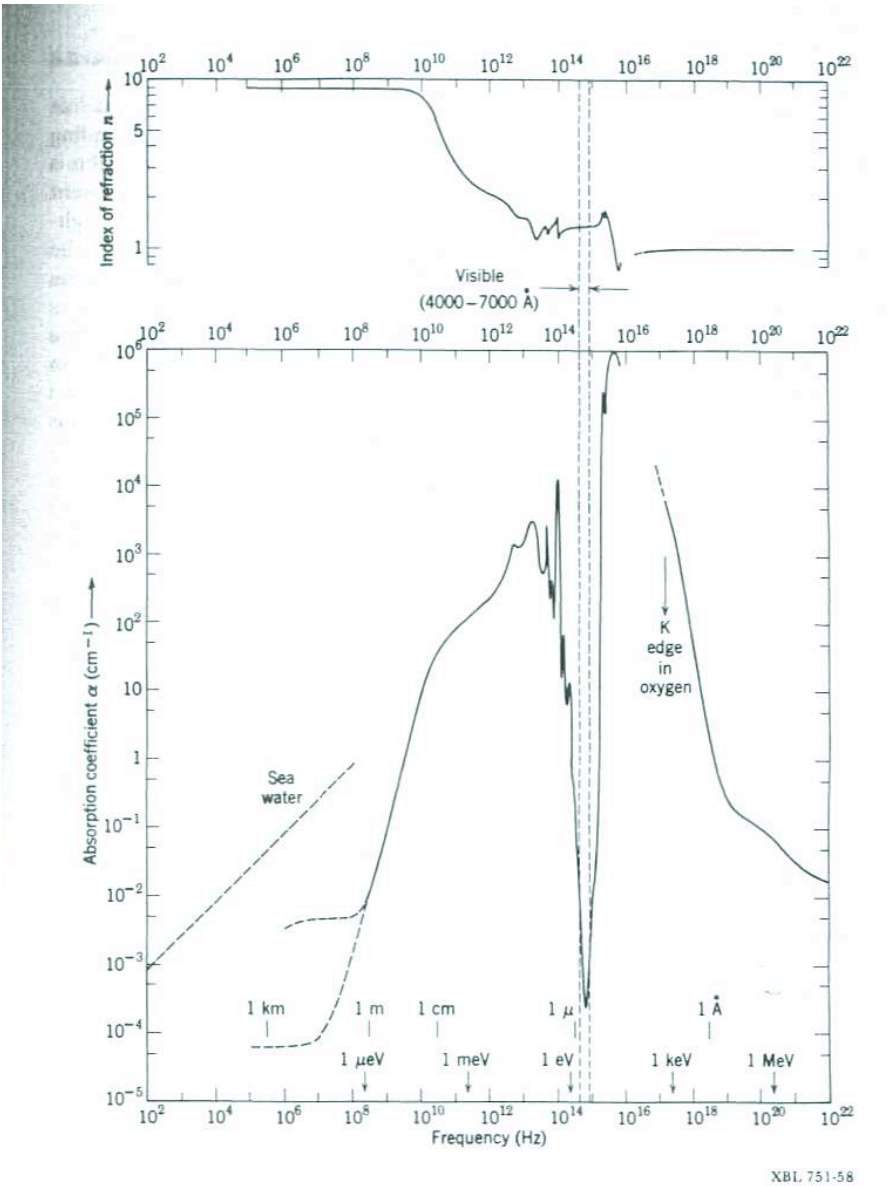


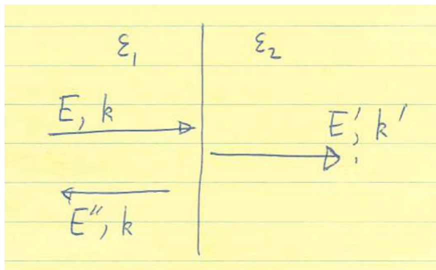
Fig. 7.9 The index of refraction (top) and absorption coefficient (bottom) for liquid water as a function of linear frequency. Also shown as abscissas are an energy scale (arrows) and a wavelength scale (vertical lines). The visible region of the frequency spectrum is indicated by the vertical dashed lines. The absorption coefficient for *sea water* is indicated by the dashed diagonal line at the left. Note that the scales are logarithmic in both directions.

2.3 Reflection and transmission at an interface

In this section, we consider what happens at the interface between two materials. To solve this, we set up incident, reflected, and transmitted waves and use the boundary conditions on the fields to relate them. This is similar to wave transmission problems that you may have seen in other courses such as mechanics (waves on a string) or quantum mechanics (particle incident on a potential barrier).

2.3.1 A wave at normal incidence between two linear dielectrics

The simplest case to consider is a wave incident perpendicular to a plane boundary between two linear dielectrics. We refer to this as "normal incidence" as the incoming wave is in the direction of the normal vector to the surface.



We assume that the permeabilities are all equal, $\mu_1 = \mu_2 = \mu_0$. The incident, reflected, and transmitted waves have amplitudes E, E'' and E' respectively, as shown in the diagram, and k-vectors k for the incident wave, $-k$ for the reflected wave, and k' for the transmitted wave. (Make sure you understand why the magnitudes of the wavevectors of the reflected and incident waves must be the same.) Note that the time-dependence of all the waves is the same $\propto e^{-i\omega t}$ (again, can you make an argument that this must be the case?).

The first boundary condition to consider is that $E_{\parallel}$ should be continuous, so the sum of the electric fields on each side must be equal,

$$E + E'' = E'. \tag{2.62}$$

As there are no free surface currents, $H_{\parallel}$ must also be continuous, so that $B + B'' = B'$. An EM wave in a dielectric obeys $E = \pm c'B = \pm cB/n$ with the sign depending on the direction of the wave. Therefore,

$$n_1 E - n_1 E'' = n_2 E'. \tag{2.63}$$

Equations (2.62) and (2.63) are two simultaneous equations which we can solve for E' and E'' given an incident amplitude E. The result is

$$\frac{E''}{E} = \frac{n_1 - n_2}{n_1 + n_2} \qquad \frac{E'}{E} = \frac{2n_1}{n_1 + n_2}. \tag{2.64}$$

The energy flux in the wave is $(1/2)c'\epsilon E^2 \propto E^2/c' \propto nE^2$. The ratio of reflected to incident intensity (the fraction of the incident power that is reflected) is therefore

$$R = \left(\frac{E''}{E}\right)^2 = \left(\frac{n_1 - n_2}{n_1 + n_2}\right)^2 \tag{2.65}$$

giving a ratio of transmitted intensity to incident intensity (fraction of energy transmitted) of

$$T = \frac{n_2}{n_1}\left(\frac{E'}{E}\right)^2 = \frac{4n_1 n_2}{(n_1+n_2)^2}. \tag{2.66}$$

Note that $R + T = 1$ so that all the energy is accounted for. Also, if $n_1 = n_2$ then $T = 1$ and $R = 0$.

2.3.2 The impedance Z

If we drop the assumption that $\mu_1 = \mu_2$, the $H_\parallel$ boundary condition would be written slightly differently:

$$\frac{B}{\mu_1} + \frac{B''}{\mu_1} = \frac{B'}{\mu_2}. \tag{2.67}$$

Everything follows through as before, but with the replacement $n \to n/\mu$. Dividing by c, we can write this factor as $n/c\mu = \sqrt{\mu\epsilon}/\mu = \sqrt{\epsilon/\mu}$. The resulting transmission and reflection coefficients can be written in terms of this quantity, and in fact we define its inverse as the *impedance* Z:

$$Z = \sqrt{\frac{\mu}{\epsilon}}, \tag{2.68}$$

giving amplitude ratios

$$r = \frac{E''}{E} = \frac{Z_2 - Z_1}{Z_1 + Z_2} \qquad t = \frac{E'}{E} = \frac{2Z_2}{Z_1 + Z_2} \tag{2.69}$$

and energies

$$R = \left(\frac{Z_2 - Z_1}{Z_1 + Z_2}\right)^2 \qquad T = \frac{4Z_1 Z_2}{(Z_1 + Z_2)^2}. \tag{2.70}$$

The units of impedance are ohms, and the quantity

$$Z_0 = \sqrt{\frac{\mu_0}{\epsilon_0}} = 377 \text{ ohms} \tag{2.71}$$

is a useful one to remember, the *impedance of free space*. If $\mu \approx 1$ as it is for most materials, then $Z \propto c'$, the speed of the wave in the material. If $Z_2 > Z_1$ (wave speeds up) then the reflected wave is in phase ($r > 0$, e.g. glass to air). If $Z_2 < Z_1$ (wave slows down), then the reflected wave is out of phase ($r < 0$, e.g. air to glass)[7].

There is a more general definition of impedance, which is the ratio of the **E** and **H** fields at any point

$$Z = \frac{E}{H}. \tag{2.72}$$

[7]You may already have some intuition for this from waves on a string. In that problem, the wave speed is $\sqrt{T/\mu}$, where T is the tension and μ is the mass per unit length, and the impedance is $\mu v = \sqrt{\mu T}$ such that the same reflection and transmission coefficients apply but with Z_1 and Z_2 swapped over. That technical detail aside, recall that a wave that encounters a very heavy string (slows down) will reflect with $r \approx -1$, e.g. a string tied to a wall at one end. A wave that encounters a very light string (speeds up) will reflect with $r = +1$, giving $t = 2$, e.g. an open end of a string.

We can work this out for a few cases. For free space, $E = cB \Rightarrow Z_0 = \mu_0 c = \sqrt{\mu_0/\epsilon_0}$. For a dielectric, $Z = \sqrt{\mu/\epsilon}$. For a good conductor,

$$\frac{B}{E} = \frac{(\mu_0 \sigma \omega)^{1/2}}{\omega} e^{i\pi/4}$$

(see earlier notes) giving

$$Z_{\text{conductor}} = Z_0 \left(\frac{\omega}{\sigma/\epsilon_0} \right)^{1/2} \frac{1-i}{\sqrt{2}} \ll Z_0 \tag{2.73}$$

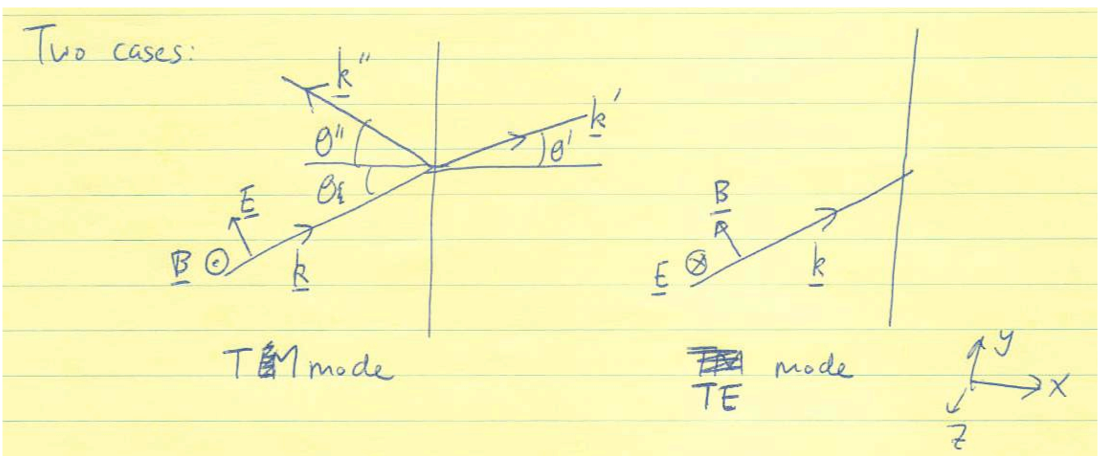
For a plasma, $Z = \mu_0 c/n$ gives

$$Z_{\text{plasma}} = \frac{Z_0}{\sqrt{1 - (\omega_p/\omega)^2}} > Z_0. \tag{2.74}$$

2.3.3 Oblique incidence

We next consider a wave incident at some angle θ_i to the normal. We'll recover some standard results from optics.

There are two cases to consider, depending on whether the magnetic field of the wave or the electric field of the wave is parallel to the interface:



First consider the TE ("transverse electric") case where the electric field is parallel to the interface. The first step is to argue that there is nothing special about the location in the (y, z) plane where we've drawn the vectors above - the boundary conditions have to match at all points. This implies that

$$k_y = k'_y = k''_y \qquad k_z = k'_z = k''_z \tag{2.75}$$

otherwise we wouldn't be able to match the boundary conditions everywhere on the surface (the waves would become out of phase as we moved along the surface)[8].

[8]This is also the answer to the question raised in the last section about why we can safely assume that all three waves have the same time-dependence $e^{-i\omega t}$. We can then arrange for the boundary conditions to be satisfied at a particular time and they will remain satisfied for all times.

We next arrange our axes so that the incident wave propagates in the y–x plane, or in other words we choose $k_z = 0$, and this implies that the reflected and transmitted waves will also propagate in the y–x plane. This is the "plane of incidence". The $\mathbf{k}$ vector components are

$$\mathbf{k} = \frac{n_1\omega}{c}\left(\hat{x}\cos\theta + \hat{y}\sin\theta\right) \tag{2.76}$$

$$\mathbf{k}'' = \frac{n_1\omega}{c}\left(-\hat{x}\cos\theta'' + \hat{y}\sin\theta''\right) \tag{2.77}$$

$$\mathbf{k}' = \frac{n_2\omega}{c}\left(\hat{x}\cos\theta' + \hat{y}\sin\theta'\right) \tag{2.78}$$

where the angles θ, θ', θ'' give the directions of the three waves, and $\hat{x}$, $\hat{y}$ and $\hat{z}$ are the unit vectors.

We can immediately obtain two laws of optics. First, setting $k_y = k''_y$ gives

$$\sin\theta = \sin\theta'' \tag{2.79}$$

or *angle of incidence = angle of reflection*. Setting $k_y = k'_y$ gives

$$n_1 \sin\theta = n_2 \sin\theta' \tag{2.80}$$

which is *Snell's law*. If $n_2 < n_1$, the wave speeds up on entering the second material, and it bends away from the normal; if $n_2 > n_1$, the wave slows down on entering the material and it bends towards the normal.

Now let's derive the reflection and transmission coefficients. This requires matching the electric and magnetic fields at the boundary. In TE mode, the electric vector is perpendicular to the plane of incidence, or parallel to the surface. Therefore the boundary condition that $E_\parallel$ is continuous is straightforward to apply, giving

$$E_0 + E''_0 = E'_0 \tag{2.81}$$

as before. The magnetic fields are more tricky because they have components perpendicular and parallel to the interface. The magnetic fields are

$$\mathbf{H}_0 = \frac{E_0}{Z_1}\left[\hat{x}\sin\theta - \hat{y}\cos\theta\right]$$

$$\mathbf{H}''_0 = \frac{E''_0}{Z_1}\left[\hat{x}\sin\theta + \hat{y}\cos\theta\right]$$

$$\mathbf{H}'_0 = \frac{E'_0}{Z_2}\left[\hat{x}\sin\theta' - \hat{y}\cos\theta'\right]. \tag{2.82}$$

A second boundary condition is $H_\parallel$ continuous, giving

$$\frac{E_0 - E''_0}{Z_1}\cos\theta = \frac{E'_0}{Z_2}\cos\theta'. \tag{2.83}$$

Equations (2.81) and (2.83) are simultaneous equations that can be solved to find E'_0 and E''_0 in terms of the incident amplitude E_0. The result is

$$\frac{E''_0}{E_0} = \frac{Z_2\cos\theta - Z_1\cos\theta'}{Z_2\cos\theta + Z_1\cos\theta'} \tag{2.84}$$

$$\frac{E_0'}{E_0} = \frac{2Z_2 \cos\theta}{Z_2 \cos\theta + Z_1 \cos\theta'}. \tag{2.85}$$

This is actually another optics result in disguise. When $\mu_1 = \mu_2 = \mu_0$ you can show that $Z_2 \to n_1$ and $Z_1 \to n_2$ in the above equations. With some manipulation, equation (2.84) becomes

$$\frac{E_0''}{E_0} = \frac{\sin(\theta' - \theta)}{\sin(\theta' + \theta)} \tag{2.86}$$

Fresnel's equation for the TE mode.

For the TM mode, the magnetic vector is perpendicular to the plane of incidence, and then the boundary conditions become

$$\frac{(E_0 + E_0'')}{Z_1} = \frac{E_0'}{Z_2} \tag{2.87}$$

($H_{\parallel}$ continuous) and then

$$\left(E_0 - E_0''\right)\cos\theta = E_0' \cos\theta' \tag{2.88}$$

($E_{\parallel}$ continuous). The result is then

$$\frac{E_0''}{E_0} = \frac{Z_1 \cos\theta - Z_2 \cos\theta'}{Z_1 \cos\theta + Z_2 \cos\theta'} \tag{2.89}$$

$$\frac{E_0'}{E_0} = \frac{2Z_2 \cos\theta}{Z_2 \cos\theta' + Z_1 \cos\theta}. \tag{2.90}$$

Again when $\mu_1 = \mu_2 = \mu_0$ a simplified version for the reflected amplitude is

$$\frac{E_0''}{E_0} = \frac{\tan(\theta - \theta')}{\tan(\theta' + \theta)}, \tag{2.91}$$

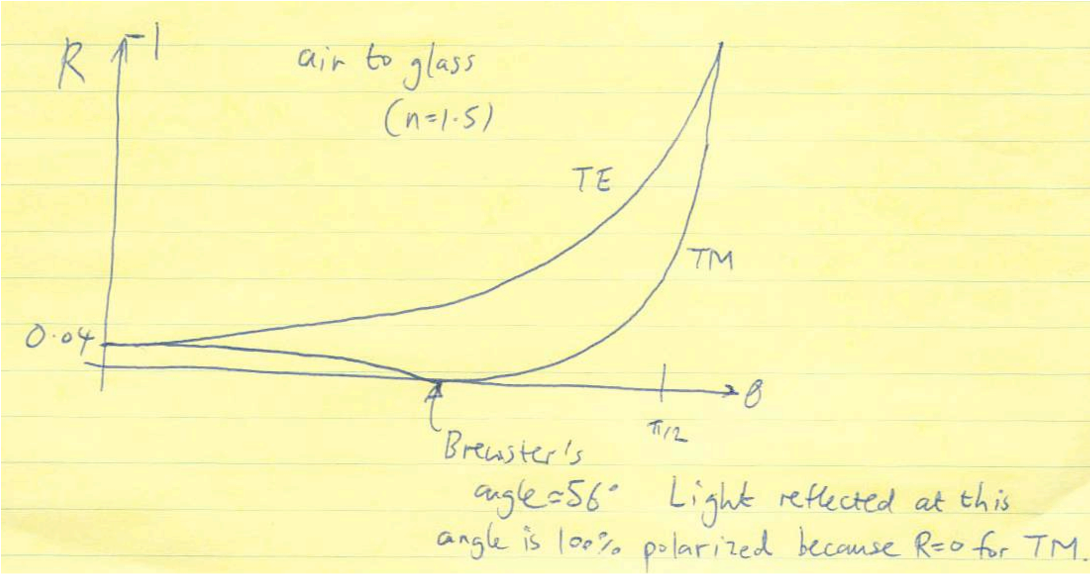
Fresnel's equation for the TM mode.

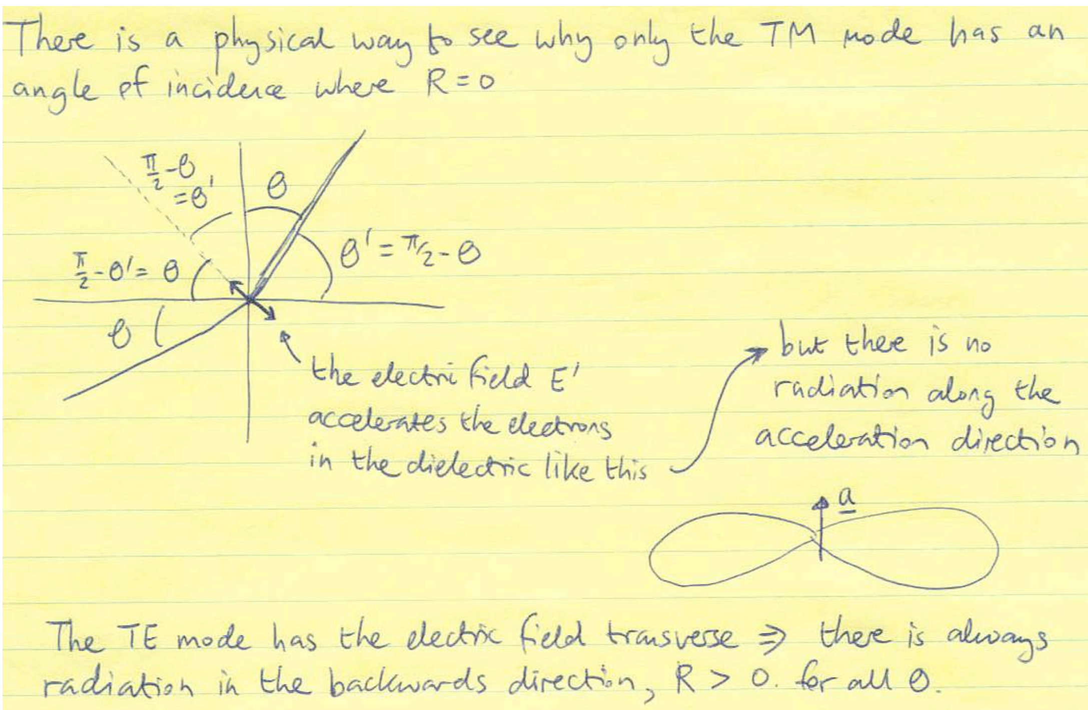
The reflectivity R in the two cases is

$$R\,(\mathrm{TM}) = \left(\frac{Z_1 \cos\theta - Z_2 \cos\theta'}{Z_1 \cos\theta + Z_2 \cos\theta'}\right)^2 \tag{2.92}$$

$$R\,(\mathrm{TE}) = \left(\frac{Z_2 \cos\theta - Z_1 \cos\theta'}{Z_2 \cos\theta + Z_1 \cos\theta'}\right)^2 \tag{2.93}$$

I've included a sketch of R against the angle of incidence θ below for the case of a wave going from air to glass. For the TM mode, the reflectivity goes to zero at a particular angle $\approx 56°$, known as *Brewster's angle*. At this angle, only the TE mode reflects, so an unpolarized beam of radiation is 100% polarized on reflection.

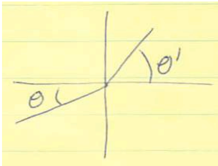




2.3.4 Optics results

Let's collect together the various optics results we've seen already, plus some new ones.

1. **angle of incidence = angle of reflection** $\theta = \theta''$
2. **Snell's law** $n_1 \sin\theta = n_2 \sin\theta'$



If $n_2 > n_1$ (wave slows down) $\theta' < \theta$ (move towards the normal).

If $n_2 < n_1$ (wave speeds up) $\theta' > \theta$ (move away from the normal).

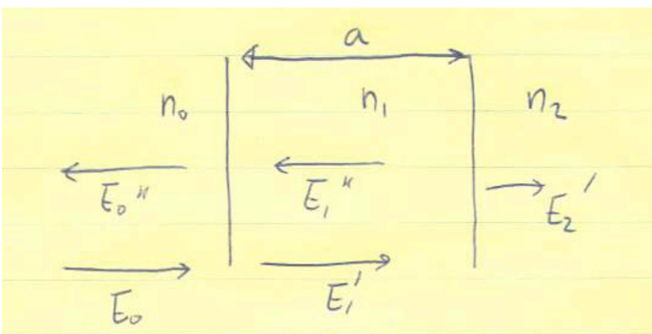
3. **Total internal reflection**

   There is a critical angle of incidence for which $\theta' = \pi/2$.

$$n_1 \sin\theta_c = n_2 \to \theta_c = \sin^{-1}\left(\frac{n_2}{n_1}\right)$$

   For glass to air, $\theta_c = 41°$. For $\theta > \theta_c$, all the incident energy is reflected and there is an evanescent wave in medium 2. This is total internal reflection. e.g. optical fibres

4. **Non-reflective coating**



   The boundary conditions at $x = 0$ are

$$E_0 + E_0'' = E_1' + E_1''$$

$$n_0(E_0 - E_0'') = n_1(E_1' - E_1'')$$

   and at $x = a$

$$E_1' e^{ik_1 a} + E_1'' e^{-ik_1 a} = E_2 \prime e^{ik_2 a}$$

$$n_1\left(E_1' e^{ik_1 a} - E_1'' e^{-ik_1 a}\right) = n_2 E_2' e^{ik_2 a}$$

   There is zero reflected wave $E_0'' = 0$ if we choose

$$n_1 = \sqrt{n_0 n_2}$$

   and

$$a = (2j+1)\frac{\lambda_1}{4}$$

   e.g. MgF_2 coating for lenses: $n_1 = 1.38$ which is $\approx \sqrt{1.5}$.

5. **Reflection** Reflectivity for normal incidence

$$R = \left(\frac{Z_2 - Z_1}{Z_1 + Z_2} \right)^2$$

For oblique incidence, Fresnel equations

$$R = \left(\frac{E_0''}{E_0} \right)^2 = \frac{\sin^2(\theta' - \theta)}{\sin^2(\theta' + \theta)} \qquad \text{TE}$$

$$R = \left(\frac{E_0''}{E_0} \right)^2 = \frac{\tan^2(\theta - \theta')}{\tan^2(\theta' + \theta)} \qquad \text{TM}$$

6. **Brewster's angle** at which the reflectivity is zero for TM. Corresponds to $\theta + \theta' = \pi/2$ so that R vanishes (denominator in Fresnel equation diverges).

   To find θ:

$$\theta' = \frac{\pi}{2} - \theta \Rightarrow \sin\theta' = \cos\theta$$

$$\frac{n_1}{n_2} \sin\theta = \cos\theta$$

   (using Snell's law)

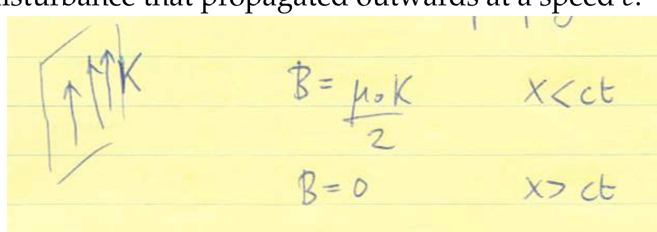
$$\Rightarrow \tan\theta = \frac{n_2}{n_1}.$$

   For $n_2 = 1.5$ and $n_1 = 1$ (air to glass), $\theta = 56^\circ$.

2.3.5 A physical explanation for the reflected wave

In §2.3.1, we assumed that there would be a reflected wave, and calculated its amplitude by matching the boundary conditions. But this begs the question, where does the reflected wave come from? Here is an argument that gives a physical explanation for the reflected wave, while at the same time deriving the correct expression for its amplitude.

We saw in Part 1 of the class that switching on a surface current K in a plane created a disturbance that propagated outwards at a speed c:



If K depends on time $K(t)$, then B at position x must depend on whatever the current K was at the sheet one light travel time ago, that is at a time $t - x/c$. In other words, the magnetic field must be

$$B(x, t) = \frac{\mu_0}{2} K \left(t - \frac{x}{c} \right).$$

The quantity $t - x/c$ is called the "retarded time" (retarded because we evaluate the current at an earlier time, based on the light travel time to whatever spatial position x we are looking at).

Now consider a transmitted wave $E_0' e^{ik'x - i\omega t}$ propagating into the dielectric. At a distance x into the dielectric, there is a current

$$K(x,t) = \sigma(\omega) E_0' e^{ik'x - i\omega t} dx,$$

where $\sigma(\omega)$ is the conductivity of the dielectric (e.g. as given by the Lorentz model). (The dx is there to convert from the volume current density $J = \sigma E$ to a surface current). Now consider a position x_0 which is in the vacuum outside the dielectric ($x_0 < 0$). The magnetic field there is a sum over all the magnetic field contributions from the currents inside the dielectric induced by the propagating wave

$$B(x_0, t) = \int \frac{\mu_0}{2} \sigma(\omega) E_0' \, dx \, e^{ik'x} e^{-i\omega(t - (x - x_0)/c)},$$

or

$$B(x_0, t) = \frac{\mu_0}{2} \sigma(\omega) E_0' e^{-i\omega(t + x_0/c)} \int_0^\infty dx \, e^{ik'x + i\omega x/c}.$$

Notice that B is a function of $t + x_0/c$, in other words it is a wave travelling to the left, as expected for the reflected wave.

To get the amplitude of the reflected wave, we need to do the integral and evaluate the prefactor:

$$B(x_0, t) = e^{-i\omega(t + x_0/c)} \frac{\mu_0 \sigma E_0'}{2} \frac{1}{i(k' + \omega/c)} \left[1 + e^{i\infty}\right].$$

We assume that we can drop the contribution from $x = \infty$, because the contributions from further and further away mix rapidly in phase and will gradually taper off depending on what happens at large distance into the dielectric (see Feynman 30-7 for a discussion of a similar $e^{i\infty}$ term)[9]. Also, we use the relations $k' + \omega/c = (\omega/c)(1+n)$ and $\epsilon_r - 1 = n^2 - 1 = (n-1)(n+1) = \sigma/(-i\omega\epsilon_0)$ (see the dispersion relation eq. [2.28]). The result is

$$\frac{cB(x_0,t)}{E_0} = \frac{E_0''}{E_0} e^{-i\omega(t + x_0/c)} = -e^{-i\omega(t + x_0/c)} \frac{E_0'}{E_0} \frac{n-1}{2}$$

or since $E_0'/E_0 = 2/(1+n)$, we see that

$$\frac{E_0''}{E_0} = \left(\frac{1-n}{1+n}\right)$$

as we found previously.

[9] And for a similar approach to understanding the reflected wave see Feynman lectures I 30-7, 31-1, 31-2, and volume 2 18-4 is also relevant.

2.4 Scattering of light from single electrons

In Part 1 of the course, we derived Larmor's formula for the power radiated by an accelerated electron

$$\text{Power} = \frac{q^2 a^2}{6\pi\epsilon_0 c^3}$$

where a is the acceleration of the electron (see eq. [1.22] from Part 1 of the notes). The angular distribution of the emitted energy is given by the Poynting flux

$$\mathbf{S} = \hat{r}\frac{q^2 a^2}{16\pi^2\epsilon_0 c^3 r^2}\sin^2\theta$$

(see eq. [1.21]), where θ is the angle of the emitted radiation with respect to the acceleration direction. The radiation is concentrated in directions perpendicular to the acceleration ($\sin\theta \sim 1$) and with no emission in the direction of the acceleration ($\sin\theta = 0$).

When an electromagnetic wave encounters a single electron, it accelerates the electron, causing it to emit radiation in all directions. The result is scattering of the incoming wave, and we can use Larmor's formula to calculate the scattering cross-section.

2.4.1 Thomson scattering

Consider first a single free electron which is accelerated by a passing wave which we write as usual $E = E_0 e^{-i\omega t}$. The acceleration of the electron is $a = -eE/m_e$ and so it radiates a power

$$\langle\text{Power}\rangle = \frac{1}{2}\frac{\left(e^2 E/m_e\right)^2}{6\pi\epsilon_0 c^3},$$

where we have used Larmor's formula and included a factor of $1/2$ coming from the time-average of $\sin^2$. The incoming energy flux in the wave is $(1/2)c\epsilon_0 E^2$. By comparing the incoming energy flux to the power reradiated, we can calculate the cross-section for scattering σ. The cross-section is defined by

(Reradiated power) = (Incident flux) (Cross-section σ)

$$\frac{1}{2}\frac{\left(e^2 E/m_e\right)^2}{6\pi\epsilon_0 c^3} = \frac{1}{2}c\epsilon_0 E^2 \times \sigma \tag{2.94}$$

which gives

$$\sigma = \frac{1}{6\pi\epsilon_0^2}\frac{e^4}{(m_e c^2)^2}.$$

A useful way to rewrite this is in terms of the classical electron radius, defined by

$$\frac{e^2}{4\pi\epsilon_0 r_e} = m_e c^2 \Rightarrow r_e = \frac{1}{4\pi\epsilon_0}\frac{e^2}{m_e c^2}.$$

The cross-section is then

$$\sigma = \sigma_T = \left(\frac{e^2}{4\pi\epsilon_0 m_e c^2}\right)^2 \frac{8\pi}{3} = \frac{8\pi}{3} r_e^2 \tag{2.95}$$

which is the famous Thomson cross-section σ_T for the scattering of EM radiation by an electron.

Putting in numbers, $\sigma_T = 6.65 \times 10^{-29}$ m^2[10]. An example is the Sun, which is a large ball of ionized plasma with average density $\rho \approx 1$ g cm^{-3}. The electron density is therefore $n_e \approx \rho/m_p \approx 10^{30}$ m^{-3}. The mean free path for scattering of photons is therefore $\lambda_{\rm mfp} \approx 1/n_e\sigma_T \approx 1$ cm, much smaller than the size of the Sun (about 10^9 m or 10^{11} mean free paths!). Thomson scattering is therefore extremely important in the solar interior and in fact controls the energy transport.

We can also derive a differential cross-section which gives the angular distribution of the scattered radiation because we know the angular distribution of the radiated power. The power that goes into solid angle $d\Omega$ in direction (θ, ϕ) is

$$dP = |\mathbf{S}|\, r^2 d\Omega = |\mathbf{S}|\, r^2 d\phi \sin\theta d\theta$$

or

$$\frac{dP}{d\Omega} = r^2 |\mathbf{S}| = \frac{q^2 a^2}{16\pi^2\epsilon_0 c^3} \sin^2\theta.$$

The differential cross-section is defined as

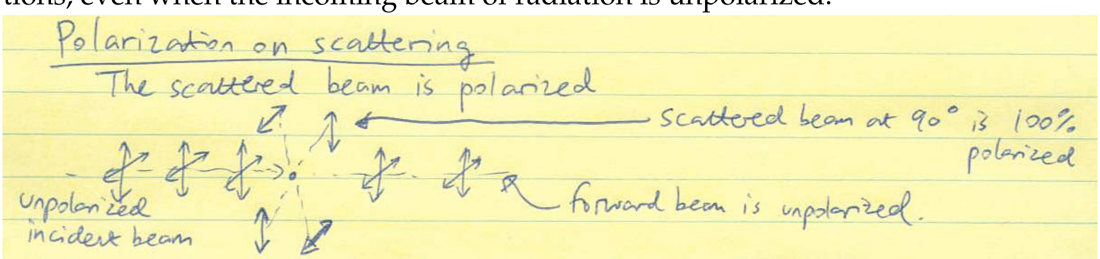
$$\frac{dP}{d\Omega} = \left(\frac{1}{2} c\epsilon_0 E^2\right) \frac{d\sigma}{d\Omega}$$

giving

$$\frac{d\sigma}{d\Omega} = \frac{(e^2 E/m_e)^2}{16\pi^2\epsilon_0 c^3} \sin^2\theta \frac{1}{(1/2)c\epsilon_0 E^2} = \left(\frac{e^2}{4\pi\epsilon_0 m_e c^2}\right)^2 \sin^2\theta$$

$$\frac{d\sigma}{d\Omega} = r_e^2 \sin^2\theta. \tag{2.96}$$

This angular dependence means that the scattered beam is polarized in some directions, even when the incoming beam of radiation is unpolarized.



A good practise question to help you think this through is Q4 from the 2012 final exam.

[10] The number I remember is that the Thomson cross-section in cm^2 is approximately the inverse of the proton mass in g. Useful if you are using cgs units.

2.4.2 Rayleigh scattering

Now consider a bound electron. Looking back at the Lorentz dielectric with $\gamma = 0$, the displacement is

$$x = \frac{eE/m}{\omega^2 - \omega_0^2}$$

and the acceleration is

$$a = -\frac{\omega^2}{\omega^2 - \omega_0^2}\frac{eE}{m_e},$$

the same as for a free electron except for the factor $\omega^2/(\omega^2 - \omega_0^2)$. Therefore the cross-section for the bound electron must be

$$\sigma = \sigma_T \frac{\omega^4}{(\omega^2 - \omega_0^2)^2} \tag{2.97}$$

$$\frac{d\sigma}{d\Omega} = r_e^2 \sin^2\theta \frac{\omega^4}{(\omega^2 - \omega_0^2)^2}. \tag{2.98}$$

Two limits are

1. $\omega \gg \omega_0$. Then $\sigma \to \sigma_T$. The bound electron acts as a free electron for short photon wavelengths.

2. $\omega \ll \omega_0$. Then

$$\sigma \to \sigma_T \left(\frac{\omega}{\omega_0}\right)^4 \propto \frac{1}{\lambda^4}.$$

   This is *Rayleigh scattering*. The strong dependence on wavelength is the reason that blue light scatters more in the atmosphere and gives blue sky/red sunsets.

SUMMARY

Here are the main ideas and results that we covered in this part of the course:

EM waves in a plasma. The current is $\pi/2$ out of phase with the electric field $\mathbf{J} = i(n_e e^2/m_e\omega)\mathbf{E}$. How to get the dispersion relation $n^2 = (ck/\omega)^2 = 1 - (\omega_p/\omega)^2$ where the plasma frequency is given by $\omega_p^2 = n_e e^2/m_e\epsilon_0$, or $f_p = 9\ \mathrm{kHz}(n_e/cm^{-3})^{1/2}$. Evanescence of low frequency waves and reflection of radio waves from the ionosphere.

Waves in a conductor. Current and electric field are in phase, $\mathbf{J} = \sigma\mathbf{E}$. The dispersion relation $n^2 = (ck/\omega)^2 = 1 + i\sigma/\epsilon_0\omega$. The limit of a good conductor $\sigma/\epsilon_0 \gg \omega$ where conduction current dominates displacement current. The complex wavevector $k = (1+i)/\delta$ in the good conductor where $\delta^2 = 2/(\mu_0\sigma\omega)$ defines the skin depth. Calculation of the Poynting flux and Ohmic dissipation in the conductor.

Waves in an LIH dielectric. Assuming instantaneous response of the material gives a wave speed $c'^2 = 1/(\mu\epsilon)$ or $c' = c/n$ where $n^2 = \epsilon_r\mu_r$ defines the refractive index. Typical values of ϵ_r and μ_r for real materials.

The Lorentz dielectric. Derivation of the frequency-dependent conductivity in this model

$$\frac{\sigma(\omega)}{\epsilon_0} = \frac{i\omega\omega_p^2}{\omega^2 - \omega_0^2 + i\omega\gamma}$$

and the dielectric constant

$$\epsilon_r = 1 - \frac{\omega_p^2}{\omega^2 - \omega_0^2 + i\omega\gamma}.$$

The real and imaginary parts of ϵ, anomalous dispersion, absorption near the resonance.

Reflection and transmission at a boundary. Decomposition into incident, reflected, and transmitted waves, and the boundary conditions on the fields at the surface for both normal and oblique incidence. For normal incidence, the reflectivity is $R = (n_1 - n_2)^2/(n_1 + n_2)^2$ and fraction of intensity transmitted is $T = 4n_1n_2/(n_1 + n_2)^2$. The definition of impedance Z as the ratio of E to H in the wave. Impedance of free space $\sqrt{\mu_0/\epsilon_0} = 377$ Ohms.

Optics results: angle of incidence equals angle of reflection; Snell's law $n_1 \sin\theta = n_2 \sin\theta'$; total internal reflection; non-reflective coating with $n_1 = \sqrt{n_0 n_2}$ and thickness a multiple of $\lambda/4$; the difference between TE and TM modes and Brewster's angle.

Scattering. The classical derivation of the Thomson cross-section,

$$\frac{d\sigma_T}{d\Omega} = r_e^2 \sin^2\theta \qquad \sigma_T = \frac{8\pi}{3} r_e^2,$$

where $r_e = (e^2/m_e c^2)/4\pi\epsilon_0$ is the classical electron radius. Rayleigh scattering from a bound electron $\sigma \propto 1/\lambda^4$ for long wavelengths.

The polarization of the scattered radiation.

You should be able to:

- know the relation between refractive index, permittivity ϵ and wavevector k, and how to interpret the real and imaginary parts of a complex wavevector or permittivity.
- Use the electron equation of motion to determine the relation between **J** and **E** for a material, under some approximation such as plasma, conductor, or Lorentz dielectric. From there you should be able to get the dispersion relation for the material.
- Use the dispersion relation for a conductor to derive an expression for the skin depth.
- Be able to calculate the reflection and transmission coefficients between two materials with different dispersion relations by matching boundary conditions at the interface.

- Calculate the scattering cross-section for an electron, including the total and differential cross-sections.
- Be able to describe the concepts of normal and anomalous dispersion, and the link between the real and imaginary parts of the permittivity.

Appendix A: Natural frequencies of a plasma or conductor

One question you may have is that when we derived the dispersion relation for a plasma or conductor, we included the current induced by the EM wave in Ampere's law, but we did not consider density perturbations in the plasma as a source term in Gauss' law. The reason is that the waves are transverse. If the wave caused a perturbation to the electron density δn_e, then Gauss' law $\nabla \cdot \mathbf{E} = \rho/\epsilon_0$ gives

$$i\mathbf{k} \cdot \mathbf{E_0} = -\frac{e\delta n_e}{\epsilon_0}. \tag{2.99}$$

But for a transverse wave $\mathbf{k} \cdot \mathbf{E}_0 = 0$ and the EM wave does not cause any perturbations to the electron density.

However, it is worth thinking about waves that would be driven by density fluctuations, as they give some insight into the two characteristic frequencies that enter into the dispersion relation, either the plasma frequency for plasmas or σ/ϵ_0 for conductors. The continuity equation is $\nabla \cdot \mathbf{J} + \partial\rho/\partial t = 0$, or

$$i\mathbf{k} \cdot \mathbf{J}_0 + i\omega e\delta n_e = 0. \tag{2.100}$$

Eliminating δn_e gives a relation between the divergences of the electric field and current density

$$i\mathbf{k} \cdot \mathbf{E_0} = \frac{\mathbf{k} \cdot \mathbf{J}_0}{\epsilon_0 \omega} \tag{2.101}$$

Physically, this just says that if there is a charge overdensity, it must have come from a non-zero divergence of the current density, but will also lead to a divergence of $\mathbf{E}$ through Gauss' law. So the divergence of the current density and the divergence of the electric field must be related.

Now consider first a conductor for which $\mathbf{J}_0 = \sigma\mathbf{E}_0$. Then equation (2.101) implies that either $\mathbf{k} \cdot \mathbf{E}_0 = 0$ or $i\omega = \sigma/\epsilon_0$. In other words, the response to a non-transverse perturbation in a conductor has a time-dependence $\propto e^{-i\omega t} \propto e^{-\sigma t/\epsilon_0}$. In a conductor, a charge excess decays on a timescale ϵ_0/σ.

Similarly, for a plasma, $\mathbf{J}_0 = (ine^2/m\omega)\mathbf{E}_0$ gives either $\mathbf{k} \cdot \mathbf{E}_0 = 0$ (transverse wave) or $\omega = \omega_p$. Unlike in the plasma case, the frequency is real: there is a longitudinal oscillation in a plasma driven by electron density perturbations whose frequency is the plasma frequency (this is known as a plasma oscillation or Langmuir wave).

In this way, we see that the plasma frequency and σ/ϵ_0 are "natural frequencies" of plasmas or conductors, respectively.

Appendix B: A note on calculating the Poynting flux

When you are using complex notation, it is important to remember that if you are evaluating a quantity that is a product, such as the Poynting flux $\mathbf{E} \times \mathbf{H}$, you should take the real parts of $\mathbf{E}$ and $\mathbf{H}$ before you evaluate the product, not afterwards. In other words

$$\mathrm{Re}\,(\mathbf{E} \times \mathbf{H}) \neq \mathrm{Re}\,(\mathbf{E}) \times \mathrm{Re}\,(\mathbf{H})$$

(or similarly $\mathrm{Re}(z^2) \neq \mathrm{Re}(z)^2$ for a complex number z). In the sections on plasmas and conductors above, I wrote the real parts of $\mathbf{E}$ and $\mathbf{B}$ first, and then used them to calculate the Poynting flux, taking the appropriate time average of $\sin^2 \omega t$ for example.

An alternative and faster method is to use the identity

$$\langle S \rangle = \frac{1}{2}\mathrm{Re}\,(EH^\star) \tag{2.102}$$

where $H^\star$ is the complex conjugate of H (and I've written E and H as scalars, assuming that they are in orthogonal directions). You'll prove this relation in one of the problem sets.

Examples:

- **Plane wave.** $E = E_0 e^{ikx-i\omega t}$, $B = (E_0/c)e^{ikx-i\omega t}$. Then equation (2.102) gives

$$\langle S \rangle = \frac{1}{2}\frac{E_0^2}{\mu_0 c} = \frac{1}{2}c\epsilon_0 E_0^2.$$

- **Plasma** with $\omega < \omega_p$. Then $E = E_0 e^{ikx-i\omega t}$ with $k = i/\delta$. Faraday's law is

$$\omega B = kE \Rightarrow B = \frac{i}{\delta\omega}E_0 e^{-x/\delta - i\omega t}.$$

  The product

$$EH^\star = \frac{E_0^2}{\delta\omega}e^{-2x/\delta}\,(-i)$$

  is imaginary and so

$$\langle S \rangle = \mathrm{Re}(EH^\star) = 0$$

  (compare eq. [2.37]).

- **Conductor.** The dispersion relation is

$$k = \frac{(1+i)}{\sqrt{2}}\,(\mu_0\sigma\omega)^{1/2} = \frac{1+i}{\delta}.$$

  Again Faraday's law relates B to E,

$$\omega B = kE = \frac{1+i}{\delta}E.$$

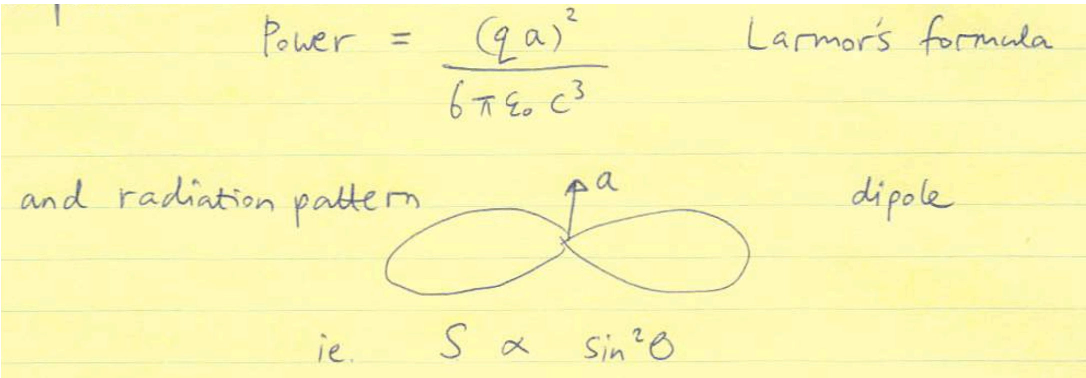
  The Poynting flux is

$$\langle S \rangle = \frac{1}{2}\mathrm{Re}\,(EH^\star) = \frac{1}{2}E_0^2 e^{-2x/\delta}\frac{1}{\delta\mu_0\omega} = \frac{1}{2\sqrt{2}}E_0^2\left(\frac{\sigma}{\omega\mu_0}\right)^{1/2} e^{-2x/\delta}$$

  (compare eq. [2.46]).

Part 3: Time-dependent Fields and Radiation

These are notes for the third part of PHYS 352 Electromagnetic Waves. In part 1, we discussed the idea that accelerating charges radiate, and we saw Thomson's geometric argument that led to Larmor's formula for the power radiated by an accelerated charge:



Here we will put that on a firmer footing by directly solving the wave equations for the potentials ϕ and $\mathbf{A}$.

3.1 Retarded potentials

We saw in part 1 that choosing the Lorentz gauge

$$\nabla \cdot \mathbf{A} = -\frac{1}{c^2}\frac{\partial \phi}{\partial t}$$

gives the following decoupled wave equations for the potentials ϕ and $\mathbf{A}$:

$$-\nabla^2\phi + \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = \frac{\rho}{\epsilon_0} \tag{3.103}$$

$$-\nabla^2\mathbf{A} + \frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2} = \mu_0\mathbf{J}, \tag{3.104}$$

where in general the source terms are functions of position and time, $\rho(\mathbf{r}, t)$ and $\mathbf{J}(\mathbf{r}, t)$. We want to solve these equations to determine the potentials in the time-dependent case. There are two approaches to this. The first is a more physical approach from Griffiths book, the second a more mathematical treatment from Pollack and Stump. We'll go through each in turn, and then apply the results to calculate the fields of a current-carrying wire after the current is turned on.

3.1.1 Derivation by analogy with statics

The idea is to make a generalization based on what we already know from electrostatics and magnetostatics. In the static case, Poisson's equation $\nabla^2\phi = -\rho/\epsilon_0$ has the solution

$$\phi(\mathbf{r}) = \int \frac{\rho(\mathbf{r}')}{4\pi\epsilon_0|\mathbf{r}-\mathbf{r}'|} d^3r',$$

and similarly Ampere's law $\nabla^2\mathbf{A} = -\mu_0\mathbf{J}$ has the solution

$$\mathbf{A}(\mathbf{r}) = \int \frac{\mu_0\mathbf{J}(\mathbf{r}')}{4\pi\epsilon_0|\mathbf{r}-\mathbf{r}'|} d^3r'.$$

This works because the Green's function that satisfies

$$\nabla^2\phi = -\frac{\delta(\mathbf{r})}{\epsilon_0}$$

is

$$\phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0|\mathbf{r}|} = G(\mathbf{r})$$

so that the solution for a general charge distribution is

$$\begin{aligned}\phi(\mathbf{r}) &= \int d^3\mathbf{r}'\rho(\mathbf{r}')G(\mathbf{r}-\mathbf{r}') \\ &= \int d^3\mathbf{r}' \frac{\rho(\mathbf{r}')}{4\pi\epsilon_0|\mathbf{r}-\mathbf{r}'.|}\end{aligned}$$

In the integral, we visit each point in space, and include the contribution from the source at that point by using the Green's function.

We can do something similar in the time-dependent case, by making one important modification. We've already introduced the idea that electromagnetic disturbances propagate at the speed of light, so that the field a given point in space depends on what the source was doing one light travel time ago – at the *retarded time* $(t - r/c)$. This suggests the solution

$$\phi(\mathbf{r}, t) = \int \frac{\rho(\mathbf{r}', t - |\mathbf{r}-\mathbf{r}'|/c)}{4\pi\epsilon_0|\mathbf{r}-\mathbf{r}'|} d^3\mathbf{r}' \tag{3.105}$$

$$\mathbf{A}(\mathbf{r}, t) = \int \frac{\mu_0\mathbf{J}(\mathbf{r}', t - |\mathbf{r}-\mathbf{r}'|/c)}{4\pi|\mathbf{r}-\mathbf{r}'|} d^3\mathbf{r}'. \tag{3.106}$$

These are the **retarded potentials**. At every point in the integral $\mathbf{r}'$, we include the contribution from the source at that location, but evaluate the source at the retarded time $t - |\mathbf{r}-\mathbf{r}'|/c$. This then takes into account the fact that the information from the source takes a light travel time to get to the location $\mathbf{r}$ that we are interested in.

We can see our guess was correct by verifying that the retarded potential solutions satisfy the wave equations. It is straightforward to do, but you have to be careful that the integral has an $\mathbf{r}$ dependence not only in the denominator as usual, but also in the numerator because of the retarded time. When you take the spatial derivative, you'll get terms like

$$\nabla\rho(\mathbf{r}', t - |\mathbf{r}-\mathbf{r}'|/c) = \nabla\rho(\mathbf{r}', t_r) = \frac{\partial\rho}{\partial t_r}\nabla t_r = -\frac{1}{c}\dot{\rho}\nabla|\mathbf{r}-\mathbf{r}'|$$

that aren't there in the static case. Try it! You should find that the wave equation is satisfied.

3.1.2 Derivation of the Green's function

The other approach is to come from the other direction and develop a Green's function for equations (3.103) and (3.104). To do this, make a Fourier decomposition and assume the sources have an $e^{-i\omega t}$ time-dependence

$$\rho(\mathbf{r},t) = \tilde{\rho}(\mathbf{r})e^{-i\omega t}.$$

We look for a solution

$$\phi(\mathbf{r},t) = \tilde{\phi}(\mathbf{r})e^{-i\omega t}$$

$$\Rightarrow -\nabla^2\tilde{\phi} - k^2\tilde{\phi} = \frac{\tilde{\rho}}{\epsilon_0}$$

where we have defined $k = \omega/c$. This is the Helmholtz equation, which has a Green's function $e^{ikr}/4\pi r$, i.e.

$$-\left(\nabla^2 + k^2\right)\frac{e^{ikr}}{r} = \delta(\mathbf{r}).$$

To see that this is true, (i) note that $\nabla^2(e^{ikr}/r) = -k^2 e^{ikr}/r$ so that the left hand side (LHS) vanishes except at the origin, and (ii) if you integrate the LHS over a spherical volume centred on the origin you get a value of 1 independent of the radius of the sphere.

Having obtained the Green's function, we can then construct the solution

$$\tilde{\phi}(\mathbf{r}) = \int d^3\mathbf{r}'\frac{\tilde{\rho}(\mathbf{r}')}{\epsilon_0}G(\mathbf{r}-\mathbf{r}') = \int d^3\mathbf{r}'\tilde{\rho}(\mathbf{r}')\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{4\pi\epsilon_0|\mathbf{r}-\mathbf{r}'|}.$$

Now for arbitrary time-dependence,

$$\rho(\mathbf{r},t) = \int_{-\infty}^{\infty}\tilde{\rho}(\mathbf{r},\omega)e^{-i\omega t}d\omega$$

and

$$\phi(\mathbf{r},t) = \int_{-\infty}^{\infty}\tilde{\phi}(\mathbf{r},\omega)e^{-i\omega t}d\omega$$

$$= \int_{-\infty}^{\infty}d\omega e^{-i\omega t}\int d^3\mathbf{r}'\tilde{\rho}(\mathbf{r}',\omega)\frac{e^{i\omega|\mathbf{r}-\mathbf{r}'|/c}}{4\pi\epsilon_0|\mathbf{r}-\mathbf{r}'|}$$

(where we use the fact that $k = \omega/c$)

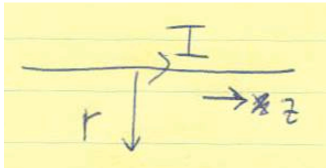
$$= \int d^3\mathbf{r}'\frac{1}{4\pi\epsilon_0|\mathbf{r}-\mathbf{r}'|}\int_{-\infty}^{\infty}d\omega e^{-i\omega(t-|\mathbf{r}-\mathbf{r}'|/c)}\tilde{\rho}(\mathbf{r}',!)$$

$$= \int d^3\mathbf{r}'\frac{1}{4\pi\epsilon_0|\mathbf{r}-\mathbf{r}'|}\rho(\mathbf{r}',t-\frac{|\mathbf{r}-\mathbf{r}'|}{c}),$$

which is the retarded potential from earlier.

3.1.3 An example: Time-dependent fields of a current-carrying wire

As a first application of the retarded potentials, consider an infinite straight wire that carries a current I for $t \geq 0$ (and $I = 0$ for $t < 0$). Use cylindrical coordinates so that r is the radial distance from the wire and z is along the wire:



The vector potential is then

$$A(r,t) = \frac{\mu_0}{4\pi}\hat{z}\int_{-\infty}^{\infty} dz \frac{I(t_r)}{\sqrt{r^2+z^2}}.$$

Notice that the current is evaluated at the retarded time $I(t_r)$, and is non-zero only for $t_r > 0$ or

$$(ct)^2 > r^2 + z^2$$
$$z^2 < (ct)^2 - r^2.$$

This tells us how to set the limits of the integral:

$$A(r,t) = \frac{\mu_0}{4\pi}\hat{z}\int_{-\sqrt{c^2t^2-r^2}}^{\sqrt{c^2t^2-r^2}} dz \frac{I}{\sqrt{r^2+z^2}}.$$

This integral can be done, giving

$$A(r,t) = \hat{z}\frac{\mu_0 I}{2\pi}\ln\left(\frac{ct+\sqrt{(ct)^2-r^2}}{r}\right) \qquad r < ct$$
$$A(r,t) = 0 \qquad r > ct.$$

For $r > ct$, none of the wire lies within a light travel time and so none of the wire contributes to the integral and $A = 0$.

The fields are

$$\mathbf{E}(\mathbf{r},t) = -\frac{\partial \mathbf{A}}{\partial t} = \frac{\mu_0 I c}{2\pi\sqrt{(ct)^2-r^2}}\hat{z}$$

and

$$\mathbf{B} = \nabla \times \mathbf{A} = -\hat{\phi}\frac{\partial A_z}{\partial r} = \hat{\phi}\frac{\mu_0 I}{2\pi r}\frac{ct}{\sqrt{(ct)^2-r^2}}.$$

It is interesting to look at the limits. At late times, $ct \gg r$, then

$$\mathbf{B} = \frac{\mu_0 I}{2\pi r}\hat{\phi} \qquad \mathbf{E} = \hat{z}\frac{\mu_0 I c}{2\pi(ct)} \to 0$$

which agrees with the static case. For $r < ct$, the ratio of E to B is

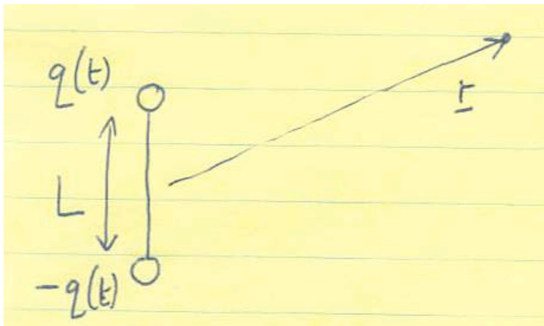
$$\frac{E}{cB} = \frac{r}{ct},$$

so that $E = cB$ at the front which moves outwards from the wire at speed c.

3.2 The Hertzian Dipole and Some Properties of Antennas

3.2.1 The Hertzian Dipole

The Hertzian dipole is the simplest example of a radiating system. We consider two metal spheres joined by a wire of length L, with charge oscillating back and forth at frequency ω. The total charge is zero, so as charge moves back and forth between the two spheres, one sphere has charge $+q(t)$ and the other has charge $-q(t)$. We're interested in the fields a large distance $r \gg L$ from the wire.



We will also assume that the dipole is short in the sense that $L \ll \lambda = 2\pi c/\omega$. This means that the light travel time across the dipole is short compared with the timescale on which the charge oscillates. Then, to a first approximation, each part of the wire has the same retarded time $t_r \approx t - r/c$ =constant. In the retarded potential integral for $\mathbf{A}$,

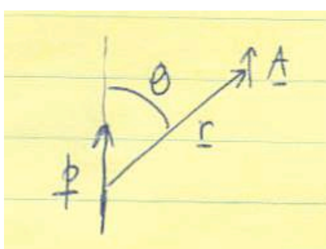
$$\int \frac{J(t_r)}{r}\, d^3r = \frac{I(t_r)L}{r} = \frac{1}{r}\frac{dq}{dt}L = \frac{1}{r}\frac{dp}{dt}$$

where $p = qL$ is the dipole moment. Therefore the vector potential is

$$\mathbf{A}(\mathbf{r}, t) = \frac{\mu_0}{4\pi r}\mathbf{I}(t_r)L = \frac{\mu_0}{4\pi r}\frac{d\mathbf{p}(t_r)}{dt} \tag{3.107}$$

with $t_r = t - r/c$.

To work out the fields, first change coordinates into spherical coordinates:



The vector $\mathbf{A}$ always points straight up (same direction as the current). In spherical coordinates,

$$A_r = A\cos\theta \qquad A_\theta = -A\sin\theta \qquad A_\phi = 0$$

where $A(r,t)$ is a function of distance r and t only. Taking the curl of $\mathbf{A}$, we see that the only non-zero component is

$$B_\phi = \frac{1}{r}\left[\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial \theta}\right] = \frac{\partial A_\theta}{\partial r} = -\sin\theta\frac{\partial A}{\partial r} \tag{3.108}$$

or

$$B_\phi = -\sin\theta\frac{\mu_0 L}{4\pi}\frac{\partial}{\partial r}\left(\frac{I(t_r)}{r}\right) \tag{3.109}$$

$$= -\sin\theta\frac{\mu_0 L}{4\pi}\left(-\frac{[I]}{r^2} + \frac{[\dot{I}]}{r}\frac{\partial t_r}{\partial r}\right) \tag{3.110}$$

where I've introduced a new notation [ ] which indicates that the quantity inside the square brackets should be evaluated at the retarded time. Then

$$B_\phi = \frac{\mu_0 L}{4\pi}\sin\theta\left(\frac{[I]}{r^2} + \frac{[\dot{I}]}{rc}\right) \tag{3.111}$$

$$= \frac{\mu_0}{4\pi}\left(\frac{[\dot{p}]}{r^2} + \frac{[\ddot{p}]}{rc}\right)\sin\theta. \tag{3.112}$$

The second term in B is known as the *radiation field*. It is proportional to $\ddot{p}$ and falls off as $1/r$ at large distance.

To get the electric field, we first need the electric potential ϕ. To find it, we can use the Lorentz gauge

$$\nabla\cdot\mathbf{A} = -\frac{1}{c^2}\frac{\partial\phi}{\partial t}$$

or

$$-\frac{1}{c^2}\frac{\partial\phi}{\partial t} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 A\cos\theta\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta(-A\sin\theta)\right) = \cos\theta\frac{\partial A}{\partial r}.$$

But we have already calculated $\partial A/\partial r = -B_\phi/\sin\theta$, which gives

$$-\frac{1}{c^2}\frac{\partial\phi}{\partial t} = -\cos\theta\frac{\mu_0}{4\pi}\left(\frac{[\dot{p}]}{r^2} + \frac{[\ddot{p}]}{rc}\right)$$

and therefore

$$\phi = \frac{\cos\theta}{4\pi\epsilon_0}\left(\frac{[p]}{r^2} + \frac{[\dot{p}]}{rc}\right).$$

The first term should look familiar — it is the usual static electric dipole potential. The second term is new, and again is a radiation field term. In the homework, you will work through deriving ϕ from the retarded potential integral directly. For now, note that whereas in the integral for $\mathbf{A}$ we were able to take the lowest order approximation $r' = r$, the electric potential is different because the lowest order terms vanish (the net charge is zero). There are two first order terms: the first with $\phi \propto 1/r^2$ comes about because the two charges are at slightly different spatial locations (the usual static dipole); the second with $\phi \propto 1/r$ is because we see the charges at slightly different retarded times, so they don't quite cancel each other out. (This is why we are able to get a monopole potential from a system with zero net charge!)

With both potentials in hand, we can evaluate $\mathbf{E} = -\nabla\phi - \dot{\mathbf{A}}$, which gives

$$\begin{aligned} E_r &= \frac{2\cos\theta}{4\pi\epsilon_0}\left(\frac{[p]}{r^3} + \frac{[\dot{p}]}{r^2 c}\right) && (3.113) \\ E_\theta &= \frac{\sin\theta}{4\pi\epsilon_0}\left(\frac{[p]}{r^3} + \frac{[\dot{p}]}{r^2 c} + \frac{[\ddot{p}]}{rc^2}\right) && (3.114) \\ E_\phi &= 0. && (3.115) \end{aligned}$$

Notice that E_θ has a radiation field $\propto 1/r$. The first terms in E_r and E_θ are the usual static dipole fields. There are also terms $\propto 1/r^2$, intermediate between the static and radiation fields that depend on $\dot{p}$, but do not contribute to radiation.

The radiation fields are the ones that dominate when $r \gg c/\omega$ or $r \gg \lambda$. They satisfy $E_\theta = cB_\phi$. The Poynting flux is

$$S = \frac{E_\theta B_\phi}{\mu_0} = \frac{\sin^2\theta}{(4\pi)^2\epsilon_0}\frac{[\ddot{p}]}{r^2 c^3}. \tag{3.116}$$

Integrating over a sphere with area element $r^2 d\Omega = r^2 \sin\theta d\theta d\phi$ gives the total power

$$P = \frac{[\ddot{p}]^2}{6\pi\epsilon_0 c^3} \tag{3.117}$$

Larmor's formula. Another way of writing it is

$$P = \frac{[\dot{I}]^2 L^2}{6\pi\epsilon_0 c^3}. \tag{3.118}$$

3.2.2 Properties of antennas

The Hertzian dipole is a simple antenna. An important concept in antenna design is *radiation resistance* which is defined as the time-averaged radiated power divided by $\langle I^2\rangle$. For the Hertzian dipole

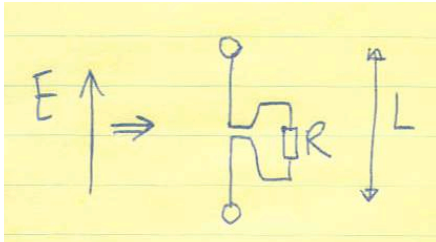
$$R_r = \frac{\langle P\rangle}{\langle I^2\rangle} = \frac{L^2}{6\pi\epsilon_0 c^3}\frac{\langle[\dot{I}]^2\rangle}{\langle I^2\rangle}.$$

Assuming we are driving the antenna with a sinusoidally-varying current with frequency ω, this is

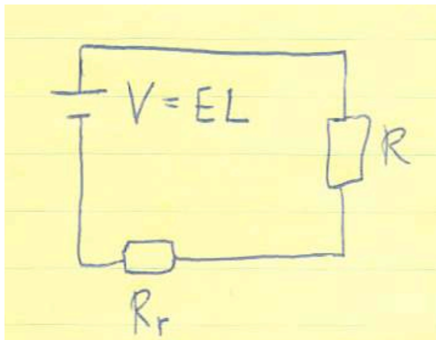
$$R_r = \frac{L^2\omega^2}{6\pi\epsilon_0 c^3} = \left(\frac{L\omega}{c}\right)^2 \frac{1}{6\pi}\sqrt{\frac{\mu_0}{\epsilon_0}} = \frac{2\pi}{3}Z_0\left(\frac{L}{\lambda}\right)^2 = 789\ \Omega\ \left(\frac{L}{\lambda}\right)^2$$

where recall we have assumed $L \ll \lambda$ so $R_r \ll 789\ \Omega$. (The impedance of free space $Z_0 = 377\ \Omega$.)

To see the importance of the radiation resistance, consider using the Hertzian dipole as a receiver. If we expose the dipole to an incoming EM wave, it will act as a receiver:



where the dipole is arranged parallel to the electric field of the wave E. The electric field causes a current to flow in the dipole, which therefore radiates. We can model this as the following circuit:



The current that flows is

$$I = \frac{EL}{R + R_r},$$

and the power absorbed in the load is

$$P_{\rm abs} = \frac{(EL)^2}{(R + R_r)^2} R.$$

The power absorbed is maximum when $R = R_r$ (matched resistance) and equal to

$$P_{\rm abs,max} = \frac{E^2 L^2}{4R_r} = \frac{E^2}{Z_0} \frac{3\lambda^2}{8\pi}.$$

But E^2/Z_0 is the incoming flux in the wave $S_{\rm inc} = EH = E^2/Z_0$, and therefore we see that the effective cross-section of the antenna is

$$\sigma = \frac{3\lambda^2}{8\pi},$$

independent of L (and again we are in the limit $L \ll \lambda$).

Another quantity that you may see discussed for antennas is the *power gain* which quantifies its directionality. The gain $G(\theta, \phi)$ is the flux in a direction (θ, ϕ) divided by the flux averaged over all directions. For example, the Hertzian dipole has $S \propto \sin^2\theta$, so

$$G = \frac{\sin^2\theta}{\int d\Omega \sin^2\theta / 4\pi} = \frac{3}{2}\sin^2\theta.$$

If you are looking at the equator of the dipole, you are receiving a flux that is 3/2 of the flux averaged over all directions.

3.3 Multipole radiation

We now develop a multipole expanson of the vector potential. First a reminder of the multipole expansion in electrostatics or magnetostatics. A large distance away from a charge distribution, the electrostatic potential can be expanded as

$$\phi(\mathbf{r}) = \int \frac{\rho(\mathbf{r}')d^3\mathbf{r}'}{4\pi\epsilon_0|\mathbf{r}-\mathbf{r}'|} = \frac{Q}{4\pi\epsilon_0 r} + \frac{\hat{r}\cdot\mathbf{p}}{4\pi\epsilon_0 r^2} + \frac{\hat{r}\cdot\mathbf{Q}_2\cdot\hat{r}}{4\pi\epsilon_0 r^3} + ...$$

where the total charge is

$$Q = \int \rho(\mathbf{r}')d^3\mathbf{r}',$$

the dipole moment is

$$\mathbf{p} = \int \rho(\mathbf{r}')\mathbf{r}'d^3\mathbf{r}',$$

and the quadrupole moment tensor is

$$(\mathbf{Q}_2)_{ij} = \int \rho(\mathbf{r}')\left[3r'_i r'_j - r'^2\delta_{ij}\right] d^3\mathbf{r}'.$$

Similarly, for a current distribution $\mathbf{J}$, the vector potential can be expanded

$$\mathbf{A}(\mathbf{r}) = \int \frac{\mu_0\mathbf{J}\,d^3\mathbf{r}'}{4\pi|\mathbf{r}-\mathbf{r}'|} = \frac{\mu_0}{4\pi r^2}\mathbf{m}\times\hat{r} + ...$$

where the magnetic dipole moment is

$$\mathbf{m} = \frac{1}{2}\int \mathbf{r}'\times\mathbf{J}(\mathbf{r}')d^3\mathbf{r}'.$$

To derive these results, use the expansion

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|} = \frac{1}{r}\left(1 + \frac{\hat{r}\cdot\mathbf{r}'}{r} + \frac{3(\hat{r}\cdot\mathbf{r}')^2 - r'^2}{r^2} + ...\right).$$

You may have seen this before but written as a sum of terms involving Legendre polynomials (see eq. [5.79] in Griffiths for example).

Now the idea is to do something similar for the time-dependent case by expanding the retarded potentials, and in particular the radiation fields, as a sum of multipole components. The key difference from the static case is that there is now a new length-scale in the problem, $\lambda = 2\pi c/\omega$. We will assume that

$$L \ll \lambda \ll r,$$

where L is the source size. In other words, we assume that the light crossing time for the source L/c is short compared to the wave period, which is short compared to the light travel time to the observer. If we look at individual Fourier components,

$$\mathbf{J} = \mathbf{J}(\mathbf{r})e^{-i\omega t}$$

etc., then the vector potential is

$$\mathbf{A} = \mathbf{A}(\mathbf{r})e^{-i\omega t} = \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}(\mathbf{r}')d^3\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|}e^{-i\omega t}e^{i\omega|\mathbf{r}-\mathbf{r}'|/c},$$

or

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}(\mathbf{r}')d^3\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|}e^{ik|\mathbf{r}-\mathbf{r}'|},$$

where $k = \omega/c$. We will expand $\mathbf{A}$ and then get the fields from $\mathbf{B} = \nabla\times\mathbf{A}$ and Faraday's law $\mathbf{E} = i(c/k)\nabla\times\mathbf{B}$. In this section I am following the approach of Pollock & Stump.

3.3.1 Electric dipole term

We first write $|\mathbf{r}-\mathbf{r}'| \approx r$ just as we did for the Hertzian dipole (we ignore the variation of retarded time across the source), giving

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\int \mathbf{J}(\mathbf{r}')d^3\mathbf{r}'.$$

Then we rewrite this using the identity

$$\nabla\cdot(r_i\mathbf{J}) = r_i\nabla\cdot\mathbf{J} + \mathbf{J}\cdot\nabla r_i = i\omega\rho r_i + J_i$$

to integrate by parts, giving

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}(-i\omega)\int \rho(\mathbf{r}')\mathbf{r}'d^3\mathbf{r}' = \frac{-i\omega\mu_0\mathbf{p}e^{ikr}}{4\pi r}.$$

To find the radiation fields, we take the curl, noting that because we want the terms that scale as $1/r$, we need only differentiate the e^{ikr} term and not the $1/r$. The answer is

$$\mathbf{B} = \frac{k^2e^{ikr}}{4\pi\epsilon_0 rc}\hat{r}\times\mathbf{p}$$

$$\mathbf{E} = -c\,\hat{r}\times\mathbf{B} = \frac{k^2e^{ikr}}{4\pi\epsilon_0 r}\hat{r}\times(\mathbf{p}\times\hat{r}).$$

The radiated power per unit steradian is

$$\frac{dP}{d\Omega} = \frac{1}{2\mu_0}\mathrm{Re}\left[r^2\hat{r}\cdot(\mathbf{E}\times\mathbf{B}^\star)\right] = \frac{k^4c}{32\pi^2\epsilon_0}\left(p^2-(\hat{r}\cdot\mathbf{p})^2\right) = \frac{k^4c}{32\pi^2\epsilon_0}p^2\sin^2\theta,$$

where $\cos\theta = \hat{p}\cdot\hat{r}$. Integrating over all space, the total power is

$$P = \frac{k^4c}{4\pi\epsilon_0}\frac{p^2}{3} = \frac{\omega^4p^2}{12\pi\epsilon_0c^3},$$

Larmor's formula for the time-averaged power from an oscillating dipole.

3.3.2 Magnetic dipole and electric quadrupole

For the next term, we expand $|\mathbf{r}-\mathbf{r}'|$ to the next order. First consider the exponent,

$$k|\mathbf{r}-\mathbf{r}'| \approx kr - k\hat{r}\cdot\mathbf{r}' + ...$$

which gives

$$e^{ik|\mathbf{r}-\mathbf{r}'|} \approx e^{ikr}e^{-k\hat{r}\cdot\mathbf{r}'} \approx e^{ikr}\left(1 - ik\hat{r}\cdot\mathbf{r}' + ...\right).$$

We had the first term already in the electric dipole piece, the new term is the second term. We could also expand the denominator $1/|\mathbf{r}-\mathbf{r}'|$, but the new term from that part is smaller by a factor of $kr \approx r/\lambda$ which we have assumed is $\gg 1$. Therefore, the next term in the expansion of $\mathbf{A}$ is

$$\mathbf{A}(\mathbf{r}) = -ik\frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\int \mathbf{J}(\mathbf{r}')\hat{r}\cdot\mathbf{r}'d^3\mathbf{r}'.$$

We again integrate by parts using

$$\nabla\cdot\left(r_i r_j \mathbf{J}\right) = r_i r_j \nabla\cdot\mathbf{J} + (\mathbf{J}\cdot\nabla r_i)r_j + (\mathbf{J}\cdot\nabla r_j)r_i = r_i r_j\nabla\cdot\mathbf{J} + J_i r_j + J_j r_i$$

or

$$\frac{1}{2}\left(J_i r_j - J_j r_i\right) = -J_j r_i - \frac{1}{2}r_i r_j \nabla\cdot\mathbf{J}.$$

The result is

$$\int \mathbf{J}(\mathbf{r}')\hat{r}\cdot\mathbf{r}'d^3\mathbf{r}' = \frac{-i\omega}{2}\int \mathbf{r}'(\mathbf{r}'\cdot\hat{r})\rho(\mathbf{r}')d^3\mathbf{r}' + \frac{1}{2}\int \hat{r}\times\left(\mathbf{J}\times\mathbf{r}'\right)d^3\mathbf{r}'.$$

So we naturally get two pieces: the first term is the electric quadrupole term, the second term is the magnetic dipole term.

The magnetic dipole term is

$$\mathbf{A}(\mathbf{r}) = \frac{ike^{ikr}}{r}\frac{\mu_0}{4\pi}\hat{r}\times\mathbf{m}.$$

The radiation fields are

$$\mathbf{B} = -\frac{k^2 e^{ikr}}{r}\frac{\mu_0}{4\pi}\hat{r}\times(\hat{r}\times\mathbf{m})$$

$$\mathbf{E} = \frac{k^2 c}{r}e^{ikr}\frac{\mu_0}{4\pi}\mathbf{m}\times\hat{r}.$$

The power is

$$\frac{dP}{d\Omega} = \frac{\mu_0}{32\pi^2 c^3}m^2\omega^4\sin^2\theta$$

$$P = \frac{\mu_0 m^2\omega^4}{12\pi c^3}.$$

The ratio of the magnetic dipole power to the electric dipole power is $(m/pc)^2 \sim (v/c)^2$ where we define a velocity $v = J/\rho$.

The quadrupole field is more complicated to deal with. The resulting fields are

$$\mathbf{B} = \frac{-i\omega^3\mu_0}{24\pi r c^2} e^{ikr} \hat{r} \times (\mathbf{Q_2} \cdot \hat{r})$$

with $\mathbf{E} = c\mathbf{B} \times \hat{r}$ and the power is

$$\frac{dP}{d\Omega} = \frac{1}{2\mu_0} \frac{\omega^6 \mu_0^2}{(24\pi)^2 c^3} |\hat{r} \times (\mathbf{Q_2} \cdot \hat{r})|^2.$$

The ratio of the power in the quadrupole term to the electric dipole term is $\sim (kL)^2 \sim (L/\lambda)^2 \ll 1$ (assuming that the size of the quadrupole moment is $\sim L \times p$).

SUMMARY

Here are the main ideas and results that we covered in this part of the course:
The retarded time and retarded potentials. The physical idea that to calculate the contribution of a point a distance r away to the electric/magnetic field, we need to evaluate the charge/current density there at the *retarded time* $t_r = t - r/c$. The retarded potentials are

$$\phi(\mathbf{r}) = \int \frac{\rho(\mathbf{r}', t_r)}{4\pi\epsilon_0 |\mathbf{r} - \mathbf{r}'|} d^3\mathbf{r}'$$

$$\mathbf{A}(\mathbf{r}) = \int \frac{\mu_0 \mathbf{J}(\mathbf{r}', t_r)}{4\pi |\mathbf{r} - \mathbf{r}'|} d^3\mathbf{r}'.$$

We also use a square bracket notation to indicate that the quantity inside should be evaluated at the retarded time. Examples: calculation of the fields of an infinite straight wire or a current sheet in which the current is turned on abruptly at $t = 0$. The static limit is reached at times much greater than a light travel time across the source.

Hertzian dipole. Two charged spheres connected by a thin wire, distance $L \ll \lambda = 2\pi c/\omega$. The fact that the current and the rate of change of the dipole moment are related: $\dot{p} = IL$. The vector potential $\mathbf{A}(\mathbf{r}, t) = \mu_0 [\dot{\mathbf{p}}] / 4\pi r$, and the three components of the fields: the radiation field $\propto 1/r$, static dipole, and intermediate fields. How to go from the fields to the Poynting vector, $dP/d\Omega$ and total power P.

Antennas. The radiation resistance (R_r = radiated power$/\langle I^2 \rangle$), power gain $G(\theta, \phi)$. Using an antenna as a receiver: effective area, load matching $R = R_r$, the idea that some of the incident power is absorbed and some reradiated (scattering) (a Hertzian dipole scatters a fraction $R_r/(R + R_r)$ of the power input). The effective area of the Hertzian dipole is $3\lambda^2/8\pi$. The radiation resistance of the Hertzian dipole is small $789\ \Omega\ (L/\lambda)^2$ (where $L \ll \lambda$). Why it matters whether the radiation resistance is large or small (power radiated for a given current).

Multipole radiation. The multipole expansion applied to radiation fields. Dipole moments

$$\mathbf{p} = \int \rho(\mathbf{r}')\, \mathbf{r}' d^3\mathbf{r}' \qquad \mathbf{m} = \frac{1}{2} \int \mathbf{r}' \times \mathbf{J}(\mathbf{r}')\, d^3\mathbf{r}'$$

and quadrupole moment tensor

$$(Q_2)_{ij} = \int \rho\left(\mathbf{r}'\right)\left\{3r'_i r'_j - r'^2 \delta_{ij}\right\} d^3\mathbf{r}'$$

The ordering of scales $L \ll \lambda \ll r$ which means that the phase factor $\exp(-ikr)$ is the piece which gives successive terms in the expansion (i.e. variations in retarded time across the source). The electric dipole term from setting $|\mathbf{r} - \mathbf{r}'| = r$ in the denominator and the phase factor. The power radiated in the electric dipole term is

$$\frac{dP}{d\Omega} = \frac{\omega^4 p^2}{32\pi^2 \epsilon_0 c^3} \sin^2\theta \qquad P = \frac{\omega^4 p^2}{12\pi\epsilon_0 c^3}.$$

The magnetic dipole and electric quadrupole terms from taking the next term in the expansion of $|\mathbf{r} - \mathbf{r}'|$ in the phase factor. The power in the magnetic dipole is $\sim (v/c)^2$ relative to the electric dipole. The power in the electric quadrupole is $\sim (kL)^2$ compared to the electric dipole emission.

Part 4: Relativity and Electromagnetism

These are notes for the fourth part of PHYS 352 Electromagnetic Waves. The question we want to answer here is how electromagnetism fits with special relativity.

We know from relativity that light moves at speed c in vacuum in all inertial frames. When we derived the speed of light from Maxwell's equations, we didn't ask what frame we were in, which suggests that they are already independent of frame. However, this doesn't seem the case at first glance. By changing frame, we can make moving charges appear stationary, implying that **E** and **B** must mix under a Lorentz transformation. How does this work, and how do we write Maxwell's equations in a relativistically covariant way?

We've already had a hint of how to do this in the symmetry between the wave equations for ϕ and **A**, which we mentioned can be written as a single equation involving 4-vectors. But we will also write down an electromagnetic field tensor that describes the electromagnetic field at a given location, with the division into **E** and **B** occurring once we choose a reference frame.

4.1 A review of some ideas from special relativity

First, we'll go over some of the ideas from special relativity that we will build on to incorporate electromagnetism.

4.1.1 Lorentz transform

We consider a frame S (the "lab" frame) and a frame S' moving with velocity v with respect to frame S. Events in the two frames are related by the Lorentz transform:

$$t' = \gamma\left(t - \frac{vx}{c^2}\right) \qquad x' = \gamma\left(x - vt\right) \qquad y' = y \qquad z' = z$$

where

$$\gamma^2 = \frac{1}{1-\beta^2} \qquad \beta = \frac{v}{c}$$

and we've taken v along the x-direction. A useful identity to remember is $\gamma^2\beta^2 = \gamma^2 - 1$.

A couple of important results follow immediately:

1. **Time dilation**. Consider two events in S' at the same location $\Delta x' = 0$, but different times $\Delta t' > 0$. Then in S, the time delay between the events is

$$\Delta t = \gamma\left(\Delta t' - \frac{v\Delta x'}{c^2}\right) = \gamma\Delta t' > \Delta t'.$$

2. **Length contraction.** Two events in S' spaced a distance $\Delta x' = L$. Make a simultaneous measurement in S:

$$\Delta t = 0 \Rightarrow \Delta t' = -v\Delta x/c^2$$

$$\Delta x = \gamma \Delta x' + \gamma v \Delta t' = \gamma \Delta x' (1 - \beta^2) = \frac{\Delta x'}{\gamma}$$
$$\Rightarrow \Delta x = \frac{L}{\gamma} < L.$$

4.1.2 4-vectors

Just as we think of events as having coordinates in 4D spacetime, we can also define 4-vectors that span the 4D spacetime (or Minkowski space). The idea is that just as 3D vectors exist independently of particular choice of coordinate axes, a 4-vector exists similarly in Minkowski space. Components of the 4-vector in one reference frame are related to those in another frame by the Lorentz transform.

In terms of the 4-vector $x^\mu = (ct, \mathbf{x})$, the Lorentz transform is

$$x'^\mu = \Lambda^\mu_{\ \nu} x^\nu, \tag{4.119}$$

with

$$\Lambda^\mu_{\ \nu} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Memorising this matrix is a good way to learn the Lorentz transform. Here we use the usual convention that greek indices run over all 4-indices from 0 to 3 (roman indices run over the spatial indices 1 to 3), and we use the Einstein summation convention.

The (frame-independent) scalar product of two 4-vectors is given by

$$g_{\mu\nu} a^\mu b^\nu = -a^0 b^0 + \mathbf{a} \cdot \mathbf{b},$$

where the metric

$$g_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1).$$

A useful way to think about this is in terms of contravariant and covariant vectors. A *contravariant* vector transforms according to equation (4.119), whereas a *covariant* vector, written with indices down, transforms according to

$$x'_\mu = (\Lambda^{-1})^\nu_{\ \mu} x_\nu. \tag{4.120}$$

Note that the components of the inverse matrix Λ^{-1} are the same as Λ except the off-diagonal components change sign. The covariant and contravariant vectors are related by contracting with the metric, for example

$$x_\mu = g_{\mu\nu} x^\nu.$$

An important example of a covariant vector is the derivative $\partial/\partial x^\mu = \partial_\mu$ (see Appendix A). Note that because of the form of $g^{\mu\nu}$ that we will be using here (flat spacetime), the contravariant and covariant vectors differ only by a minus sign in the time-component. The scalar product is then

$$a_\mu b^\mu = g_{\mu\nu} a^\mu b^\nu = -a^0 b^0 + \mathbf{a} \cdot \mathbf{b}.$$

Going in the other direction, we can construct a tensor which has a more complicated transformation

$$T'^{\mu\nu} = \Lambda^{\mu}{}_{\rho}\Lambda^{\nu}{}_{\sigma}T^{\rho\sigma}.$$

Note that each index gets a separate Lorentz transformation.

A general principle is that if we can write down a physical equation in one frame in terms of scalars, 4-vectors or tensors, we know how it transforms from one frame to another. This is equivalent to writing down a vector equation in 3D like $\mathbf{F} = m\mathbf{a}$ where we don't have to worry about the coordinate system or vector components. We'll use this idea to figure out how we should write the electromagnetic field.

4.1.3 Dynamics in special relativity

Now consider a particle moving with velocity $\mathbf{u}$ in some frame. We want to describe its dynamics.

1. **Proper time.** We've seen that the norm of a 4-vector is a Lorentz scalar, e.g. $x^\mu x_\mu = -(ct)^2 + x^2 + y^2 + z^2$ is the same in all frames. Along a particle's trajectory in spacetime, the interval

$$ds^2 = -c^2dt^2 + dx^2 + dy^2 + dz^2$$

is a scalar. In the rest frame, the spatial part is zero (the particle is not moving), and so the interval has a contribution only from the dt term. We therefore write ds^2 in terms of the time experienced in the rest frame $d\tau$:

$$d\tau^2 = -\frac{ds^2}{c^2}$$

where τ is known as the proper time. The corresponding time in the lab frame is (using the Lorentz transform with $dx' = 0$, $dt' = d\tau$)

$$dt = \gamma d\tau$$

(longer because of time dilation).

2. **4-velocity**. We define the 4-velocity as

$$\eta^\mu = \frac{dx^\mu}{d\tau}.$$

Because it is composed of a 4-vector and a scalar, it is another 4-vector (it transforms in the same way as x^μ). For a particle moving at speed u, the components are

$$\eta^i = \frac{dx^i}{d\tau} = \gamma\frac{dx^i}{dt} = \gamma u^i \qquad \eta^0 = c\frac{dt}{d\tau} = \gamma c.$$

Therefore

$$\eta^\mu = \gamma(u)(c, \mathbf{u})$$

with

$$\gamma^2(u) = \frac{1}{1-(u/c)^2}.$$

You can check that $\eta^\mu \eta_\mu = -c^2$ is indeed invariant as it should be.

3. **4-momentum**. We define the 4-momentum

$$p^\mu = m\eta^\mu$$

where m is the rest mass of the particle. The components are

$$p^0 = \gamma mc = \frac{E}{c} \qquad p^i = \gamma m u^i = p^i$$

where the particle energy and momentum are $E = \gamma mc^2$ and $p = \gamma mu$. The dot product $p^\mu p_\mu = -E^2/c^2 + p^2 = -m^2c^2$. In collision problems, conserving the 4-momentum is equivalent to conserving both momentum and energy.

4. **Equation of motion**. We write the equation of motion of a particle as

$$\frac{dp^\mu}{d\tau} = K^\mu$$

where K^μ is the Minkowski force. Consider a particle in frame S subject to force **F**. Then

$$\mathbf{K} = \frac{d\mathbf{p}}{d\tau} = \gamma \frac{d\mathbf{p}}{dt} = \gamma \mathbf{F} \qquad K^0 = \frac{1}{c}\frac{dE}{d\tau} = \frac{\gamma}{c}\frac{dE}{dt} = \frac{\gamma}{c}\mathbf{F}\cdot\mathbf{u}$$

so that

$$K^\mu = \gamma\left(\frac{\mathbf{F}.\mathbf{u}}{c}, \mathbf{F}\right). \tag{4.121}$$

Note that $K^\mu \eta_\mu = 0$.

With these results in place, we'll now consider how to incorporate electromagnetism into a Lorentz-covariant form in the next section.

4.2 The Lorentz force and the electromagnetic field tensor

One way to see how we should write the electromagnetic field in special relativity is to consider the Minkowski force (eq. [4.121]) that arises from the Lorentz force on a particle, and try to write it in terms of 4-vectors, i.e. in a frame-independent way. We'll see that it leads us to the conclusion that the electromagnetic field should be written as a tensor.

Consider a particle in frame S subject to a Lorentz force

$$\mathbf{F} = q\left(\mathbf{E} + \mathbf{v}\times\mathbf{B}\right).$$

The spatial part of K^μ is therefore

$$\mathbf{K} = \gamma_u q \left(\mathbf{E} + \mathbf{u} \times \mathbf{B}\right).$$

The idea is to write this in terms of the components of a 4-vector that we already know, in particular the 4-velocity $\eta^\mu = \gamma_u(c, \mathbf{u})$, and so

$$\mathbf{K} = q\frac{\eta^0}{c}\mathbf{E} + q\eta \times \mathbf{B}$$

or in components

$$K^i = q\eta^0 \frac{E^i}{c} + q\epsilon_{ijk}\eta^j B^k. \tag{4.122}$$

The time part of K^μ is

$$K^0 = \gamma_u q \frac{\mathbf{u} \cdot \mathbf{E}}{c} = q\frac{\eta \cdot \mathbf{E}}{c} = q\frac{\eta^i E^i}{c}. \tag{4.123}$$

Now we define the electromagnetic field tensor $F^{\mu\nu}$ with components

$$F^{00} = 0 \qquad F^{0i} = -F^{i0} = \frac{E^i}{c} \qquad F^{ij} = \epsilon_{ijk}B^k$$

or written as a matrix

$$F^{\mu\nu} = \begin{pmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & B_z & -B_y \\ -E_y/c & -B_z & 0 & B_x \\ -E_z/c & B_y & -B_x & 0 \end{pmatrix}.$$

Then we can write equations (4.122) and (4.123) as

$$K^\mu = q\eta_\nu F^{\mu\nu}. \tag{4.124}$$

Because η^μ and K^μ are both 4-vectors, then $F^{\mu\nu}$ must be a tensor, ie. it transforms according to

$$F'^{\mu\nu} = \Lambda^\mu{}_\sigma \Lambda^\nu{}_\tau F^{\sigma\tau} \tag{4.125}$$

where the equation is evaluated at the same space-time position, ie. on the left hand side F' is evaluated at x'^μ and on the right F is evaluated at x^μ, where the components x'^μ and x^μ are related by the usual Lorentz transforms.

The equation of motion of a particle under the Lorentz force is therefore

$$\frac{dp^\mu}{d\tau} = q\eta_\nu F^{\mu\nu}. \tag{4.126}$$

In frame S the components are

$$\frac{d\mathbf{p}}{dt} = q\left(\mathbf{E} + \mathbf{u} \times \mathbf{B}\right) \qquad \frac{dE}{dt} = q\mathbf{u} \cdot \mathbf{E}$$

as expected. (Note that I'm using the symbol E to refer to the particle energy E and the electric field **E** here. It should be clear from the context which is which!)

4.3 Transformation laws for E and B

We can use the Lorentz transformation of $F^{\mu\nu}$ (eq. [4.125]) to figure out how **E** and **B** transform between frames. Consider the usual setup in which S' moves along the x-axis of S with velocity v, and

$$\Lambda = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

First, think about the electric field. From the definition of $F^{\mu\nu}$, we pull out the piece that gives the electric field in frame S':

$$E'^{i} = cF'^{0i} = c\Lambda^{0}{}_{\rho}\Lambda^{i}{}_{\tau}F^{\rho\tau}.$$

Then it is just a matter of working out which terms contribute on the right hand side. For example, note that ρ must be either 0 or 1 to give a non-zero element of $\Lambda^{0}{}_{\rho}$. Then we look at each value of i in turn:

- $i = 1$: $\tau = 0$ or 1 for non-zero $\Lambda^{1}{}_{\tau}$, but $F^{00} = 0$ and $F^{11} = 0$ so there are only two terms, either $\rho\tau = 01$ or $\rho\tau = 10$. Writing these out gives

$$E'_x = c\gamma^2 \frac{E_x}{c} + c(-\beta\gamma)^2 \frac{E_x}{c} = \gamma^2 E_x (1 - \beta^2) = E_x.$$

- $i = 2$: this time τ must be 2, and ρ can be 0 or 1 so there are two terms again involving F^{02} and F^{12}:

$$E'_y = c\gamma \frac{E_y}{c} + c(-\beta\gamma)B_z = \gamma(E_y - vB_z).$$

- $i = 3$: similar to $i = 2$:

$$E'_z = \gamma(E_z + vB_y).$$

The B field in S' is $B'^{i} = F'^{jk}$ where ijk is a cyclic permutation. This is left as an exercise which is covered in one of the problems. The result is

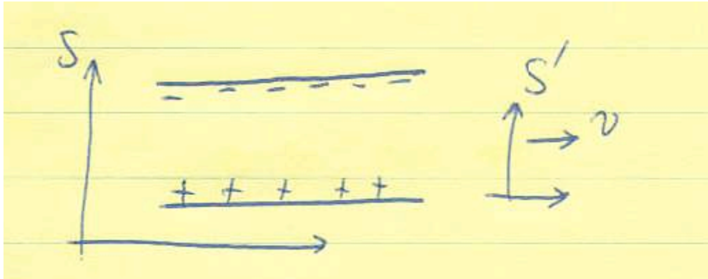
$$B'_x = B_x \qquad B'_y = \gamma\left(B_y + \frac{vE_z}{c^2}\right) \qquad B'_z = \gamma\left(B_z - \frac{vE_y}{c^2}\right).$$

More generally, the transformation of **E** and **B** is

$$\begin{aligned} E'_{\parallel} &= E_{\parallel} \qquad \mathbf{E}'_{\perp} = \gamma(\mathbf{E}_{\perp} + \mathbf{v} \times \mathbf{B}_{\perp}) \\ B'_{\parallel} &= B_{\parallel} \qquad \mathbf{B}'_{\perp} = \gamma\left(\mathbf{B}_{\perp} - \frac{\mathbf{v} \times \mathbf{E}_{\perp}}{c^2}\right). \end{aligned} \tag{4.127}$$

For inverse transforms, swap $E' \leftrightarrow E$, $v \leftrightarrow -v$ and $B' \leftrightarrow B$. As we anticipated, the fields **E** and **B** mix when we make a Lorentz transformation, specifically the fields perpendicular to the boost direction.

To see these transformations at work, consider two examples. The first is a parallel plate capacitor as viewed from a frame moving along the plates:



In S, $B = 0$ and $E_z = \sigma/\epsilon_0$ (standard result for plane parallel capacitor), $E_x = E_y = 0$. In S', using the transforms gives

$$E'_z = \gamma \frac{\sigma}{\epsilon_0} \qquad E'_x = E'_y = 0$$

and

$$B'_y = \frac{\gamma v}{c^2} \frac{\sigma}{\epsilon_0} \qquad B'_x = B'_z = 0.$$

Physically, the observer in S' sees an increased surface charge density because of length contraction $\sigma' = \gamma\sigma$, and the electric field is as expected $E' = \sigma'/\epsilon_0$. The B field arises because there is now a surface current $K' = \sigma' v = \gamma\sigma v$ and the B field is the expected value $B' = \mu_0 K' = \mu_0 \gamma\sigma v = \gamma\sigma v/\epsilon_0 c^2$. (And you can check that the direction of B' makes sense given the direction of motion and the right hand rule).

The second example is an electromagnetic wave travelling in the x-direction. Its perpendicular E and B fields will mix under a boost in the direction of propagation, and it is interesting to look at what happens. Write the wave as

$$E_z = E_0 e^{i(kx - \omega t)} \qquad B_y = -\frac{E_0}{c} e^{i(kx-\omega t)}.$$

There are two pieces to transform: the coordinates x and t, and the electric and magnetic field amplitudes. First the coordinates:

$$kx - \omega t = k\gamma(x' + vt') - \omega\gamma\left(t' + \frac{vx'}{c^2}\right)$$

which can be written as $k'x' - \omega' t'$ with

$$k' = \gamma\left(k - \frac{v\omega}{c^2}\right) \qquad \omega' = \gamma(\omega - vk).$$

Two things to note are (1) we can define a 4-vector $k^\mu = (\omega/c, \mathbf{k})$ (which is consistent with the photon 4-momentum $p^\mu = \hbar k^\mu$), and (2) the dispersion relation $\omega = ck$ in frame S also holds in frame S', $\omega' = ck'$. However, the frequency and wavelength are different in the new frame: using the dispersion relation to eliminate k gives

$$\omega' = \gamma(1-\beta)\omega = \sqrt{\frac{1-\beta}{1+\beta}}\omega$$

which is the relativistic Doppler shift.

Now look at the field amplitude:

$$E_0' = \gamma E_0 + \gamma v B_y = \gamma E_0 - \gamma\beta E_0 = E_0\gamma(1-\beta) = E_0\sqrt{\frac{1-\beta}{1+\beta}}.$$

Similarly, you can show that

$$E_z' = E_0' e^{i(k'x'-\omega' t')} \qquad B_y' = -\frac{E_0'}{c} e^{i(k'x'-\omega' t')}.$$

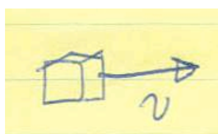
In the moving frame the wave has the same polarization, but its frequency/wavelength is Doppler shifted and the amplitude changes. For $v > 0$ the observer is moving with the wave and the frequency is redshifted, $\omega' < \omega$, and $E_0' < E_0$. For $v < 0$ the observer is moving against the wave and the frequency is blueshifted, $\omega' > \omega$, and $E_0' > E_0$.

In each of the two examples, an important point is that we would get the same answer if we had just applied Maxwell's equations in the new frame, as long as we account for length contraction of the sources (in the capacitor example, $\sigma' = \gamma\sigma$). This suggests that Maxwell's equations are already Lorentz covariant. In the next section, we will show this explicitly by writing them in tensor notation in terms of $F^{\mu\nu}$.

4.4 Maxwell's equations in covariant form

4.4.1 The 4-current J^μ

Before we can write down Maxwell's equations, we need to know how to write the sources ρ and $\mathbf{J}$ as a 4-current J^μ. To motivate this, think about an element of charge moving in frame S:



In the rest frame, the charge density is ρ_0 and there is no current density. In S, the box is contracted in the x-direction but contains the same number of charges, so the charge density in S must be $\gamma\rho_0$. The current density in S is $\mathbf{J} = \mathbf{v}\gamma\rho_0$. We see that ρ and $\mathbf{J}$ naturally mix under Lorentz transformations. This suggests that we should define a 4-current

$$J^\mu = \begin{pmatrix} \rho c \\ \mathbf{J} \end{pmatrix} = \rho_0 \eta^\mu. \tag{4.128}$$

To get a feeling for this, think about a couple of examples:

1. in frame S', $\mathbf{J}' = 0$, charge density $= \rho$. Then, in S we have

   $$c\rho = \gamma(c\rho' - \beta\gamma J_x') \Rightarrow \rho = \gamma\rho'$$

   which is as expected (length contraction increases the charge density). The current is

   $$J_x = \beta\gamma c\rho' + \gamma J_x' \Rightarrow J_x = v\gamma\rho'$$

   again as expected from our earlier arguments.

2. Now consider $\rho = 0$, $\mathbf{J} = \hat{x} J_x$ in frame S. Boost into frame S':

$$c\rho' = -\beta\gamma J_x \Rightarrow \rho' = -\frac{\gamma v J_x}{c^2}$$

and

$$J'_x = \gamma J_x.$$

In this example, a charge density appears in frame S'. We can understand this from the different length contractions for the positive and negative charges that are carrying the current, but moving in opposite directions. How should we understand the change in the current density from one frame to another?

The 4-current allows a compact expression for charge conservation,

$$\frac{\partial}{\partial x^\mu} J^\mu = \partial_\mu J^\mu = 0,$$

which you should be able to show is equivalent to $\nabla \cdot \mathbf{J} = -\partial\rho/\partial t$.

4.4.2 Maxwell's equations in terms of the EM field tensor

We might guess that Maxwell's equations are going to come from a derivative of $F^{\mu\nu}$, so let's look at $\partial_\nu F^{\mu\nu}$. The time component is

$$\frac{\partial}{\partial x^\nu} F^{0\nu} = \frac{\partial}{\partial x^i} F^{0i} = \frac{1}{c} \nabla \cdot \mathbf{E}$$

which is the left hand side of Gauss' law. The spatial component is

$$\frac{\partial}{\partial x^\nu} F^{i\nu} = \frac{1}{c}\frac{\partial}{\partial t} F^{i0} + \frac{\partial}{\partial x^j} F^{ij} = -\frac{1}{c^2}\frac{\partial E^i}{\partial t} + \epsilon_{ijk}\frac{\partial}{\partial x^j} B^k = \left[-\frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} + \nabla \times \mathbf{B} \right]_i$$

which looks like Ampere's law without the source term. This suggests that we add the 4-current as a source:

$$\frac{\partial}{\partial x^\nu} F^{\mu\nu} = \mu_0 J^\mu. \tag{4.129}$$

Looking again at the spatial and time components, the new term adds a term $\mu_0 \mathbf{J}$ to Ampere's law, and a term $\mu_0 \rho c = \rho/\epsilon_0 c$ in the time component, as needed for Gauss' law. Therefore equation (4.129) is half of what we were looking for - a Lorentz covariant way of writing two of Maxwell's equations, the equations with the source terms.

The other two Maxwell equations come from the *dual field*

$$G^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta} F_{\alpha\beta}.$$

The components are

$$G^{00} = 0 \qquad G^{ij} = \frac{1}{2}\epsilon^{ij\alpha\beta} F_{\alpha\beta} = -\epsilon^{ijk}\frac{E^k}{c}$$

and

$$G^{i0} = -G^{0i} = -\frac{1}{2}\epsilon^{0ijk}F_{jk} = -\frac{1}{2}\epsilon^{0ijk}\epsilon_{jkl}B^l = -B^i.$$

(I use the fact that the contraction of ϵ's with two common indices gives a factor of 2, i.e. $\epsilon_{ijk}\epsilon_{ijl} = 2\delta_{kl}$). As a matrix,

$$G^{\mu\nu} = \begin{pmatrix} 0 & B_x & B_y & B_z \\ -B_x & 0 & -E_z/c & E_y/c \\ -B_y & E_z/c & 0 & -E_x/c \\ -B_z & -E_y/c & E_x/c & 0 \end{pmatrix}.$$

To get $G^{\mu\nu}$ from $F^{\mu\nu}$, replace E/c with B and B with $-E/c$.

Then

$$\frac{\partial}{\partial x^\nu}G^{\mu\nu} = 0 \tag{4.130}$$

has components

$$\nabla \cdot \mathbf{B} = 0 \qquad \nabla \times \mathbf{E} - \frac{\partial \mathbf{B}}{\partial t} = 0.$$

(Try it!)

Equations (4.129) and (4.130) are Maxwell's equations written in covariant form.

4.4.3 The 4-potential

We earlier derived the wave equations for the potentials ϕ and $\mathbf{A}$,

$$-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\mathbf{A} + \nabla^2\mathbf{A} = -\mu_0\mathbf{J}$$

$$-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\phi + \nabla^2\phi = -\frac{\rho}{\epsilon_0}.$$

One the left hand side, we now see that we have the operator

$$\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x_\mu} = \partial_\mu\partial^\mu = -\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \nabla^2 = \Box^2$$

and on the right hand side we have components of $\mu_0 J^\mu$. This suggests we define a 4-vector

$$A^\mu = \left(\frac{\phi}{c}, \mathbf{A}\right)$$

and then

$$\Box^2 A^\mu = -\mu_0 J^\mu.$$

Some notes:

1. To obtain the fields from the potentials previously, we wrote $\mathbf{B} = \nabla \times \mathbf{A}$ and $\mathbf{E} = -\nabla V - \partial\mathbf{A}/\partial t$. The equivalent here is

$$F_{\mu\nu} = \frac{\partial A_\nu}{\partial x^\mu} - \frac{\partial A_\mu}{\partial x^\nu}.$$

   (You should be able to show this gives the right answer).

2. The wave equations were derived for the Lorentz gauge choice $\nabla \cdot \mathbf{A} = -(1/c^2)(\partial\phi/\partial t)$ which can be written

$$\partial_\mu A^\mu = 0,$$

so that we are back to the simple choice of choosing a zero-divergence vector potential, but now in the world of 4-vectors.

3. Note that when

$$F_{\mu\nu} = \frac{\partial A_\nu}{\partial x^\mu} - \frac{\partial A_\mu}{\partial x^\nu},$$

then

$$G^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}\left(\frac{\partial A_\beta}{\partial x^\alpha} - \frac{\partial A_\alpha}{\partial x^\beta}\right)$$

and

$$\frac{\partial}{\partial x^\nu}G^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}\left(\partial_\nu\partial\alpha A_\beta - \partial_\nu\partial_\beta A_\alpha\right) = 0.$$

So writing the field in terms of potentials guarantees that the Maxwell equations without source terms are satisfied, as expected.

4.4.4 Invariants

Finally, we can use $F^{\mu\nu}$ and $G^{\mu\nu}$ to find invariants. The quantity $F^{\mu\nu}F_{\mu\nu}$ is a scalar (we have contracted all indices). Lowering both indices using the metric, you can show that

$$F_{\mu\nu} = \begin{pmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & B_z & -B_y \\ E_y/c & -B_z & 0 & B_x \\ E_z/c & B_y & -B_x & 0 \end{pmatrix}$$

(as you may have expected the time components have changed sign). Then

$$F^{\mu\nu}F_{\mu\nu} = 2\left(-\frac{E^2}{c^2} + B^2\right)$$

so that

$$X = -\frac{1}{2}F^{\mu\nu}F_{\mu\nu} = \frac{E^2}{c^2} - B^2$$

is a scalar. Similarly, you can show that

$$Y = \frac{1}{4}G_{\mu\nu}F^{\mu\nu} = -\mathbf{B}\cdot\mathbf{E}$$

is a scalar. Both $E^2 - c^2B^2$ and $\mathbf{E}\cdot\mathbf{B}$ are invariant under Lorentz transforms.

This has some interesting implications:

1. An EM wave has $X = 0$ and $Y = 0$, and so in all frames $\mathbf{E}$ and $\mathbf{B}$ are perpendicular and $|\mathbf{E}| = c\,|\mathbf{B}|$.

2. Is there a frame in which the electromagnetic field is pure **E** or pure **B**? For this to happen, $\mathbf{E} \cdot \mathbf{B}$ must vanish (since it will vanish in the pure frame and so must vanish in all frames), so only if $Y = 0$. Then if $X > 0$ there is a frame where B vanishes (pure E is the only pure field that can make X positive and it must be the same in all frames), while if $X < 0$ there is a frame in which E vanishes (pure B).

4.5 Energy and momentum conservation in relativity

In part 1 of the course, we derived the equation for energy conservation

$$\frac{\partial}{\partial t}\left(\frac{B^2}{2\mu_0} + \frac{\epsilon_0 E^2}{2}\right) + \nabla \cdot \left(\frac{\mathbf{E} \times \mathbf{B}}{\mu_0}\right) = -\mathbf{J} \cdot \mathbf{E}. \tag{4.131}$$

At this stage, you might notice that on the left hand side there is a time-derivative and a spatial derivative. Could we write the left hand side as a 4-divergence? The answer is yes, we can write energy and momentum conservation as

$$\partial_\nu T^{\mu\nu} = -J_\nu F^{\mu\nu} \tag{4.132}$$

where $T^{\mu\nu}$ is the *energy-momentum tensor*. The $\mu = 0$ component of equation (4.132) is

$$\frac{1}{c}\frac{\partial}{\partial t} T^{00} + \frac{\partial}{\partial x^i} T^{0i} = -\frac{\mathbf{J} \cdot \mathbf{E}}{c} \tag{4.133}$$

where we've used $F^{00} = 0$ and $F^{0i} = E^i/c$. Comparing with equation (4.131), we see that T^{00} is the energy density in the fields, and cT^{0i} is the i-th component of the Poynting flux S^i.

What about the i-th (spatial) component of equation (4.132)? Using the definition of $F^{\mu\nu}$ on the right hand side, it is

$$\frac{1}{c}\frac{\partial}{\partial t} T^{i0} + \frac{\partial}{\partial x^j} T^{ij} = -\left(\rho E^i + \epsilon_{ijk} J^j B^k\right). \tag{4.134}$$

The right hand side is the Lorentz force per unit volume. A force per unit volume gives a rate of change of momentum per unit volume, or a rate of change of momentum density. Therefore we interpret T^{i0}/c as the momentum density, and we already know how to write that – it is S^i/c – and therefore $T^{i0} = T^{0i} = S^i/c$.

What about T^{ij}? We see from the form of equation (4.134) that T^{ij} must be a momentum flux. Specifically, T^{ij} is the flux of the i-th component of momentum in the j-direction. Equation (4.132) represents both energy and momentum conservation in a single equation. Integrating the momentum equation (4.134) over a volume gives

$$-\int dV\,(\rho\mathbf{E} + \mathbf{J} \times \mathbf{B}) = \frac{\partial}{\partial t}\left(\int dV \frac{T^{i0}}{c}\right) + \int dV \frac{\partial}{\partial x^j} T^{ij} \tag{4.135}$$

or, using the divergence theorem to transform the last term into a surface integral,

$$-\int dV\,(\rho\mathbf{E} + \mathbf{J} \times \mathbf{B}) = \frac{\partial}{\partial t}\left(\int dV \frac{T^{i0}}{c}\right) + \int dS\, n_j T^{ij} \tag{4.136}$$

where n_j is the j-th component of the normal vector to the surface. We see that the diagonal parts of T^{ij}, e.g. T^{xx} give the forces perpendicular to the surface – they act like the pressure of a gas which is always perpendicular to a surface. The off-diagonal components represent forces that are parallel to a surface. For example, T^{xz} represents a force in the z direction on a surface whose normal vector is in the x-direction. These off-diagonal pieces are shearing forces. T^{ij} for electromagnetic fields is known as the Maxwell stress tensor.

Given the fact that we have already identified T^{00} with the energy density in the fields $\sim E^2 + B^2$ and cT^{i0} with the Poynting flux $\propto \mathbf{E} \times \mathbf{B}$, it is possible to deduce the form of $T^{\mu\nu}$ written in terms of the field $F^{\mu\nu}$. It must be quadratic in the fields, and in fact is

$$T^{\mu\nu} = \frac{1}{\mu_0}\left[F^{\mu\rho}F^{\nu}_{\ \rho} - \frac{1}{4}g^{\mu\nu}F^{\rho\sigma}F_{\rho\sigma}\right]. \tag{4.137}$$

In terms of $\mathbf{E}$ and $\mathbf{B}$,

$$T^{00} = \frac{B^2}{2\mu_0} + \frac{1}{2}\epsilon_0 E^2 \tag{4.138}$$

$$T^{0i} = T^{i0} = \frac{\epsilon_{ijk}E^j B^k}{c\mu_0} = \frac{S^i}{c} \tag{4.139}$$

$$T^{ij} = \left(\frac{B^2}{2\mu_0} + \frac{1}{2}\epsilon_0 E^2\right)\delta^{ij} - \frac{E^i E^j}{\mu_0 c^2} - \frac{B^i B^j}{\mu_0}. \tag{4.140}$$

For a time-independent situation, equation (4.136) gives a way to evaluate the force on a system of charges and currents (left hand side) as a surface integral of the Maxwell stress tensor (right hand side). You'll see some examples in the old exams. The strategy is to first calculate the **E** and **B** fields due to the currents and/or charges, use them to evaluate T^{ij}, and then choose a suitable surface and compute the surface integral of T^{ij}. Whereever possible in these problems, use symmetry to simplify the calculation, e.g. by choosing the right surface you may only have to calculate one or two components of T^{ij}.

4.6 The fields of a moving charge

4.6.1 Derivation using Lorentz transform

In frame S, a charge q moves with velocity v in the x-direction. What are the **E** and **B** fields? We can answer this by moving into the rest frame S' of the particle, where the charge is at the origin and

$$\mathbf{E}'(\mathbf{x}') = \frac{q}{4\pi\epsilon_0}\frac{\mathbf{x}'}{|\mathbf{x}'|^3} \qquad \mathbf{B}' = 0 \tag{4.141}$$

where $\mathbf{x}' = (x', y', z')$.

We are interested in the field at position (x, y, z, t) in frame S, or $x' = \gamma(x - vt)$, $y' = y$, $z' = z$. At that point, the electric field components in S are $E_x = E'_x$, $E_y = \gamma E'_y$,

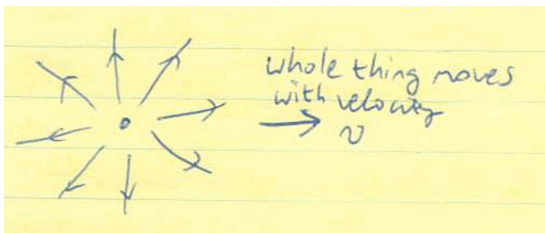
$E_z = \gamma E'_z$ (using the Lorentz transforms for E and B). Putting this all together gives

$$\mathbf{E}(\mathbf{x}, t) = \frac{q\gamma}{4\pi\epsilon_0} \frac{(x - vt)\hat{x} + y\hat{y} + z\hat{z}}{[\gamma^2(x - vt)^2 + y^2 + z^2]^{3/2}}. \tag{4.142}$$

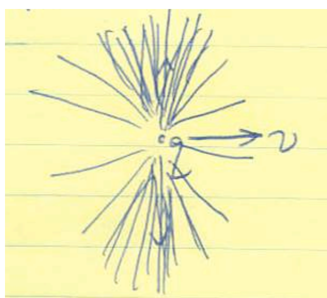
The x coordinate of the moving charge is $x = vt$.

Equation (4.142) is actually a little wierd. The field is radial and points away from the current charge position. This means that the field everywhere "knows" the instantaneous position of the charge, despite the fact that the information from the charge is communicated at the speed of light, i.e. depends on what the charge was doing one light travel time ago. We used this when we used Thomson's geometric argument for the radiation from an accelerated charge in Part 1 – we assumed that the field pointed radially back at the charge even for a moving charge.

If $v \ll c$ so that $\gamma \approx 1$, then the electric field is the Coulomb field centered on the current position of the charge.



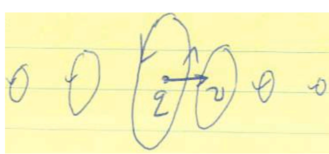
When v is large enough that γ becomes significant, $\mathbf{E}$ falls off more quickly in the x-direction than in the y or z directions. This means that the electric field lines are compressed into a disk perpendicular to the direction of motion.



The magnetic field is $B_x = 0$, $B_y = -\gamma v E'_z/c^2$, and $B_z = \gamma v E'_y/c^2$ (i.e. $\mathbf{B} = \gamma \mathbf{v} \times \mathbf{E}/c^2$) or

$$\mathbf{B}(\mathbf{x}, t) = \frac{\mu_0 \gamma q v}{4\pi} \frac{(y\hat{z} - z\hat{y})}{[\gamma^2(x - vt)^2 + y^2 + z^2]^{3/2}}. \tag{4.143}$$

These are loops of magnetic field around the direction of motion,



We can rewrite B as

$$\mathbf{B} = \frac{\mu_0 \gamma q v}{4\pi} \hat{\phi} \frac{r}{[r^2 + \gamma^2 (x - vt)^2]^{3/2}}, \tag{4.144}$$

where we use a cylindrical coordinate system with x being along the symmetry axis.

Note that

$$\frac{c|\mathbf{B}|}{|\mathbf{E}|} \approx \frac{v}{c}. \tag{4.145}$$

4.6.2 Larmor's formula

Thomson's geometric argument for the radiation from an accelerated charge gave Larmor's formula:

$$P = \frac{q^2 a^2}{6\pi\epsilon_0 c^3} \tag{4.146}$$

where P is the power radiated by a charge q with acceleration a. Earlier, we derived the power radiated by an electric dipole

$$P = \frac{(\ddot{p})^2}{6\pi\epsilon_0 c^3} \tag{4.147}$$

from the retarded potential. If we write the dipole moment of a single charge as $\mathbf{p} = q\mathbf{r}$ and therefore $\ddot{p} = qa$, we get Larmor's formula (eq. [4.146]). Note that the electric dipole formula assumes that the source size is much smaller than the wavelength of the radiation. For a moving charge, it means that in one wave period, the charge should move much less than a wavelength, $v \times \text{Period} \ll \lambda$ or $v \ll c$. The electric dipole formula can only be applied to non-relativistic particles. This is all consistent because in the geometric argument, we used the standard Coulomb field which we see from equation (4.142) is only true for $v \ll c$.

For relativistic charges, we must recompute the potentials and the fields using the moving charge as a source term (a moving delta function). We're out of time, so won't cover this here, but if you want to look this up, the potentials of a moving charge are the Lienard-Wiechart potentials (see Griffiths for example). These potentials give two contributions to the fields of a moving charge: the first term is the "velocity field" (same as eq. [4.142]) which applies for a charge moving at constant velocity; the second term is a "radiation field" that leads to radiation and gives Larmor's formula (eq. [4.146]).

There is one more thing we can do using ideas from this chapter. We can get the total power radiated by a relativistic electron by transforming from the particle rest frame. In the particle rest frame, the charge is non-relativistic (not moving) and so Larmor's formula applies. First, define the 4-acceleration

$$a^\mu = \frac{d\eta^\mu}{d\tau} \tag{4.148}$$

where $\eta^\mu = \gamma_u(c, \mathbf{u})$ is the particle's 4-velocity. In the particle rest frame, $d\tau = dt'$. Also, note that $d\gamma'_u/dt' = 0$ in the particle rest frame because γ'_u is quadratic in u':

$$\frac{d\gamma'}{dt'} = \frac{d}{dt'}\left(1 - \frac{\mathbf{u}' \cdot \mathbf{u}'}{c^2}\right)^{-1/2} = \frac{\gamma'^3}{2c^2}\frac{d}{dt'}\left(\mathbf{u}' \cdot \mathbf{u}'\right) = \frac{\gamma'^3}{c^2}\mathbf{u}' \cdot \frac{d\mathbf{u}'}{dt'} = 0 \tag{4.149}$$

since $\mathbf{u}' = 0$ in the rest frame. This means that in the rest frame,

$$a'^{\mu} = (0, \mathbf{a}') \tag{4.150}$$

Therefore, we can write the power radiated as

$$P = \frac{q^2}{6\pi\epsilon_0 c^3}\mathbf{a}' \cdot \mathbf{a}' = \frac{q^2}{6\pi\epsilon_0 c^3} a'^{\mu} a'_{\mu}. \tag{4.151}$$

The second of these is written entirely in terms of Lorentz scalars: it must be true in all reference frames. The relativistic generalization of Larmor's formula is therefore

$$P = \frac{q^2}{6\pi\epsilon_0 c^3} a^{\mu} a_{\mu}. \tag{4.152}$$

In the lab frame,

$$a^{\mu} a_{\mu} = -\left(c\gamma \frac{d\gamma}{dt}\right)^2 + \left(\gamma \frac{d}{dt}(\gamma \mathbf{u})\right)^2. \tag{4.153}$$

Using $d\gamma/dt = \gamma^3 \mathbf{u} \cdot \mathbf{a}/c^2$, this is

$$a^{\mu} a_{\mu} = \gamma^4 |\mathbf{a}|^2 + \frac{\gamma^6 (\mathbf{u} \cdot \mathbf{a})^2}{c^2}. \tag{4.154}$$

Dividing the acceleration vector into components parallel and perpendicular to the velocity $\mathbf{a}_{\|} = \mathbf{u}(\mathbf{a} \cdot \mathbf{u})/u^2$ and $\mathbf{a}_{\perp} = \mathbf{a} - \mathbf{a}_{\|} = \mathbf{u} \times (\mathbf{a} \times \mathbf{u})/u^2$, Larmor's formula in the rest frame is

$$P = \frac{2q^2}{3c^3}\gamma^4 \left(a_{\perp}^2 + \gamma^2 a_{\|}^2\right). \tag{4.155}$$

We see that for a relativistic particle, the power is boosted by either γ^4 or γ^6 depending on whether the acceleration is along the velocity direction or perpendicular.

A classic example of this is synchrotron radiation from relativistic particles in a magnetic field. Consider a particle with charge q moving in a magnetic field $\mathbf{B}$. The equations of motion are

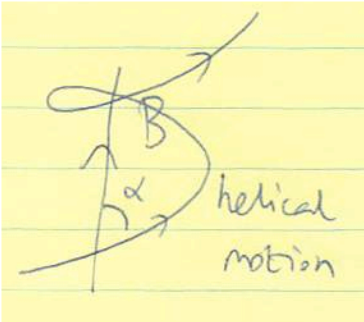
$$\frac{d}{dt}(\gamma m \mathbf{u}) = q\frac{\mathbf{u} \times \mathbf{B}}{c} \tag{4.156}$$

and

$$\frac{d}{dt}(\gamma m c^2) = q\mathbf{u} \cdot \mathbf{E}. \tag{4.157}$$

If the electric field is $\mathbf{E} = 0$, usually the case in astrophysical applications for example, then the energy of the particle is constant (γ is a constant) (recall that the magnetic field does no work on the particle because the force is always perpendicular to the velocity). The solution to equation (4.156) is *helical motion*: a constant velocity parallel to the magnetic field, $u_{\|} = \mathbf{u} \cdot \mathbf{B}/B$, and uniform circular motion in a plane perpendicular to $\mathbf{B}$, with gyration frequency

$$\omega_B = \frac{qB}{\gamma m c}. \tag{4.158}$$



The angle α is known as the *pitch angle*. The velocity perpendicular to the magnetic field is $u_\perp = u \sin\alpha$, so that $\alpha = \pi/2$ for pure circular motion ($u_\parallel = 0$).

The acceleration is $a_\perp = u_\perp \omega_B$, so that the total power is

$$P = \frac{2e^2}{3c^3}\gamma^4 \omega_B^2 u_\perp^2 = \frac{2}{3c} r_0^2 u_\perp^2 \gamma^2 B^2 \tag{4.159}$$

where $r_0 = (e^2/mc^2)$ is the classical electron radius. For a uniform distribution of pitch angles, the total power is

$$P = \frac{2}{3} r_0^2 c \gamma^2 \beta^2 B^2 \int \sin^2\alpha \, \frac{d\Omega}{4\pi}. \tag{4.160}$$

The integral is $2/3$, and the Thomson cross-section is $8\pi r_0^2/3$, giving the famous result

$$P = \frac{4}{3}\sigma_T c \beta^2 \gamma^2 U_B \tag{4.161}$$

where $U_B = B^2/8\pi$ is the magnetic energy density. Synchrotron radiation is observed across our Galaxy and is the source of radiation from the bright radio lobes that are huge galaxy-size bubbles blown by jets from the central black holes of radio galaxies.

SUMMARY

Here are the main ideas and results that we covered in this part of the course:
Lorentz transforms and 4-vectors. How 4-vectors transform between frames, e.g. position vector

$$x' = \gamma\,(x - vt) \qquad t' = \gamma\left(t - \frac{vx}{c^2}\right)$$

$$x^{\mu\prime} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} x^\mu$$

The difference between contravariant vectors $x'^\mu = \Lambda^\mu{}_\sigma x^\sigma$, and covariant vectors $x'_\mu = \left(\Lambda^{-1}\right)^\nu{}_\mu x'_\nu$. Useful identities

$$\gamma^2 = 1/(1-\beta^2) \qquad \gamma^2\beta^2 = \gamma^2 - 1$$

Raising and lowering with the metric tensor. Scalar product

$$a^\mu b_\mu = g_{\mu\nu} a^\mu b^\nu = -a^0 b^0 + \mathbf{a} \cdot \mathbf{b}$$

The gradient as a covariant vector.

Kinematics and dynamics in special relativity. Time dilation $\Delta t = \gamma \Delta t'$. Length contraction $\Delta x = \Delta x'/\gamma$. Velocity addition

$$u_\parallel = \frac{u'_\parallel + v}{1 + u'_\parallel v/c^2} \qquad u_\perp = \frac{u'_\perp}{\gamma \left(1 + u'_\parallel v/c^2\right)}$$

Time-like and space-like intervals. Proper time $d\tau = dt/\gamma$, 4-velocity, 4-momentum, Minkowski force,

$$\eta^\mu = \frac{dx^\mu}{d\tau} = \gamma_u (c, \mathbf{u}) \qquad p^\mu = m\eta^\mu = (E/c, \mathbf{p}) \qquad K^\mu = \gamma_u \left(\frac{\mathbf{F} \cdot \mathbf{u}}{c}, \mathbf{F}\right)$$

and equation of motion $dp^\mu/d\tau = K^\mu$.

The electromagnetic field tensor and transformation of E and B. The EM field tensor

$$F^{00} = 0 \qquad F^{0i} = -F^{i0} = \frac{E^i}{c} \qquad F^{ij} = \epsilon_{ijk} B^k$$

$$F^{\mu\nu} = \begin{pmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & B_z & -B_y \\ -E_y/c & -B_z & 0 & B_x \\ -E_z/c & B_y & -B_x & 0 \end{pmatrix}$$

The Lorentz force $K^\mu = q\eta_\nu F^{\mu\nu}$. Transformation of **E** and **B**

$$E'_x = E_x \qquad E'_y = \gamma(E_y - vB_z) \qquad E'_z = \gamma(E_z + vB_y)$$

$$B'_x = B_x \qquad B'_y = \gamma(B_y + \frac{vE_z}{c^2}) \qquad B'_z = \gamma(B_z - \frac{vE_y}{c^2})$$

The invariants $E^2 - c^2 B^2 = 0$ and $\mathbf{E} \cdot \mathbf{B}$, and what they tell you about whether there is a frame in which **E** or **B** vanish.

Maxwell's equations in covariant form. 4-current $J^\mu = (\rho c, \mathbf{J})$ and continuity $\partial_\mu J^\mu = 0$. The 4-potential $A^\mu = (\phi/c, \mathbf{A})$, and the wave equation

$$\Box^2 A^\mu = \partial_\nu \partial^\nu A^\mu = -\frac{1}{c^2}\frac{\partial^2 A^\mu}{\partial t^2} + \nabla^2 A^\mu = -\mu_0 J^\mu.$$

The Lorentz gauge can be written as $\partial_\mu A^\mu = 0$. The field tensor is obtained from the vector potential by $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. The dual field

$$G^{\mu\nu} = \begin{pmatrix} 0 & B_x & B_y & B_z \\ -B_x & 0 & -E_z/c & E_y/c \\ -B_y & E_z/c & 0 & -E_x/c \\ -B_z & -E_y/c & E_x/c & 0 \end{pmatrix}$$

and Maxwell's equations $\partial_\nu F^{\mu\nu} = \mu_0 J^\mu$, $\partial_\nu G^{\mu\nu} = 0$.

Energy and momentum. Energy and momentum conservation in the single equation $J_\nu F^{\mu\nu} = -\partial_\nu T^{\mu\nu}$ where the energy-momentum tensor is

$$T^{\mu\nu} = \frac{1}{\mu_0}\left[F^{\mu\rho}F^{\nu}_{\ \rho} - \frac{1}{4}g^{\mu\nu}F^{\rho\sigma}F_{\rho\sigma}\right]$$

or in terms of the fields

$$T^{00} = \frac{B^2}{2\mu_0} + \frac{1}{2}\epsilon_0 E^2 \qquad T^{i0} = T^{0i} = \frac{S^i}{c}$$

$$T^{ij} = \delta^{ij}\left(\frac{B^2}{2\mu_0} + \frac{1}{2}\epsilon_0 E^2\right) - \epsilon_0 E^i E^j - \frac{B^i B^j}{\mu_0}.$$

The interpretation of the different components: T^{00} is the energy density, cT^{0i} the Poynting flux, T^{i0}/c the momentum density, and T^{ij} the Maxwell stress tensor. How to use the Maxwell stress tensor to derive the force on a charge distribution by doing a surface integral.

The fields of a moving charge. A charge moving with constant velocity has an **E** field that is radial, but compressed into a pancake perpendicular to the direction of motion. The **B** field circulates around the velocity direction, and has a magnitude $\sim vE$.

Radiation from an accelerated charge. For relativistic particles, generalized Larmor's formula is

$$P = \frac{q^2}{6\pi\epsilon_0 c^3}a^\mu a_\mu = \frac{q^2}{6\pi\epsilon_0 c^3}\left(\gamma^4 a_\perp^2 + \gamma^6 a_\parallel^2\right).$$

Appendix A: Transformation properties of the gradient $\partial/\partial x^\mu$

We know that x^μ transforms according to $x'^\mu = \Lambda^\mu_{\ \nu} x^\nu$. The Lorentz transform is a linear combination of the coordinates, so we can write

$$\Lambda^\mu_{\ \nu} = \frac{\partial x'^\mu}{\partial x^\nu}.$$

For example, consider $ct' = \gamma(ct - \beta x)$, which implies that $\partial t'/\partial t = \gamma$ and $\partial ct'/\partial x = -\beta\gamma$ which are the first two elements of the Lorentz transform matrix.

This means that we can write

$$\frac{\partial}{\partial x^\mu} = \left(\frac{\partial x'^\nu}{\partial x^\mu}\right)\frac{\partial}{\partial x'^\nu} = \Lambda^\nu_{\ \mu}\frac{\partial}{\partial x'^\nu}$$

$$(\Lambda^{-1})^\mu_{\ \sigma}\frac{\partial}{\partial x^\mu} = (\Lambda^{-1})^\mu_{\ \sigma}\Lambda^\nu_{\ \mu}\frac{\partial}{\partial x'^\nu} = \frac{\partial}{\partial x'^\sigma}.$$

Therefore

$$\frac{\partial}{\partial x'^\sigma} = (\Lambda^{-1})^\mu_{\ \sigma}\frac{\partial}{\partial x^\mu}.$$

The gradient transforms as a covariant vector, opposite to x^μ which is a contravariant vector. Note that the notation is to write $\partial/\partial x^\mu$ (index up in the denominator) or ∂_μ (index down in the numerator). Also, the components of Λ^{-1} are the same as Λ except the $-\beta\gamma$ terms have their signs changed to $+\beta\gamma$.

Computational Exercise 1: Numerical solution of Laplace's equation

These notes describe the Gauss-Seidel method for solving Laplace's equation $\nabla^2 V(x,y) = 0$. To obtain a numerical solution, we divide the region of interest into a grid of discrete points labelled by i, j at which we will compute the potential $V_{i,j}$. We will assume that the points are uniformly spaced in x and y.

First, consider the Taylor expansion which relates the potentials at neighbouring points in the x-direction. This is

$$V(x + \Delta x, y) = V_{i+1,j} = V_{i,j} + \Delta x \left.\frac{\partial V}{\partial x}\right|_{i,j} + \frac{(\Delta x)^2}{2} \left.\frac{\partial^2 V}{\partial x^2}\right|_{i,j} + ... \tag{4.162}$$

or in the opposite direction

$$V(x - \Delta x, y) = V_{i-1,j} = V_{i,j} - \Delta x \left.\frac{\partial V}{\partial x}\right|_{i,j} + \frac{(\Delta x)^2}{2} \left.\frac{\partial^2 V}{\partial x^2}\right|_{i,j} + ... \tag{4.163}$$

By either summing or subtracting equations (4.162) and (4.163), we obtain second order accurate expressions for the first or second derivatives

$$\left.\frac{\partial V}{\partial x}\right|_{i,j} = \frac{V_{i+1,j} - V_{i-1,j}}{2\Delta x} + \mathcal{O}\left(\Delta x^2\right) \tag{4.164}$$

$$\left.\frac{\partial^2 V}{\partial x^2}\right|_{i,j} = \frac{V_{i+1,j} + V_{i-1,j} - 2V_{i,j}}{(\Delta x)^2} + \mathcal{O}\left(\Delta x^2\right). \tag{4.165}$$

The finite difference representation of Laplace's equation is therefore

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = \frac{V_{i+1,j} + V_{i-1,j} - 2V_{i,j}}{(\Delta x)^2} + \frac{V_{i,j+1} + V_{i,j-1} - 2V_{i,j}}{(\Delta y)^2} = 0. \tag{4.166}$$

For simplicity, we consider the case where the spacings in the x and y directions are the same, $\Delta x = \Delta y$, and therefore

$$V_{i,j} = \frac{V_{i+1,j} + V_{i-1,j} + V_{i,j+1} + V_{i,j-1}}{4}. \tag{4.167}$$

This equation has a simple interpretation: Laplace's equation implies that the potential at any grid point i, j is equal to the average of the potentials at the four nearest neighbours.

Equation (4.167) is the basis of the Gauss-Seidel method. We first make a guess at the solution (it doesn't have to be very accurate, e.g. we can just set $V =$ constant). We then visit each grid point in turn, and set the potential at that grid point equal to the average of the four nearest neighbours. After making a pass over the whole grid (one iteration), we repeat, and the solution will converge on the correct answer. The grid points on the boundaries must be set appropriately according to the boundary conditions. Interior grid points may also be held fixed, e.g. to model conducting surfaces held at constant potential.

The convergence is quite slow for this algorithm (the number of iterations required is $\propto N^2$ where N is the number of grid points in each direction), but it is simple to program and so a good starting point for investigating numerical methods.

I have supplied a starting code that implements the Gauss-Siedel method in python. See if you can get it running and then try the following exercises:

- Look at the convergence of the solution as a function of number of iterations. For example, you could find the maximum change in V across the grid from one iteration to the next, and plot it against iteration number.
- The example I gave is a simple example with a constant potential on the upper boundary and periodic boundary conditions in x. Try to implement a more complex geometry. For example, you could calculate the potential of a finite parallel-plate capacitor.
- Here is an idea to make the code run a lot faster: because the update to a given grid point depends on the nearest neighbours only, it is possible to divide the grid into two interwoven grids like a checkerboard. The updates to the white squares relies only on the black sqaures, for example. This means that you could do the update in a vectorized form rather than looping through each grid point. This should run a lot faster. Try it and measure the speed up.

Computational Exercise 2: Numerical Solution of the Wave Equation

The goal of this exercise is to solve Maxwell's equations for a propagating wavepacket.

Maxwell's equations. We start with Maxwell's equations in the form

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E} \qquad \frac{\partial \mathbf{E}}{\partial t} = \frac{c^2}{\epsilon_r}\nabla \times \mathbf{B} \tag{4.168}$$

where we assume that $\mu = \mu_0$ but allow for $\epsilon \neq \epsilon_0$. For a wave propagating in the x-direction, and writing $ct \to t$, $cB \to B$, and $\epsilon_r = n^2$, these equations are

$$\frac{\partial B}{\partial t} = -\frac{\partial E}{\partial x} \tag{4.169}$$

$$\frac{\partial E}{\partial t} = -\frac{1}{n(x)^2}\frac{\partial B}{\partial x}, \tag{4.170}$$

where we allow the refractive index $n(x)$ to be a function of position.

Finite differencing. In Exercise 1, we wrote down finite difference approximations for derivatives, in particular the second order accurate derivative

$$\left.\frac{\partial f_i}{\partial x}\right|_{x=x_i} = \frac{f_{i+1} - f_{i-1}}{2\Delta x}.$$

Using this for the time-derivatives and spatial-derivatives gives the update scheme

$$B_i^{n+1} = B_i^{n-1} - \frac{\Delta t}{\Delta x}\left(E_{i+1}^n - E_{i-1}^n\right)$$

$$E_i^{n+1} = E_i^{n-1} - \frac{\Delta t}{\Delta x}\left(B_{i+1}^n - B_{i-1}^n\right)\frac{1}{n_i^2}.$$

Here subscript i labels the grid cell and superscript n the time step. To find the values at time $n+1$ requires storing both values from the previous two timesteps, n and $n-1$. This is known as a leapfrog method.

The simplest boundary conditions to use are periodic, so for example if you have N grid cells from $i = 1$ to N, then you take $E_{N+1} = E_1$ and $E_0 = E_N$ when updating the points at the edge of the grid.

Exercise 1: Getting a wave to propagate. Code up the algorithm above. Set the refractive index to $n = 1$ across the grid and start with a Gaussian profile for the electric field centered on your grid. Can you choose an appropriate profile for B to get the pulse to move either to the left or to the right? Does it move without changing shape? Try a sine-wave multiplied by a Gaussian (as in HW3 question 1). Observe the wavepacket propagating and again check to see if it propagates without changing shape.

Exercise 2: Reflection at an interface. Now set $n = n_1$ on the left half of your grid and $n = n_2$ on the right half of the grid. Start a pulse in the left half moving towards the right. Do you see the expected behaviour when the pulse encounters the change in n at the middle of the grid? (Things to check are: the amplitudes of reflected and transmitted waves, relative velocities, behaviour with different ratios of n_2 to n_1.)

Exercise 3: Dispersion. Now add an extra term to Maxwell's equations to introduce dispersion as in a plasma. [Hint: in the plasma, $dJ/dt \propto E$ so you can do this by integrating a third equation for J.] Set up a simulation of a wavepacket travelling from air into plasma. Do you see the expected behavior of the wavepacket in a dispersive medium? Explore how the transmission and reflection change as you increase the current and interpret what you see in terms of what you know about waves propagating in a plasma (with ω larger or smaller than ω_p).