PHYS 432 Physics of Fluids

Winter Term 2023

Introduction

These are notes for PHYS 432 Physics of Fluids, Winter term 2023. The was a “flipped” format class, with notes to read outside class and class time devoted to exercises. Each week covers a different topic. The beginning of class each week was spent discussing questions that test understanding of the notes; they can be found at the end of each topic’s notes. The remaining class time was spent on Practice Problems, which are more traditional analytic problems, and Computational Exercises, which cover an application of that week’s topic and over the term give an introduction to computational techniques in fluids.

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Week 1: The fluid equations and Bernoulli's principle

These are notes for the first week of PHYS 432 Physics of Fluids. We first discuss what we mean by a fluid, and then introduce the fluid equations as a series of conservation laws. Finally, we look at a famous consequence of the fluid equations, Bernoulli's principle.

What is a fluid?

We start by asking, "what is a fluid?" The obvious answer to this is "something that flows" such as a liquid or a gas. A solid has a non-zero shear modulus and can statically support a shear stress, so we don't think of it as a fluid. But we'll see that solids can be handled by adding a shear modulus to the fluid equations.

In fact, by fluid we mean a material that we can treat as a **continuous substance** or a **continuum**– ie. we don't have to worry about the fact that it is made up of atoms. The requirement is that the mean free path λ is $\ll L$, the scale on which macroscopic properties such as velocity or temperature vary. In this limit, we are doing **continuum mechanics**.

For example, let's estimate the **mean free path in air**. To do this, we imagine the air molecule sweeping out a cylinder as it moves with cross-section σ. Any other molecule that falls within the cylinder will result in a scattering. The mean free path is then given by the cylinder that contains (on average) one other molecule,

$$n\sigma\lambda = 1,$$

where n is the number density of molecules. For the cross-section, we can assume it is set by the size of an air molecule $\sigma \sim 10^{-20}\ \mathrm{m}^2$, and the number density is $n = \rho/Am_p$ where $\rho \approx 1\ \mathrm{kg\ m^{-3}}$ is the density of air, and we'll take the typical mass of a molecule (mostly N_2 in air) as $Am_p \approx 28m_p$. Putting this together gives $n \approx 3 \times 10^{25}\ \mathrm{m^{-3}}$ and

$$\lambda \approx 3 \times 10^{-6}\ \mathrm{m} \approx \text{a few } \mu\mathrm{m}.$$

The mean free path in air is therefore $\ll$ than macroscopic lengthscales. So the flow of air at atmospheric pressure can be studied by treating the air as a fluid, a continuum. Locally, at any given point in space, the short mean free path means that the different particles in the gas collide a lot, and the gas is in **local thermodynamic equilibrium**. It has a well-defined temperature, and we can write for example $P = nk_BT$ for an ideal gas. The temperature at each location measures the random velocities of the particles; we will track the **bulk velocity** (the average velocity of the particles) as the vector field $\mathbf{u}(\mathbf{r})$. Similarly the density, temperature, and pressure are also functions of position $\mathbf{r}$, this time scalar fields $T(\mathbf{r})$, $\rho(\mathbf{r})$, and $P(\mathbf{r})$.

The fluid treatment requires that, for example,

$$\frac{T}{dT/dx} \gg \lambda.$$

The route to the fluid equations

The fluid equations describe the evolution of the velocity, pressure, density, and temperature fields over time.

One route to the fluid equations is via statistical mechanics, in which we start with the microscopic description of the material and average over lengthscales $\ll L$ (expand in the small parameter λ/L). [In case you want to look this up, the names are Liouvilles theorem $\rightarrow$ Boltzmann equation $\rightarrow$ moments of the Boltzmann equation.]

Instead, we are going to take a short cut and use conservation laws to derive the fluid equations.

Continuity equation (mass conservation)

First, consider a small volume of fluid with total mass

$$M = \int \rho \, dV.$$

If we keep the boundary of the volume fixed in space as the fluid evolves, the rate of change of mass within the volume is

$$\frac{dM}{dt} = \frac{d}{dt} \int \rho \, dV = \int \frac{\partial \rho}{\partial t} \, dV.$$

Any mass change must come about because fluid moves into or out of the volume, so we can also write

$$\frac{dM}{dt} = -\int \rho \mathbf{u} \cdot d\mathbf{S},$$

adding up the mass flowing across the surface of the fluid element. The quantity $\rho \mathbf{u}$ is the **mass flux** (units of kg m^{-2} s^{-1}).

Equating these two expressions for dM/dt gives

$$\int \frac{\partial \rho}{\partial t} \, dV = -\int \rho \mathbf{u} \cdot d\mathbf{S}.$$

We can write this in a simpler way by applying the divergence theorem to convert the surface integral to a volume integral,

$$\int \frac{\partial \rho}{\partial t} \, dV = -\int \nabla \cdot (\rho \mathbf{u}) \, dV.$$

But the choice of volume we are using is arbitrary which implies that at each point in space we must have

$$\boxed{\frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho \mathbf{u})}$$

This is the **continuity equation**, a local expression of mass conservation.

Eulerian vs Lagrangian; the advective derivative

The continuity equation can be rewritten as

$$\left(\frac{\partial}{\partial t} + \mathbf{u}\cdot\nabla\right)\rho = -\rho\nabla\cdot\mathbf{u}$$

or

$$\frac{D\rho}{Dt} = -\rho\nabla\cdot\mathbf{u}$$

where we define the **Lagrangian derivative** or **advective derivative**

$$\boxed{\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \mathbf{u}\cdot\nabla}$$

This derivative comes up a lot: D/Dt describes the rate of change of a quantity **following the fluid element**, whereas $\partial/\partial t$ is the rate of change at a fixed point in space.

There are two different points of view when describing a fluid:

- **Eulerian** in which we describe the fluid properties at fixed points in space – think about standing at a fixed spot as the fluid flows past you.
- **Lagrangian** in which we describe the fluid properties following a fluid element as it moves – now you jump into the fluid and go with the flow.

Let's check that D/Dt is indeed the Lagrangian derivative. Consider a quantity f (eg. it could be density or temperature or a component of velocity), that could depend on both position and time. Write the path of a fluid element as $\mathbf{r}(t) = (x(t), y(t), z(t))$. The velocity of the fluid element is therefore

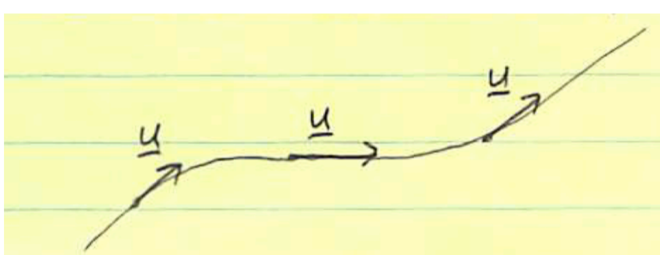
$$\mathbf{u} = \frac{d\mathbf{r}}{dt} = \left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right).$$

The value of f moving with the fluid element is $f(\mathbf{r}(t), t)$. Its rate of change is

$$\frac{\partial f}{\partial t} + \frac{dx}{dt}\frac{\partial f}{\partial x} + \frac{dy}{dt}\frac{\partial f}{\partial y} + \frac{dz}{dt}\frac{\partial f}{\partial z} = \left(\frac{\partial}{\partial t} + \mathbf{u}\cdot\nabla\right)f = \frac{Df}{Dt}.$$

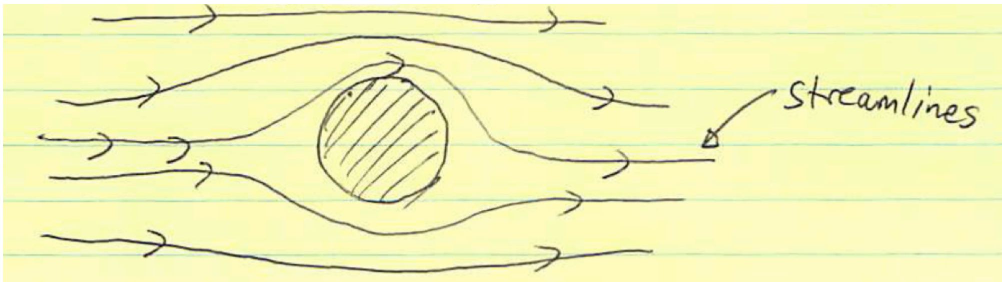
Streamlines

A useful idea when sketching fluid flows is the **streamline**, a curve that follows the direction of $\mathbf{u}$ from one position to another. The tangent to the streamline is always in the direction $\mathbf{u}$ at each position $\mathbf{r}$.



[These are equivalent to magnetic field lines for a magnetic field $\mathbf{B}$.]

For a **steady flow** ($\partial/\partial t = 0$), the fluid elements follow the streamlines. For example, steady flow around a cylinder:



In that case, a quantity f that is constant along a streamline $(\mathbf{u}\cdot\nabla)f = 0$ is also constant for a fluid element since then $D/Dt = 0$.

Momentum equation

Now consider the momentum of our small volume of fluid. Since momentum is a vector, consider the rate of change of momentum in the x-direction as an example:

$$\frac{d}{dt}\left(\int dV\, \rho u_x\right) = -\int \rho u_x\, \mathbf{u}\cdot d\mathbf{S} + (\text{forces})_x$$

The quantity $\rho\mathbf{u}$ is the **momentum density**[1]. On the left hand side, we integrate the x-component of this over the volume to get the total momentum in the x-direction. The first term on the right hand side is the flux of x-momentum across the boundary (into or out of the fluid element). We also include an extra term on the right hand side in case there are forces acting on the fluid element: momentum is conserved only if no forces are acting. Note that the units of the force term are force density (force per unit volume). The same equation applies for the y or z components of momentum, we just have to substitute u_y or u_z instead of u_x.

We can again use the divergence theorem to turn the surface integral into a volume integral. Writing the dot product in component form

$$\int \rho u_x u_i dS_i = \int dV\ \partial_i\left(\rho u_x u_i\right)$$

where the repeated index i means a sum over x, y, and z (Einstein notation) and the vector $\partial = (\partial/\partial x, \partial/\partial y, \partial/\partial z)$. We use the same argument from before that the volume over which we integrate is arbitrary, and therefore we must have locally

$$\boxed{\frac{\partial}{\partial t}\left(\rho u_x\right) = -\partial_i\left(\rho u_x u_i\right) + (\text{forces})_x}$$

[1] Yes, $\rho\mathbf{u}$ is also a mass flux (as we used it in our derivation of the continuity equation), but in this context we are using it as the momentum density.

It is instructive to simplify this equation using the continuity equation. Expand the derivatives on both sides:

$$u_x\frac{\partial\rho}{\partial t}+\rho\frac{\partial u_x}{\partial t}=-u_x\partial_i\left(\rho u_i\right)-\rho u_i\partial_i u_x+(\text{forces})_x$$

The first term on the left hand side cancels the first term on the right hand side because of the continuity equation. The remaining terms are

$$\rho\frac{\partial u_x}{\partial t}+\rho u_i\partial_i u_x=(\text{forces})_x$$

or

$$\boxed{\rho\frac{Du_x}{Dt}=(\text{forces})_x}$$

which is just **Newtons second law** $F=ma$ but written for the fluid element.

We see the **non-linearity** of the fluid equations in the term $(\mathbf{u}\cdot\nabla)\mathbf{u}$.

For example, if we expand the velocity in Fourier modes e^{ikx}, this term is $(\mathbf{u}\cdot\nabla)\mathbf{u}\propto e^{i2kx}$. Different spatial modes are coupled to each other, they don't evolve independently as in linear systems (for example solving the diffusion equation using a Fourier expansion). We see this clearly in turbulence, where stirring on a large scale (eg. stir your coffee) generates a lot of small scale fluid motion (the milk mixes on very small scales).

This non-linearity also means that qualitatively different flows arise as we change the size of the velocity. We will see this later.

Different kinds of forces

Now let's think about what the force term might look like. There are two kinds of forces that could act on the fluid element:

- **Body forces**. These act on each particle in the fluid element. The total force is $\int\mathbf{f}\,dV$ where $\mathbf{f}$ is the force per unit volume. An example is gravity $\mathbf{f}=\rho\mathbf{g}$, where $\mathbf{g}$ is the (vector) acceleration due to gravity.

- **Surface stress**. A force that acts on the surface of the fluid element. We write the total force as

  $$\int T_{ij}dS_j,$$

  where T_{ij} is the **stress tensor**. The simplest example is

  $$\text{pressure}\qquad T_{ij}=-P\delta_{ij}$$

  The delta-function is there because pressure always acts in the same direction as the normal to a surface. The minus sign is there because the normal vector points outwards from a surface, but the pressure force pushes inwards. In this case, the stress tensor is diagonal (only non-zero components are for $i = j$). Other kinds of forces will not be diagonal, e.g. a sideways shearing force applied to a surface. We will see examples later (see viscosity).

Once again, we can convert the surface integral to a volume integral using the divergence theorem:

$$\int T_{ij} dS_j = \int \partial_j T_{ij} \, dV$$

which for pressure is

$$-\int \delta_{ij} \partial_j P \, dV = -\int \partial_i P \, dV.$$

Including both types of forces, the momentum equation becomes

$$\boxed{\frac{\partial}{\partial t}(\rho u_i) = -\partial_j(\rho u_i u_j) + f_i + \partial_j T_{ij}}$$

or

$$\boxed{\rho \frac{D\mathbf{u}}{Dt} = \mathbf{f} + \nabla \cdot \mathbf{T}}$$

If the forces acting are pressure and gravity only, then

$$\rho \frac{D\mathbf{u}}{Dt} = \rho \mathbf{g} - \nabla P.$$

The static version of this equation $\nabla P = \rho \mathbf{g}$ is the condition of **hydrostatic balance**.

Bernoulli's principle

To end this part of the notes, we are going to put the fluid equations to work and will derive a famous result known as **Bernoulli's principle**.

Take the momentum equation with pressure and gravity forces. If we write the gravity as the gradient of the gravitational potential $\mathbf{g} = -\nabla \chi$ and if we have a constant density fluid so that $\nabla P/\rho = \nabla(P/\rho)$, then the right hand side of the momentum equation can be written as a gradient:

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\nabla \left(\frac{P}{\rho} + \chi \right)$$

Now use the vector identity

$$(\mathbf{u} \cdot \nabla)\mathbf{u} = -\mathbf{u} \times (\nabla \times \mathbf{u}) + \nabla \left(\frac{1}{2} u^2 \right)$$

⇒

$$\frac{\partial \mathbf{u}}{\partial t} - \mathbf{u} \times (\nabla \times \mathbf{u}) = -\nabla \left(\frac{P}{\rho} + \chi + \frac{1}{2}u^2 \right) = -\nabla H$$

where we define

$$H = \frac{P}{\rho} + \chi + \frac{1}{2}u^2.$$

Now, the important result is that for a steady flow ($\partial/\partial t = 0$), taking the dot product with $\mathbf{u}$ gives

$$\boxed{\mathbf{u} \cdot \nabla H = 0}$$

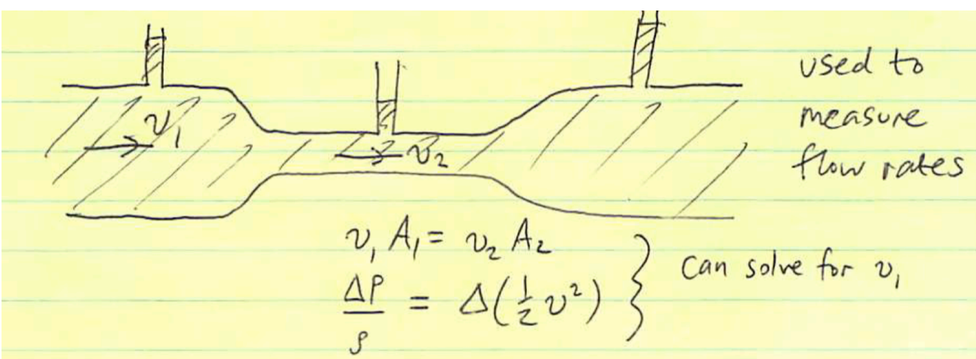
⇒ H **is a constant along streamlines in a steady flow.** This Bernoulli's principle, and in this context H is known as the **Bernoulli constant**.

There is an even stronger version, which applies when the flow is **irrotational** $\nabla \times \mathbf{u} = 0$. In that case, $\nabla H = 0$ which implies that H **is the same constant on all streamlines for an irrotational flow**.

Example applications:

- A classic example is to calculate how quickly water flows out of a hole at the bottom of a container. Applying Bernoulli's theorem you can show that the velocity of the water flowing out is $v^2 = 2gH$, where H is the height of the water in the container. (You should think about why it's okay to apply Bernoulli's theorem here, after all this is a time-dependent situation!)

- Venturi tube. Measuring the pressure drop when fluid is forced through a narrow passage allows the fluid velocity to be determined. Here is my sketch of the set up:



- If you take two sheets of paper, hold them close together and blow into the space between them, the sheets of paper will move together. The fast moving flow between the sheets has a lower pressure (perhaps a non-intuitive result!). This is an example of a "lift force" arising from a pressure difference between two sides of a body arising from different velocities on each side.

Questions for this week

1. Give a physical interpretation of the following equations

$$\frac{D\rho}{Dt} = -\rho\nabla\cdot\mathbf{u}$$

and

$$\rho\frac{D\mathbf{u}}{Dt} = \rho\mathbf{g} - \nabla P.$$

2. What is meant by each of the following:

- the Eulerian and Lagrangian descriptions of a fluid
- The advective derivative
- A streamline

3. If we have some scalar property of a fluid $f(\mathbf{r})$ (eg. temperature), what does the expression $\mathbf{u}\cdot\nabla f = 0$ tell you?

4. What is the difference between a body force and a surface stress, and how are they represented mathematically?

5. What quantity is constant according to Bernoulli's theorem, and under what conditions?

Week 1 Computational exercise - Modelling the Earth

In this exercise, you will integrate the equation of hydrostatic balance to make a model of the Earth.

(a) The equations of planetary structure are

$$\frac{dP}{dr} = -\frac{Gm\rho}{r^2} \tag{1}$$

and

$$\frac{dm}{dr} = 4\pi r^2 \rho \tag{2}$$

where $P(r)$ and $\rho(r)$ are the pressure and density at radius r, $m(r)$ is the mass contained within radius r, and it has been assumed that the planet is spherically-symmetric.

Explain where these equations come from.

(b) To integrate the equations, we need a relation between the pressure and density $P(\rho)$. The table below from Seager et al. (2007, Astrophysical Journal 669, 1279) gives fits to the equation-of-state (pressure-density relation) for different materials in the form $\rho = \rho_0 + cP^n$ for constants ρ_0, n and c as given in the table.

Let's first assume that the Earth has a uniform silicate composition ($MgSiO_3$). Assuming a value for the pressure at the centre of the planet P_c, integrate equations (1) and (2) from the centre to the surface (where the pressure drops to a value $\ll P_c$). You will need to try the integration a few times with different values of P_c until you get the mass (the value of m at the surface) to be roughly the mass of the Earth ($M_\oplus = 6.0 \times 10^{24}$ kg).

What is the radius of your Earth model? How does it compare with Earth's radius $R_\oplus \approx$ 6400 km?

Some hints:

- We want to integrate these equations from the centre of the planet at $r = 0$ to the surface at $r = R$. However, if we start at $r = 0$, we will get a divide-by-zero error if we try to evaluate the right hand side of equation (1). Instead, we can analytically step away from the origin by a small amount $r_0 \ll R$ (e.g. try $r_0 = 0.01\ R$) and start the integration at $r = r_0$ rather than $r = 0$. If the density at the centre of the planet is ρ_c (corresponding to the pressure P_c), then the starting value of m at $r = r_0$ is $m = 4\pi r_0^3 \rho_c / 3$.

- You can integrate the two coupled ODEs using the integrator `scipy.integrate.solve_ivp` from python (see the documentation here).

TABLE 3

FITS TO THE MERGED VINET/BME AND TFD EOS OF THE FORM $\rho(P) = \rho_0 + cP^n$

Material	ρ_0 (kg m^{-3})	c (kg m^{-3} Pa^{-n})	n
Fe(α)	8300.00	0.00349	0.528
$MgSiO_3$ (perovskite)	4100.00	0.00161	0.541
(Mg, Fe)SiO_3	4260.00	0.00127	0.549
H_2O	1460.00	0.00311	0.513
C (graphite)	2250.00	0.00350	0.514
SiC	3220.00	0.00172	0.537

NOTE.—These fits are valid for the pressure range $P < 10^{16}$ Pa.

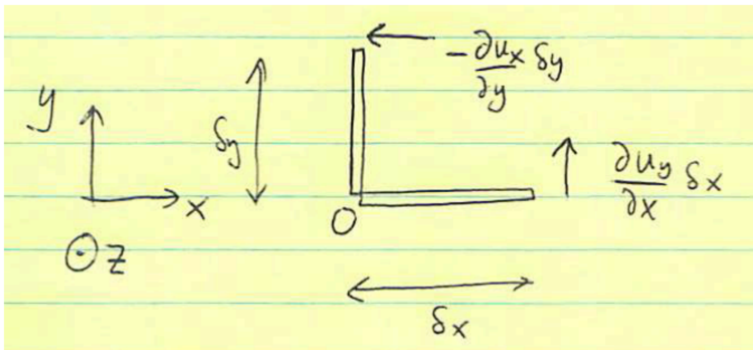
(c) The Earth actually has an iron core surrounded by a silicate mantle. Include an iron core in your model. What mass of the iron core do you need to match the Earth's radius?

How does the density profile $\rho(r)$ of your model compare with the density profile of the Earth? (e.g. see this plot).

Week 2: Vorticity

This week we will discuss an important quantity in fluids, the **vorticity** $\boldsymbol{\omega} \equiv \nabla \times \mathbf{u}$.

The vorticity measures the **local** rotation of the fluid at a given point. A way to see this is to consider a "vorticity meter", two infinitesimal rigid rods connected at right angles, initially placed so that one rod lies along the x-axis and the other along the y-axis.



Now consider how the vorticity meter will be advected by the fluid flow. If there is a gradient of u_x in the y-direction, $\partial u_x/\partial y \neq 0$, then the end of the vertical part of the rod will be advected in the x-direction relative to the origin (the origin is the point O where the two rods connect). Similarly, if there is a gradient of u_y in the x-direction, $\partial u_y/\partial x \neq 0$, then the end of the horizontal part of the rod will be advected in the y-direction relative to the origin. In this way, the vorticity meter will begin to rotate.

The mean angular velocity about the point O is

$$\frac{1}{2}\left(\frac{(\partial u_y/\partial x)\delta x}{\delta x} + \frac{(-\partial u_x/\partial y)\delta y}{\delta y}\right) = \frac{1}{2}\left(\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y}\right) = \frac{1}{2}\omega_z.$$

This shows that **the magnitude of the vorticity vector is** $2\times$ **the instantaneous local rotation rate of the fluid**.

It is very important to note that the **vorticity measures the local rotation rate which is not the same as the global rotation**. To see this, consider the following two flows which involve a rotating fluid:

1. Rigid body rotation (uniform rotation) with angular velocity Ω. In cylindrical coordinates, this has $\boldsymbol{\omega} = \Omega\hat{z}$ and $\mathbf{u} = \hat{\phi}\, r\Omega$, where $\hat{z}$ and $\hat{\phi}$ are unit vectors. The vorticity is[2]

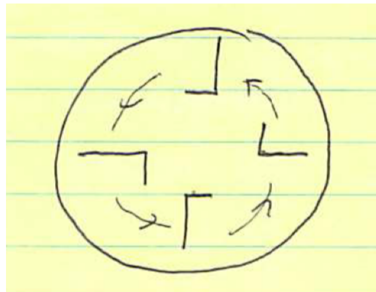
$$\boldsymbol{\omega} = \nabla \times \mathbf{u} = \hat{z}\,\frac{1}{r}\frac{\partial}{\partial r}\left(r u_\phi\right) = \hat{z}\,\frac{1}{r}\frac{\partial}{\partial r}\left(r^2\Omega\right) = \hat{z}\,2\Omega$$

or

$$\boldsymbol{\omega} = 2\boldsymbol{\Omega}.$$

[2]For reference, I've attached a page at the end which gives grad, curl etc. in cylindrical and spherical coordinates.

Therefore we infer that there is a local rotation with angular velocity $\omega = 2\Omega$ at any point. In fact, this makes sense because **the vorticity meter must rotate as it moves around the rotation axis so that the system is stationary in the rotating frame**:



This is just like the Moon, which rotates with the same angular velocity as its orbit, meaning that from the Earth we always see the same face of the Moon. If we moved into a frame rotating with the Moon's orbit, everything would appear stationary.

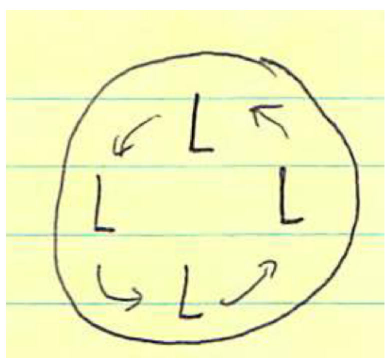
2. Line vortex flow. Now contrast this with the flow

$$\mathbf{u} = \hat{\phi}\,\frac{k}{r}$$

for some constant k. This flow has $\nabla \times \mathbf{u} = 0$ everywhere except at the origin, since

$$\nabla \times \mathbf{u} = \hat{z}\,\frac{1}{r}\frac{\partial}{\partial r}\,(r u_\phi) = 0.$$

The vorticity meter keeps the same orientation as it moves around the axis:



This is an example of a flow that is globally rotating, but there is no local rotation. It is called a line vortex flow because the vorticity is concentrated along the line that runs along the z-axis in cylindrical coordinates.

Vortices are common in real life. For example, if you have been canoeing you will have seen vortices generated as the paddle moves through the water. A simple idealized model for a vortex like this can be made from a combination of the two flows above, known as a **Rankine vortex**

$$u_\phi = \begin{cases} \Omega r & \text{for } r < a \\ \Omega a^2/r & \text{for } r > a \end{cases} \qquad u_r = u_z = 0$$

This has $u_\phi \propto r$, $\omega_z = 2\Omega$ for $r < a$, and $u_\phi \propto 1/r$, $\omega_z = 0$ for $r > a$.

The vorticity equation

The **vorticity equation** describes the time evolution of $\boldsymbol{\omega}$. We take the curl of the momentum equation in the form we were using when we derived Bernoulli's equation:

$$\nabla \times \left[\frac{\partial \mathbf{u}}{\partial t} - \mathbf{u} \times (\nabla \times \mathbf{u}) = -\nabla H \right]$$

$\Rightarrow$

$$\boxed{\frac{\partial \boldsymbol{\omega}}{\partial t} - \nabla \times (\mathbf{u} \times \boldsymbol{\omega}) = 0}$$

Using the vector identity

$$\nabla \times (\mathbf{u} \times \boldsymbol{\omega}) = (\boldsymbol{\omega} \cdot \nabla)\mathbf{u} - (\mathbf{u} \cdot \nabla)\boldsymbol{\omega} + \mathbf{u}(\nabla \cdot \boldsymbol{\omega}) - \boldsymbol{\omega}(\nabla \cdot \mathbf{u})$$

with $\nabla \cdot \boldsymbol{\omega} = 0$ and $\nabla \cdot \mathbf{u} = 0$ for a constant density (incompressible) fluid, we can also write this as

$$\boxed{\frac{D\boldsymbol{\omega}}{Dt} = (\boldsymbol{\omega} \cdot \nabla)\, \mathbf{u}}$$

There are two ways we can think about this equation:

1. We can see that the left hand side describes advection of vorticity by the flow. If the right hand side vanishes, then the vorticity $\boldsymbol{\omega}$ of a given fluid element is conserved as the fluid element moves around. The term on the right hand side is therefore responsible for changes in the local angular velocity of the fluid.

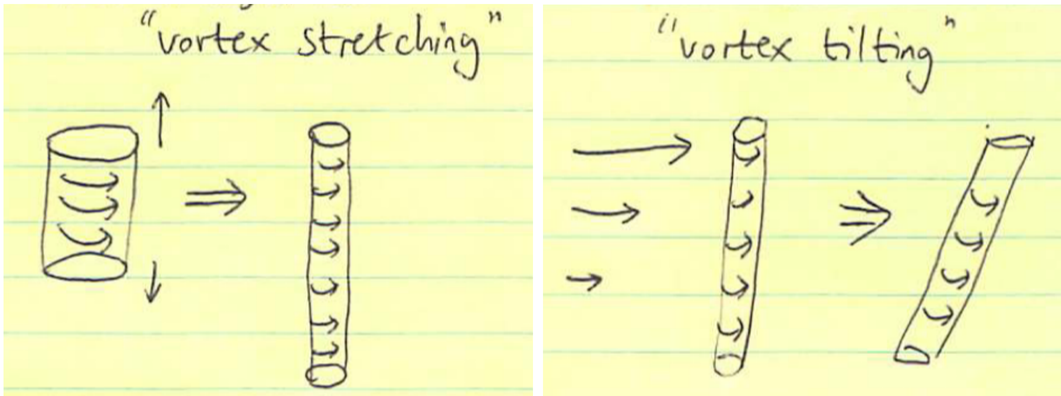
   To see the physics underlying this term, we can align the z-axis with the local direction of $\boldsymbol{\omega}$, i.e. $\boldsymbol{\omega} = \omega\, \hat{z}$. Then

   $$\frac{D}{Dt}(\omega\, \hat{z}) = \hat{z}\, \omega \frac{\partial \mathbf{u}}{\partial z}.$$

   Now write the fluid velocity as $\mathbf{u} = u\hat{x} + v\hat{y} + w\hat{z}$

   $$\Rightarrow \frac{D}{Dt}(\omega\, \hat{z}) = \hat{z}\, \omega \frac{\partial w}{\partial z} + \hat{x}\, \omega \frac{\partial u}{\partial z} + \hat{y}\, \omega \frac{\partial v}{\partial z}.$$

   The first term describes **vortex stretching**. If the velocity in the direction of the vortex has a gradient along the vortex, the advection of vorticity stretches the vortex. Conservation of mass means that the stretched vortex is also squeezed, i.e. its cross-sectional area will drop. Angular momentum conservation then results in the vortex spinning faster. So stretching out a vortex amplifies the vorticity. This is illustrated in the sketch below.



The second and third terms involving the velocities perpendicular to the vortex describe **vortex tilting**. If the perpendicular velocity varies along the length of the vortex, the shear in the flow will tilt the vortex (as shown in the sketch above). This generates components of $\boldsymbol{\omega}$ in the flow direction.

2. Another way to interpret the vorticity equation is to consider the separation between two neighbouring fluid elements at positions $\mathbf{r_1}$ and $\mathbf{r_2} = \mathbf{r_1} + \boldsymbol{d\ell}$.

   A time δt later, they are located at $\mathbf{r_1}' = \mathbf{r_1} + \mathbf{u_1}\delta t$ and $\mathbf{r_2}' = \mathbf{r_2} + \mathbf{u_2}\delta t$

   $$\Rightarrow \boldsymbol{d\ell}' = \boldsymbol{d\ell} + (\mathbf{u_2} - \mathbf{u_1})\delta t.$$

   But, by a Taylor expansion, $\mathbf{u_2} = \mathbf{u_1} + (\boldsymbol{d\ell}\cdot\nabla)\mathbf{u_1}$, and therefore

   $$\frac{\delta(\boldsymbol{d\ell})}{\delta t} = (\boldsymbol{d\ell}\cdot\nabla)\mathbf{u},$$

   where we write $\mathbf{u_1}$ as $\mathbf{u}$. The left hand side is a Lagrangian time derivative, since the δ's apply to particular fluid elements. Therefore,

   $$\boxed{\frac{D}{Dt}\boldsymbol{d\ell} = (\boldsymbol{d\ell}\cdot\nabla)\,\mathbf{u}}.$$

   This equation has the same form as the vorticity equation! This implies that if at some time $\boldsymbol{\omega}$ is parallel to the separation between two fluid elements, then it will always be so because $\boldsymbol{\omega}$ and $\boldsymbol{d\ell}$ evolve in the same way.

   $\Rightarrow$ The **vortex lines** (the lines that follow the direction of $\boldsymbol{\omega}$ at each point) move with the fluid — we say that they are "frozen" into the fluid.

   [We will see later that in a magnetized fluid the same thing also holds for the magnetic field lines, and the magnetic field $\mathbf{B}$ obeys the same form of equation as $\boldsymbol{\omega}$ and $\boldsymbol{d\ell}$. Indeed, in "magnetohydrodynamics", one of the basic principles is that magnetic field lines are frozen into the fluid.]

Circulation

The integral quantity

$$\Gamma = \oint \mathbf{u} \cdot \boldsymbol{d\ell} = \int \boldsymbol{\omega} \cdot \mathbf{dS}$$

is known as the **circulation**. Note that the loop integral is over a closed material curve, i.e. a curve that goes through particular fluid elements (Lagrangian as opposed to Eulerian). Similarly, the surface integral is over the material surface bounded by the loop.

The circulation is conserved under certain conditions. To see this, we can evaluate

$$\frac{D\Gamma}{Dt} = \frac{D}{Dt}\oint \mathbf{u} \cdot \boldsymbol{d\ell} = \oint \frac{D\mathbf{u}}{Dt} \cdot \boldsymbol{d\ell} + \oint \mathbf{u} \cdot \frac{D\boldsymbol{d\ell}}{Dt}.$$

The **first term** on the right hand side will vanish if $D\mathbf{u}/Dt$ is the gradient of a scalar (ie. we are dealing with **conservative forces**). An example is a constant density fluid with gravity, which has

$$\frac{D\mathbf{u}}{Dt} = -\nabla\left(\frac{P}{\rho} + \chi\right).$$

For the **second term**, use the result from the previous page:

$$\frac{D\boldsymbol{d\ell}}{Dt} = \delta\mathbf{u},$$

where $\delta\mathbf{u}$ is the difference in velocities between two fluid elements that lie next to each other on the curve. We can use this to change integration variables to velocity:

$$\oint \mathbf{u} \cdot \frac{D\boldsymbol{d\ell}}{Dt} = \oint \mathbf{u} \cdot d\mathbf{u} = \oint \frac{1}{2} d(u^2) = 0$$

which vanishes because we are integrating over a closed curve.

We therefore have **Kelvin's theorem**

$$\boxed{\frac{D\Gamma}{Dt} = 0}$$

Circulation is conserved around a material curve if the forces are conservative.

Vorticity generation and destruction

Kelvin's theorem only holds if the forces are conservative. Similarly, if you go back and look again at the derivation of the vorticity equation, you'll see that we assumed that the right hand side of the momentum equation was curl-free, i.e. we wrote $\mathbf{F} = -\nabla H + \nabla(u^2/2)$ where $\mathbf{F}$ is the force per unit mass $D\mathbf{u}/Dt = \mathbf{F}$ — e.g. for pressure and gravity forces,

$$\mathbf{F} = -\frac{\nabla P}{\rho} + \mathbf{g}.$$

For a constant density fluid, $\mathbf{F}$ can then be written as the gradient of a scalar: these forces are conservative.

More generally, $\nabla \times \mathbf{F}$ may not vanish, and then if you repeat the derivation of the vorticity equation you will find

$$\boxed{\frac{D\boldsymbol{\omega}}{Dt} = (\boldsymbol{\omega} \cdot \nabla)\mathbf{u} + \nabla \times \mathbf{F}}$$

This shows that a force with a non-zero curl can induce fluid rotation and therefore generate (or destroy) vorticity. This hopefully makes physical sense if you think about a force which has a non-zero integral around a closed loop — it will apply a net torque to the fluid and induce rotation.

Examples:

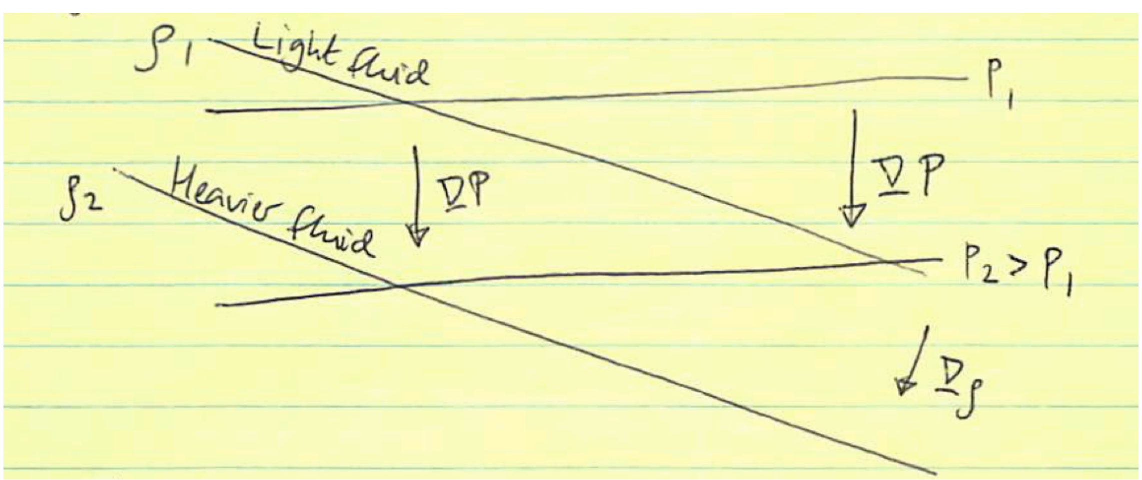
- **Viscous force**. We'll look at this in detail next week. The viscous force leads to the diffusion of vorticity and can be a source or a sink.
- **Baroclinicity**. If the density is not constant, then we have a term

$$\nabla \times \mathbf{F} = -\nabla \times \left(\frac{\nabla P}{\rho}\right) = -\frac{\nabla P \times \nabla \rho}{\rho^2},$$

known as the **baroclinic vector**.

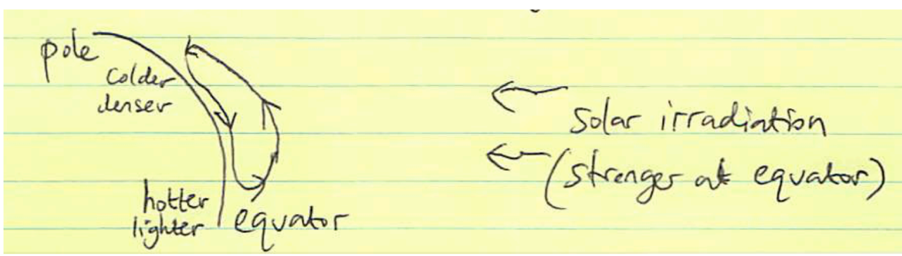
⇒ **Vorticity changes when the surfaces of constant pressure and density are misaligned**.

Here is a sketch that tries to illustrate this physically:



Here, the pressure gradients on the left and right are the same (because they connect the same two isobars with pressures P_1 and P_2). However, the lines of constant density are misaligned with the lines of constant pressure. The pressure gradient on the left is in a region with a higher density than the pressure gradient on the right. Because the acceleration of the fluid is $\propto -\nabla P/\rho$, the acceleration is therefore greater on the right compared to the left. The net effect is to start an anti-clockwise rotation (you can check for yourself that this is in the direction of the baroclinic vector $\nabla\rho \times \nabla P$).

This effect is important in geophysical fluid dynamics, e.g. in **Hadley cells** which are large scale circulations from the equator to mid-latitudes on Earth caused by differential heating of the Earth's surface. Because the solar irradiation is stronger at the equator, the density at the equator is lower than at mid-latitudes: the surfaces of constant density become misaligned with the surfaces of constant pressure.



Questions for this week

1. Sketch an example of a flow with
   - circular streamlines and $\nabla \times \mathbf{u} \neq 0$
   - circular streamlines and $\nabla \times \mathbf{u} = 0$
   - parallel streamlines and $\nabla \times \mathbf{u} \neq 0$
   - parallel streamlines and $\nabla \times \mathbf{u} = 0$

2. A uniformly rotating fluid has vorticity $\omega = 2\Omega$. Why is the factor of 2 there?

3. Explain the physical interpretation of the vorticity equation

$$\frac{D\omega}{Dt} = (\omega \cdot \nabla)\,\mathbf{u}.$$

4. What is Kelvin's theorem and when does it apply?

5. Give example(s) of when you have encountered vortices in everyday life. Describe what is happening physically to create or amplify the vortex.

Week 2 Computational exercise - Fun with vortices

Write a code to model the motion of a collection of vortices in the x-y plane. Assume that the vortices are line vortices, so that the velocity field associated with a given vortex falls off as $1/r$ where r is the distance from the centre of the vortex.

A given vortex moves with a velocity given by the sum of the velocities from all the other vortices. So a simple algorithm is to loop through each of the vortices, and for each one add up the velocity contributions from all the other vortices, and sum them to find the velocity of that vortex. You can then update the vortex position $\mathbf{x} \rightarrow \mathbf{x} + \mathbf{v}\Delta t$ for some timestep Δt.

Hints and tips:

- It might help to implement a minimum distance between two vortices when you calculate the velocity, to avoid very large velocities when two vortices approach each other very closely.
- You can animate the motion of the vortices with `matplotlib` in python by doing something like this:

```
# plot the starting values and keep a handle 'x1'
x1, = plt.plot(x,y,'ro')
plt.xlim([-200,200])
plt.ylim([-200,200])
fig.canvas.draw()

while count < nsteps:
    # ... calculate velocities and update positions...
    # and then update the data for 'x1':
    x1.set_xdata(x)
    x1.set_ydata(y)
    # redraw the figure
    fig.canvas.draw()
    # pause for a short time to give the plot time to redraw
    plt.pause(0.001)
```

- If you include some vortices with $\omega = 0$, they will act as "tracer particles" so you can see what the fluid flow looks like. (Maybe plot the tracer particles with a different symbol or colour).

Play around with different numbers and arrangements of vortices and see if you get the behaviour you expect. Here are some configurations you can try:

1. Two vortices with either parallel or antiparallel rotation.

2. Four vortices arranged in a square.
3. A line of vortices.
4. A circle of vortices.
5. A random collection of vortices with the same sign of vorticity.
6. A random collection of vortices with random signs or magnitudes of vorticity.
7. A line of vortices but with one vortex displaced slightly.

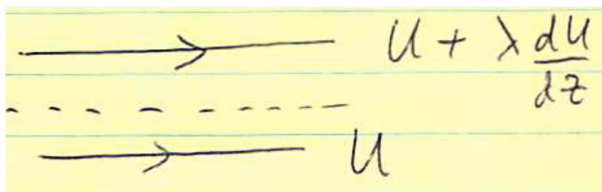
In each case, try putting in some tracer particles (especially for number 5) and see if they behave the way you expect.

Week 3: Viscosity and viscous flows

Basic idea and estimates of viscosity

In a viscous fluid, **the random motions of molecules transport momentum between adjacent layers that are moving with different bulk velocities.**

For example, consider a plane-parallel shear flow $\mathbf{u} = U(z)\,\hat{x}$. The diagram below shows two layers of fluid separated vertically by a mean free path λ, which is the average distance over which molecules move before scattering:



In this example, the upper layer has a larger x-velocity than the lower layer. Therefore, the molecules coming from the lower layer into the upper layer on average have a smaller momentum in the x-direction compared to the molecules in the upper layer. After scattering, the net effect is to slow down the upper layer, reducing the velocity contrast. The same argument applies in the opposite direction for molecules from the upper layer that move down in the lower layer and scatter.

The net flux of momentum across the dashed line is

$$-\frac{1}{3} n m v_{th} \left(\lambda \frac{dU}{dz}\right),$$

where n is the number density of molecules, m is the mass of each molecule, and v_{th} is the thermal velocity $v_{th} \approx (k_B T/m)^{1/2} \approx c_s$ (the sound speed). Note that in the example shown, where $dU/dz > 0$, the momentum flux is downwards, as encoded by the minus sign in the expression. The factor of $1/3$ comes from averaging over directions (the usual $1/3$ that comes up in kinetic theory).

A momentum flux (momentum per unit area per second) is also a force per unit area or a stress. The momentum flux we are looking at here is the flux of x-momentum in the z-direction. This corresponds to a stress in the x-direction on a surface whose normal vector points in the z-direction. This is a **tangential stress**, giving an **off-diagonal term T_{xz} in the stress tensor**.

The **shear viscosity** (often just called "viscosity") is the constant of proportionality between the stress and the velocity gradient:

$$\text{stress} = -\mu \frac{dU}{dz}$$

From our expression above, we see that

$$\mu = \text{(dynamical) viscosity} = \frac{1}{3}nmv_{th}\lambda = \rho\frac{1}{3}v_{th}\lambda$$

has units of kg m^{-1} s^{-1} = Pa s. We can also define the

$$\text{kinematic viscosity } \nu = \frac{\mu}{\rho} = \frac{1}{3}v_{th}\lambda$$

which has units m^2 s^{-1}.

A fluid that has stress $\propto$ velocity gradient is known as a **Newtonian fluid**. Not all fluids are Newtonian – a famous example which you can make at home is a mixture of corn starch and water, which behaves as a fluid on long timescales but as a solid on short timescales (search YouTube for "Non-Newtonian fluid" for some fun videos!)

Here are some values of viscosity for different substances (these are at 20°C):

	μ (g cm^{-1} s^{-1})	ν (cm^2 s^{-1})
water	0.01	0.01
air	1.8×10^{-4}	0.15
alcohol	0.018	0.022
glycerine	8.5	6.8
mercury	0.0156	0.0012
molasses		$\sim$ 50-100

Exercise: Have a think about whether these relative values make sense in terms of your own experience of fluids. You can also use the formula above for the viscosity in terms of the mean free path to check the value for air.

Viscous stress tensor

We already have the machinery to deal with these tangential kind of surface forces. Recall that we wrote the momentum equation as

$$\rho\frac{D\mathbf{u}}{Dt} = \nabla \cdot \boldsymbol{T}$$

where T_{ij} is the stress tensor. We can add an additional term to this to account for the viscous stresses:

$$T_{ij} = -P\delta_{ij} + \sigma_{ij}$$

where the **viscous stress tensor** σ_{ij} is

$$\boxed{\sigma_{ij} = \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) - \frac{2}{3}\mu\delta_{ij}\nabla \cdot \mathbf{u} + \xi\delta_{ij}\nabla \cdot \mathbf{u}}$$

which can also be written as

$$\sigma_{ij} = 2\mu e_{ij} - \frac{2}{3}\mu\delta_{ij}\nabla\cdot\mathbf{u} + \xi\delta_{ij}\nabla\cdot\mathbf{u},$$

where

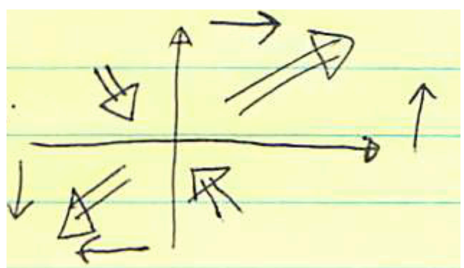
$$e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

is the **strain rate tensor**.

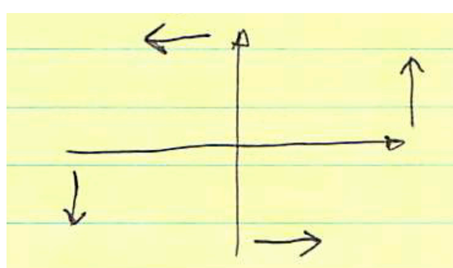
The viscous stress tensor has different pieces. The first term represents the symmetric part of the velocity gradient

$$\frac{\partial u_i}{\partial x_j} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) + \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right)$$

The symmetric term represents the deformation of the fluid element and generates viscous stress. An example is sketched below: the velocity increases in both directions such that the fluid element is stretched and squeezed:



The antisymmetric term is $\nabla\times\mathbf{u}$; we saw last week that this represents rotation of the fluid element, which doesn't generate any viscous stress. The sketch below reverses the velocity gradient on one of the axes; the result is now a rotation:



The second and third terms in the viscous stress tensor are both $\propto \delta_{ij}\nabla\cdot\mathbf{u}$, so they are diagonal like the pressure term in the stress tensor. The reason that the $\nabla\cdot\mathbf{u}$ terms are written as two separate terms like this is to separate out the traceless part of the viscous stress. If you take the trace $\sigma_{ii} = \sum_i \sigma_{ii}$, you'll see that the terms containing μ cancel, leaving

$$\sigma_{ii} = 3\xi\nabla\cdot\mathbf{u}.$$

We can think of ξ as the dynamical correction to the equilibrium pressure P. It is known as the **coefficient of bulk viscosity**. We can define a mean pressure

$$\bar{P} = -\frac{1}{3}T_{ii} = P - \xi\nabla\cdot\mathbf{u}.$$

The "Stokes assumption" is that $\xi = 0$ (σ_{ij} is traceless) so that volume changes do not lead to dissipation. This is true for a monatomic ideal gas for example. Bulk viscosity arises in cases where some irreversible process happens on compressing a fluid element. For example, a irreversible chemical reaction that happens on squeezing the fluid would give a bulk viscosity. It's often safe to ignore the bulk viscosity ξ and focus on the shear viscosity μ.

Momentum equation with a viscous term

In the case of an incompressible fluid, $\nabla\cdot\mathbf{u} = 0$, you can check that the momentum equation reduces to

$$\rho\frac{Du_i}{Dt} = -\frac{\partial P}{\partial x_i} + \frac{\partial}{\partial x_j}\sigma_{ij} = -\frac{\partial P}{\partial x_i} + \frac{\partial}{\partial x_j}\left(\mu\frac{\partial u_i}{\partial x_j}\right).$$

Just as the pressure force arises from differences in pressure from one side of a fluid element to the other, the same is true for the viscous term. It is the difference in the viscous stress from one side of the fluid element to the other that gives the net force.

For $\mu =$ constant, we therefore have

$$\boxed{\rho\frac{D\mathbf{u}}{Dt} = -\nabla P + \mu\nabla^2\mathbf{u}}$$

Focusing on the viscous term only, we see that the evolution of one component of velocity is

$$\frac{\partial u_i}{\partial t} = \nu\nabla^2 u_i$$

a **diffusion equation**. Viscosity causes diffusion of the velocity field. Just as thermal diffusion acts to equalize the temperature, so viscosity acts to equalize the velocity.

The characteristic timescale on which viscosity acts is

$$t_{\mathrm{visc}} \sim \frac{L^2}{\nu}$$

where L is the characteristic scale of the flow (lengthscale on which velocity varies).

The Reynolds number and different types of flow

The Reynolds number compares the relative sizes of the inertia and viscous terms in the momentum equation

$$\begin{array}{ccc} \mathbf{u}\cdot\nabla\mathbf{u} & \text{vs} & \nu\nabla^2\mathbf{u} \\ \sim \dfrac{U^2}{L} & & \sim \dfrac{\nu U}{L^2} \end{array}$$

where we use a typical velocity U and lengthscale L for the flow.

The ratio of the two terms gives the **Reynolds number**

$$\boxed{\mathrm{Re} = \frac{UL}{\nu}}$$

$$\begin{array}{ll} \mathrm{Re} \ll 1 & \text{viscous term dominates} \\ \mathrm{Re} \gg 1 & \text{inertia term dominates} \end{array}$$

In fluid dynamics, there are many such *dimensionless numbers* that characterize a fluid flow. They are important because of the idea of *dynamical similarity* – two flows can have dramatically-different velocity, length or timescales, but they will evolve similarly if the underlying dimensionless numbers are the same.

Increasing Re leads to flows with different properties:

- low Re$\ll 1$ — the flow is **reversible** (e.g. the experiment with the blobs of dye between two concentric cylinders that we mentioned in the first week). Microscopic biological flows are in this regime.

- moderate Re> 1 — for moderate values of Re the flow is **laminar** (well-defined streamlines) and viscous effects occur in thin **boundary layers**.

- high Re$\gg 1$ — for high values of Re (typical greater than thousands), the flow becomes **turbulent**. We'll discuss turbulence in detail in a future week.

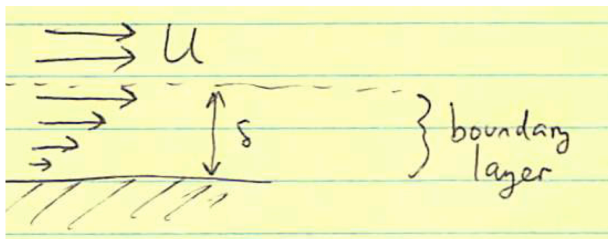
Viscous boundary layers

At the boundary with a solid surface, a viscous fluid obeys the

$$\boxed{\textbf{no slip condition} \qquad u_\parallel = 0}$$

where $u_\parallel$ is the comoponent of velocity parallel to the boundary. Because the perpendicular velocity $u_\perp = 0$ also (fluid can't penetrate a solid boundary), we have that **the total velocity $\mathbf{u} = 0$ at a solid boundary**.

This may seem counter-intuitive, because we are used to the idea that fluid flows past a solid surface. However, microscopically right at the solid wall, interactions between molecules will typically not allow a differential velocity to be maintained. But, you are probably asking, what about situations where we see fluids flowing past a solid boundary, e.g. think of water flowing in a container? In those cases, there is actually a thin **boundary layer** in which the velocity falls from the bulk velocity to zero right at the boundary. Viscous effects dominate in this thin layer (whereas inertia dominates in the bulk flow at Re>1).



We can estimate the width of the boundary layer by equating the viscous time across the boundary layer thickness $\sim \delta^2/\nu$ to the flow time along a characteristic length $\sim L/U$. This gives a boundary layer thickness

$$\delta \sim \left(\frac{L\nu}{U}\right)^{1/2} \sim \frac{L}{\mathrm{Re}^{1/2}}.$$

The typical size of a boundary layer is $\sim \mathrm{Re}^{1/2}$ times smaller than the scale of the flow.

Often when modelling laminar flows with Re>1, the thin boundary layers are replaced by a **free slip** boundary condition which says that $u_\perp = 0$ at the solid boundary, but $u_\parallel$ may take any value. The viscous terms are then dropped from the fluid equations, since they operate only in the thin boundary layers that have been absorbed into the boundary condition. We'll see some example of this later.

Questions for this week

1. You are having a conversation with a friend about fluids, and they make the statement "viscosity gives rise to off-diagonal terms in the stress tensor". Explain whether or not you agree.

2. What are the differences between high and low Reynolds number flows?

3. You stir your cup of coffee. How long would you expect it to take for the coffee to slow down due to viscosity? Does your answer make sense?

4. Estimate the Reynolds number for (i) flow past the wing of a jumbo jet, (ii) flow around a canoe paddle, (iii) a layer of maple syrup draining off a spoon, (iv) a bacteria of size 1 μm

moving at a speed of 30 μm s^{-1}. Where appropriate, estimate the thickness of the boundary layer.

Useful equations:

$$\mathrm{Re} = \frac{UL}{\nu}$$

$$\nu_{\mathrm{water}} \approx 10^{-2}\ \mathrm{cm}^2\ \mathrm{s}^{-1}$$

$$\frac{\delta}{L} \approx \frac{1}{\mathrm{Re}^{1/2}}$$

Week 3 Computational exercise - Viscous flow above an oscillating plate

In this exercise, we will model the flow of a viscous fluid above a rigid plate. The plate moves with some specified motion in the x-direction, eg. oscillating from side to side with some frequency, and we want to calculate the fluid velocity $u(z)\hat{x}$ as a function of distance z from the plate.

(a) First show that the motion of the fluid obeys a diffusion equation

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial z^2},$$

where ν is the viscosity.

(b) The way we will solve this numerically is by following the velocity at N grid points separated by Δz in height, ie. at locations $z_i = (i-1) * \Delta z$, with $i = 1, ...N$. In the technique of finite differences, the spatial derivative at grid point i is written

$$\frac{\partial^2 u_i}{\partial z^2} = \frac{1}{\Delta z}\left[\frac{u_{i+1} - u_i}{\Delta z} - \frac{u_i - u_{i-1}}{\Delta z}\right] = \frac{u_{i+1} - 2u_i + u_{i-1}}{(\Delta z)^2},$$

and we write the time-derivative as

$$\frac{\partial u_i}{\partial t} = \frac{u_i^{n+1} - u_i^n}{\Delta t},$$

where the superscript n labels the timestep.

Put this together to derive the following scheme for updating the velocities

$$u_i^{n+1} = u_i^n + \alpha\left(u_{i+1}^n - 2u_i^n + u_{i-1}^n\right)$$

where $\alpha = \nu\Delta t/(\Delta x)^2$. This is known as an *explicit* scheme since the velocities at the new timestep $\left\{u_i^{n+1}\right\}$ are written explicity in terms of the currently known velocities $\{u_i^n\}$.

(c) Write a code that implements this explicit scheme. You will need a boundary condition at the top of the fluid (at $z = (N-1)\Delta z$). Since we are imagining a finite thickness layer of fluid sitting on top of the plate, with a free surface at the top, the appropriate boundary condition is zero stress $\partial u/\partial z = 0$. You can implement this by setting $u_N = u_{N-1}$ at each timestep. The bottom velocity u_1 is set by the assumed velocity of the plate. Velocities u_2 to u_{N-1} are updated each timestep by the scheme above.

Important note: you should choose $\alpha \leq 1/2$. For larger values of α, this method is *numerically unstable* — try it and see! This is a limitation of explicit methods, we will discuss this more in a future exercise.

As a starting problem, set the velocities everywhere on the grid to be $u_i = 0$ except at the bottom $z = 0$ where $u_0 = 1$. Evolve the solution forwards in time and see whether you get the behavior you expect.

(d) Now implement an oscillating lower boundary and see whether the solution behaves as predicted analytically.

(e) *Optional extension:* By adding a gravitational acceleration term to your code, solve for the flow of a viscous fluid on an inlined plane.

Week 4: Sound waves

This week we are going use sound waves to explore some aspects of waves in fluids.

Linear adiabatic sound waves

Consider a constant density gas at rest, i.e. the fluid has $\mathbf{u} = 0$ and $\rho = \rho_0 =$ constant. If we make a small perturbation in density $\delta\rho(\mathbf{r})$, how does the fluid respond?

The continuity and momentum equations are

$$\left[\frac{\partial}{\partial t} + \delta\mathbf{u}\cdot\nabla\right]\delta\rho = -(\rho_0 + \delta\rho)\nabla\cdot\delta\mathbf{u}$$

$$\left[\frac{\partial}{\partial t} + \delta\mathbf{u}\cdot\nabla\right]\delta\mathbf{u} = -\frac{\nabla\delta P}{\rho_0 + \delta\rho},$$

where $\delta\mathbf{u}$ and δP are the perturbations to the velocity and pressure.

For small perturbations $\delta\rho \ll \rho_0$, we can solve these equations by keeping only the **linear terms** in the perturbations. This gives

$$\boxed{\frac{\partial}{\partial t}\delta\rho = -\rho_0\nabla\cdot\delta\mathbf{u}} \tag{3}$$

$$\boxed{\frac{\partial}{\partial t}\delta\mathbf{u} = -\frac{\nabla\delta P}{\rho_0}} \tag{4}$$

Note in particular that the non-linear term $(\mathbf{u}\cdot\nabla)\mathbf{u}$ has gone away because it is second order (quadratic) in $\delta\mathbf{u}$.

To close these equations, we need a relation between δP and $\delta\rho$. If the perturbations are **adiabatic** then

$$\frac{\delta P}{P} = \gamma\frac{\delta\rho}{\rho}$$

since adiabatic changes obey $P \propto \rho^\gamma$ where γ is the adiabatic index. Then

$$\frac{\partial}{\partial t}\delta\mathbf{u} = -\frac{\gamma P_0}{\rho_0}\frac{\nabla\delta\rho}{\rho_0}. \tag{5}$$

Combining equations (10) and (5) (take the time-derivative of (10) and the divergence of (5) and eliminate $\delta\mathbf{u}$) gives

$$\frac{\partial^2\delta\rho}{\partial t^2} = \frac{\gamma P_0}{\rho_0}\nabla^2\delta\rho \tag{6}$$

(you can check by eliminating $\delta\rho$ instead that $\delta\mathbf{u}$ obeys the same equation). This is a **wave equation** with wave speed given by

$$\boxed{c_s^2 = \frac{\gamma P_0}{\rho_0}}$$

where c_s is the **adiabatic sound speed**. In air at room temperature, the speed of sound is ≈ 330 m/s.

Dispersion relation, phase and group velocities

Since we have a linear equation, we can decompose the solutions into modes

$$\delta\rho, \delta\mathbf{u} \propto e^{i\mathbf{k}\cdot\mathbf{r}} e^{-i\omega t}$$

Substituting this solution into the wave equation (6) gives

$$-\omega^2 \delta\rho = c_s^2(-k^2)\delta\rho$$
$$\Rightarrow \boxed{\omega^2 = c_s^2 k^2}$$

which is the **dispersion relation** (relation between frequency and wavelength) for sound waves.

The wave frequency (in Hz) is given by $\omega/2\pi$; the wavelength is $\lambda = 2\pi/k$ where $k = |\mathbf{k}|$.

For any wave, we can define two different velocities. The first one is the velocity at which a given mode $\mathbf{k}$ remains at stationary phase, i.e. the velocity of the frame in which $\mathbf{k}.\mathbf{r} - \omega t$ is constant. This is the **phase velocity** which is in the direction of $\mathbf{k}$ and has a magnitude

$$v_p = \frac{\omega}{k} = c_s.$$

The **group velocity** is the velocity of the peak of a wavepacket, given by

$$\mathbf{v}_g = \frac{\partial\omega(\mathbf{k})}{\partial\mathbf{k}} = \left(\frac{\partial}{\partial k_x}, \frac{\partial}{\partial k_y}, \frac{\partial}{\partial k_z}\right)\omega(k_x, k_y, k_z).$$

For example, writing the dispersion relation $\omega^2 = c_s^2(k_x^2 + k_y^2 + k_z^2)$, the x-component of the group velocity is $v_{g,x} = \partial\omega/\partial k_x = c_s k_x/|\mathbf{k}|$, and similarly for the y and z-components. Therefore the group velocity for sound waves is in the direction of $\mathbf{k}$ and has a magnitude c_s.

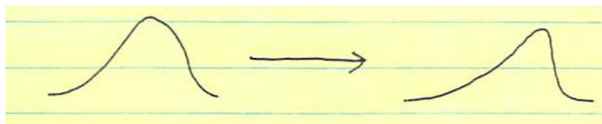
Therefore for linear sound waves we have

$$\boxed{v_g = v_p = c_s}$$

with both phase and group velocities in the direction of $\mathbf{k}$. Because of the linear dispersion relation, the velocity is independent of k: the waves are **non-dispersive**. Because all modes travel with the same phase velocity, a wavepacket keeps its shape as it moves.

Non-linear steepening of waves

The derivation above shows that small-amplitude waves are non-dispersive and propagate without changing shape. For larger amplitude waves, however, the non-linear terms become significant, leading to **wave steepening**.



The sketch shows the evolution of the velocity/density profile of a wave packet travelling to the right. The leading edge steepens as the wave packet travels. In the computational exercise this week, we'll solve the fluid equations directly to calculate the evolution of perturbations and you'll see this steepening directly in your solutions.

There are a few different ways to think about wave steepening:

- A physical way to think about it is that in the peaks of the wave, the density is larger and therefore the sound speed is larger. A peak of the wave therefore moves faster and tries to "catch up" with the trough ahead of it.

- Generation of harmonics: although we might start with a pure mode with frequency ω and wavevector $\mathbf{k}$, the non-linear term leads to generation of harmonics

$$\mathbf{u} \propto e^{ikx} \Rightarrow \mathbf{u}\cdot\nabla\mathbf{u} \propto e^{i2kx}.$$

  The shorter wavelength components result in small-lengthscale features in the profile.

- We can look at solutions of the advection terms in 1D:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0 \qquad \text{Burgers}' \text{ equation}$$

  Burgers' equation has a solution of the form $u = f(x - ut)$ for some function f. We can see this by writing $\xi = x - ut$ and then

$$\frac{\partial u}{\partial t} = f'\frac{\partial \xi}{\partial t} = -f'\left(u + t\frac{\partial u}{\partial t}\right) \Rightarrow \frac{\partial u}{\partial t} = -\frac{ff'}{1+f't},$$

$$\frac{\partial u}{\partial x} = f'\frac{\partial \xi}{\partial x} = f'\left(1 - t\frac{\partial u}{\partial x}\right) \Rightarrow u\frac{\partial u}{\partial x} = \frac{ff'}{1+f't}, \tag{7}$$

  where f' is the derivative of the function f with respect to its argument.

  Equation (7) shows that the velocity gradient steepens in the part of the wave that has $f' < 0$ (and becomes more shallow in the part with $f' > 0$). Indeed, the gradient diverges (becomes infinite) in a time $t = -1/f' = -(\partial u/\partial x)^{-1}$, the "turnover time" associated with the velocity profile.

As the steepening continues, the wave forms a **shock**: a thin interface in which the fluid properties change almost discontinuously. In reality, the shock has a small thickness which is set by diffusion which becomes very efficient when the gradient is very steep and stops the wave from steepening further. However, this interface is extremely thin. For a shock thickness Δ, momentum balance gives

$$\mathbf{u}\cdot\nabla\mathbf{u}\sim\nu\nabla^2\mathbf{u}\Rightarrow\frac{U^2}{\Delta}\sim\frac{\nu U^2}{\Delta^2}\Rightarrow\Delta\sim\frac{\nu}{U}.$$

But recall that $\nu\sim c_s\lambda$ for an ideal gas,

$$\Rightarrow\Delta\sim\lambda\left(\frac{c_s}{U}\right)$$

which implies that the interface thickness is comparable to the mean free path!

A classical example of shock formation is that of a piston moving into a tube of gas. Even if the piston moves subsonically, a shock moves ahead of the piston (at speed $\sim c_s$) to let the gas ahead of the piston know it is coming.

Questions for this week

1. Explain physically what is happening in a sound wave. Why does the wave propagate? What is the restoring force?

2. Evaluate the adiabatic sound speed in air and check the number given in the notes.

3. Is the sound speed in water larger or smaller than in air? Why?

4. According to Wikipedia, the sound intensity corresponding to lowest sound intensity detectable by the human ear is of order $I_0\sim 1$ pW/m^2. Calculate the velocity amplitude of the sound wave with this intensity and a frequency of 1 kHz. How long would it take for the wave to steepen to a shock? Does this match your everyday experience of sound waves? [Hint: the energy flux in a wave is the energy density times the wave speed]. The intensity in decibels (dB) is given by $10\ \log_{10}(I/I_0)$. How loud in dB would a sound need to be to form a shock? Does your answer make sense (look up some dB values and compare)?

Week 4 Computational exercise - Sound waves

In this exercise, we will solve the fluid equations to evolve a sound wave in 1D.

Fluid equations in flux-conservative form. For motion in 1D, the continuity and momentum equations can be written as (you should check this yourself)

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho v) = 0$$

$$\frac{\partial}{\partial t}(\rho v) + \frac{\partial}{\partial x}\left(\rho v^2\right) = -\frac{\partial P}{\partial x}.$$

This form of the equations is known as *flux-conservative form* because they are written in the form

(rate of change of a density) + (divergence of a flux) = source terms.

This is a useful form for numerical solutions because it guarantees that quantities are conserved – whatever flux leaves one cell is the same flux that enters the neighbouring cell (and since the fluid equations are based on conservation of mass, momentum, or energy, this is perfect for fluids!)

Finite-volume methods and donor cell advection. We divide the volume into cells such that the grid points x_i are the locations of the cell centres, and the cell boundaries are at the mid-point locations $x_{i\pm 1/2} = (1/2)(x_i + x_{i\pm 1})$. We then solve the equation

$$\frac{\partial f}{\partial t} = -\frac{\partial J}{\partial x},$$

in discretized form

$$\frac{f_i^{n+1} - f_i^n}{\Delta t} = -\frac{J_{i+\frac{1}{2}} - J_{i-\frac{1}{2}}}{\Delta x},$$

where n refers to the current values and $n+1$ to the future values, and we write the flux of quantity f at the cell boundaries $(i \pm 1/2)$ as

$$J_{i+\frac{1}{2}} = \begin{cases} v^n_{i+\frac{1}{2}} f^n_i, & \text{if } v^n_{i+\frac{1}{2}} > 0 \\ v^n_{i+\frac{1}{2}} f^n_{i+1}, & \text{if } v^n_{i+\frac{1}{2}} < 0 \end{cases}$$

$$J_{i-\frac{1}{2}} = \begin{cases} v^n_{i-\frac{1}{2}} f^n_{i-1}, & \text{if } v^n_{i-\frac{1}{2}} > 0 \\ v^n_{i-\frac{1}{2}} f^n_{i}, & \text{if } v^n_{i-\frac{1}{2}} < 0 \end{cases}$$

This choice for the fluxes is known as *donor cell advection* or *upwind differencing*. Depending on the sign of the velocity, the contents are either advected out of cell i or into cell i from the left or right neighbour.

Algorithm. A simple algorithm that you can use to solve the equations is as follows. First define quantities

$$f \equiv \rho$$

$$g \equiv \rho v$$

(the mass and momentum densities) and assume that $P = c_s^2 \rho$ with constant sound speed c_s. The equations to solve are then

$$\frac{\partial f}{\partial t} + \frac{\partial}{\partial x}(vf) = 0$$

$$\frac{\partial g}{\partial t} + \frac{\partial}{\partial x}(vg) = -c_s^2 \frac{\partial f}{\partial x}.$$

These are in flux-conservative form with the pressure gradient acting as a source term for the momentum density g. Note that given f and g, the velocity can be obtained from the ratio g/f.

The algorithm has two steps:

1. Use donor-cell advection to update f and g. To calculate the velocity at the cell boundaries, you can take an average of the velocity at the cell centres

$$v_{i+\frac{1}{2}} = \frac{1}{2}(v_i + v_{i+1}).$$

2. Add an additional update to the value of g from step 1 to take into account the source term. You can do this by using the new values of f you found in step 1 to calculate the derivative

$$\left.\frac{\partial f}{\partial x}\right|_i \approx \frac{f_{i+1} - f_{i-1}}{2\Delta x}.$$

The simplest boundary conditions to use are periodic boundaries (so the flux off the right hand side is the same as the flux onto the left hand side and vice versa).

Questions

1. Choose an initial condition that has a sinusoidal variation in density and/or velocity. Check that for small amplitudes, the wave propagates as expected.
2. Do you see steepening at larger wave amplitudes?
3. How large a timestep can you take and still be numerically stable?
4. Do you form a shock in your simulation? What sets its thickness?

Week 5: Turbulence

Last week we saw the effect of the non-linear term in the momentum equation $\mathbf{u} \cdot \nabla \mathbf{u}$ on sound waves. By coupling together different wavelengths, it leads to steepening of the wave profile. The non-linearities in the fluid equations also lead to **turbulence**, the topic for this week.

First watch the movie!

First watch the movie on turbulence from the National Committee for Fluid Mechanics. The movie is quite dated now (made in the 1960s), but keep watching because it gives an excellent overview of turbulence. We'll elaborate on some of the points made in the movie in the notes below. The other movies in this series are also worth watching and cover many different topics in fluids.

You can stop the movie once it gets to timestamp 22:30. There is also a set of Film Notes that might be helpful to look at.

Characteristics of turbulence

The movie highlights the following characteristics (or "symptoms") of turbulence:

- irregularity
- diffusivity
- large Re numbers
- 3D vorticity fluctuations
- dissipation

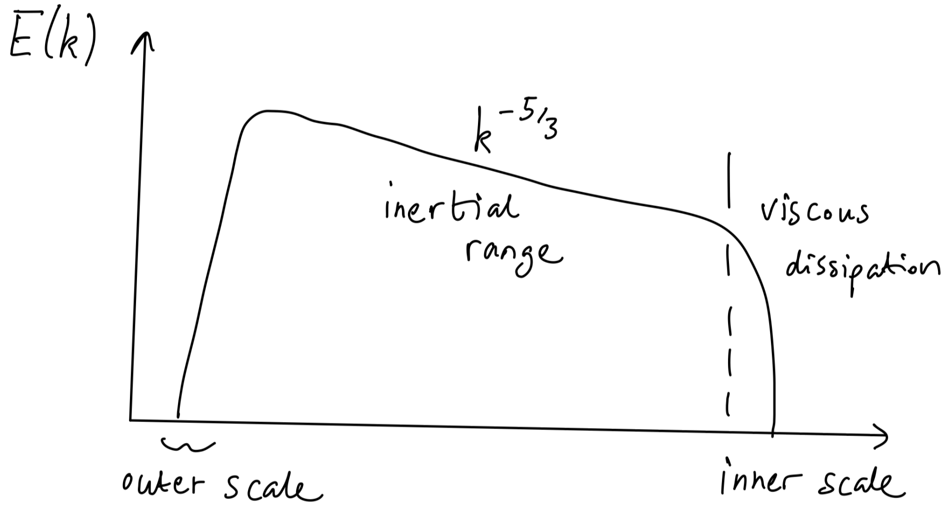
One important point is that turbulence is a property of the *flow*, not the *fluid*.

Energy cascade

Turbulence involves a cascade of energy from large scales to small scales where viscosity dissipates the energy.

Perhaps the most famous result is the $-5/3$ scaling of the energy spectrum for *isotropic homogeneous incompressible* turbulence. Let's see how that works.

As mentioned in the movie, the behaviour of the flow at a particular point is not predictable, but statistical quantities/averages are. One of these is the energy spectrum $E(k)$ were $E(k)dk$ is the kinetic energy density in modes of wavelength $\lambda = 2\pi/k$. A log-log plot of $E(k)$ against k looks like:



The **outer scale** is where the fluid is being stirred, i.e. where energy is being injected with Re= $UL/\nu \gg 1$. The **inner scale** is where viscous dissipation occurs. The typical velocity v_d and lengthscale ℓ_d there are such that $v_d \ell_d/\nu \sim 1$.

In a steady cascade, the energy transfer rate ε from scale to scale must be constant. From dimensional arguments we can write

$$\varepsilon \sim \frac{v^3}{\ell}$$

at any scale ℓ, where

$$\boxed{v \sim (\varepsilon \ell)^{1/3}}$$

is a typical velocity at scale ℓ.

In particular, this applies at each end of the inertial range

$$\varepsilon \sim \frac{U^3}{L} \sim \frac{v_d^3}{\ell_d}$$

But we also know that at the dissipation scale $v_d \ell_d/\nu \sim 1$, giving

$$\boxed{\ell_d \sim \left(\frac{\nu^3}{\varepsilon}\right)^{1/4}} \qquad \boxed{v_d \sim (\nu\varepsilon)^{1/4}}$$

These are the size and velocity of turbulent eddies for which the turnover time ℓ_d/v_d is equal to the viscous time ℓ_d^2/ν. Viscosity efficiently damps motions on these scales, terminating the cascade.

The eddy turnover time is

$$\frac{\ell}{v} \sim \varepsilon^{-1/3} \ell^{2/3}$$

which is faster and faster as we go to smaller scales.

We can also get the range of lengthscales involved in the cascade by writing

$$\left(\frac{L}{\ell_d}\right)^4 = L^3 \times L \times \ell_d^{-4} = L^3 \times \frac{U^3}{\varepsilon} \times \frac{\varepsilon}{\nu^3} = \left(\frac{LU}{\nu}\right)^3 = \mathrm{Re}^3$$

$$\Rightarrow \qquad \boxed{\ell_d = \frac{L}{\mathrm{Re}^{3/4}}} \qquad \boxed{v_d \sim \frac{U}{\mathrm{Re}^{1/4}}}$$

In the questions, you'll estimate the range of scales when you stir a cup of coffee. To give an astrophysical example, the outer convection zone of the Sun has a size $\sim 10^{10}$ cm (about 20% of the radius of the Sun), with velocities $U \sim 10^3$ cm s^{-1}. The corresponding Reynold's number is Re $\sim 10^{12}$ which then gives $\ell_d \sim 10$ cm and $v_d \sim 1$ cm s^{-1}. Note the enormous range of scales! The turnover time at the dissipation scale is $\ell_d/v_d \sim 10$ seconds compared to $L/U \sim 10^7$ seconds or months. This makes this a difficult problem to simulate!

The scaling for $E(k)$ now follows. Since $E(k)dk$ is the kinetic energy density at scale k, we must have

$$E(k)dk \sim v^2 \frac{dk}{k} \sim \varepsilon^{2/3} k^{-2/3} \frac{dk}{k} \propto \varepsilon^{2/3} k^{-5/3} dk.$$

This shows that $E(k) \propto k^{-5/3}$ which is the famous **Kolmogorov spectrum** (1941).

The 5/3 scaling was confirmed for turbulent flow in the "Seymour Narrows", a tidal channel where there is a large Re$\sim 10^8$ by Grant et al. (1961).

If the stirring is kept the same but the viscosity varied, the inertial range remains fixed but with a different scale for the viscous cutoff. This is illustrated in the movie with two turbulent jets with different Re – they look identical at large scales, but the larger Re jet has much finer small scale structure. Another point in the movie is that smaller scales decay first in freely-decaying turbulence, consistent with the picture above.

Turbulent transport

The other property of turbulence emphasized in the movie was the large increase in the transport of momentum and scalars such as temperature in a turbulent flow. Let's try to understand that.

Decompose the fluid motion into a **mean flow U** and a **fluctuating flow u′**

$$\mathbf{u} = \mathbf{U} + \mathbf{u}'$$

We do this in such a way that

$$\langle \mathbf{u} \rangle = \mathbf{U} \qquad \langle \mathbf{u}' \rangle = 0,$$

where the time-average is

$$\langle \mathbf{u} \rangle = \frac{1}{\tau} \int_{t_0}^{t_0+\tau} dt\ \mathbf{u}$$

for some suitably large τ. This is called a **Reynold's decomposition**.

Now think about the fluid equations with this decomposition. For an incompressible flow, $\nabla \cdot \mathbf{u} = 0$. Averaging this gives $\nabla \cdot \mathbf{U} = 0$, ie. the mean flow and fluctuating flow are separately incompressible.

The momentum equation is (in component form)

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = -\frac{1}{\rho} \frac{\partial P}{\partial x_i}$$

Splitting the velocity into mean and fluctuating parts and taking the average gives

$$\frac{\partial U_i}{\partial t} + U_j \frac{\partial U_i}{\partial x_j} = -\left\langle u'_j \frac{\partial u'_i}{\partial x_j} \right\rangle - \frac{1}{\rho} \frac{\partial P}{\partial x_i}$$

where P is now the mean part of the pressure. The velocity term on the right hand side can be rewritten using incompressibility as

$$-\left\langle u'_j \frac{\partial u'_i}{\partial x_j} \right\rangle = -\frac{\partial}{\partial x_j} \left(\langle u'_i u'_j \rangle \right).$$

The momentum equation for the mean flow is therefore

$$\rho \left(\frac{\partial}{\partial t} + \mathbf{U}.\nabla \right) \mathbf{U} = \nabla \cdot \mathbf{T}$$

where

$$\boxed{T_{ij} = -\delta_{ij} P - \rho \langle u'_i u'_j \rangle}$$

is the stress tensor. We see that the effect of the turbulence is to give a new term in the stress tensor known as the **Reynolds stress**.

The Reynolds stress describes the transport of momentum by the turbulent fluctuations. In particular, we see that *correlated velocity fluctuations* lead to a transport of momentum. For example, in the flow in a pipe shown in the movie, the average $\langle u'_z u'_x \rangle$ is non-zero because upwards moving fluid tends to carry an excess of horizontal momentum compared to downwards-moving fluid. The net effect is that the turbulence transports horizontal momentum towards the wall of the pipe.

In trying to solve these equations for a turbulent flow we face the "closure problem" – we need a **closure relation** to relate the Reynolds stress (fluctuating part) with the mean flow.

It is often assumed for simplicity that

$$\langle u_i' u_j' \rangle = -D_T \left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \right),$$

i.e. the same kind of relation as for microscopic viscosity. The coefficient D_T is the **Eddy viscosity**. Note that there is a crucial difference compared to a viscous fluid however: even if a relation like this were valid (probably not – the dependence of the Reynolds stress on the mean flow is likely much more complex), the Eddy viscosity D_T is a property of the *flow* unlike the microscopic viscosity ν which is a property of the *fluid*!

We can also treat the transport of a scalar using the Reynolds decomposition. An example is heat transport. The equation that describes heat transport is

$$\rho c_P \left(\frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \right) T = \frac{\partial}{\partial x_j} \left(K \frac{\partial T}{\partial x_j} \right)$$

where c_P is the specific heat capacity at constant pressure and K is the thermal conductivity. The left hand side describes advection of heat and the right hand side thermal diffusion.

The Reynolds decomposition for the heat equation gives

$$\rho c_P \left(\frac{\partial}{\partial t} + \mathbf{U} \cdot \nabla \right) \langle T \rangle = \frac{\partial}{\partial x_j} \left(-\rho c_P \langle T' u_j' \rangle + K \frac{\partial \langle T \rangle}{\partial x_j} \right)$$

The term $\rho c_P \langle T' u_j' \rangle$ is the **turbulent heat flux** – correlated fluctuations in temperature and velocity lead to transport of heat. For example, in turbulence with a background temperature gradient in the vertical direction, rising fluid elements are hotter than sinking fluid elements, leading to a net heat transport.

Questions for this week

1. Why and when are fluids turbulent?
2. What are the "symptoms" of turbulence?
3. What is the energy spectrum $E(k)$ and what does it look like for incompressible, isotropic turbulence?
4. When you stir a cup of coffee, what do you expect to be the smallest lengthscale associated with the motion?
5. What is meant by the terms *Reynold's stress* and *Eddy viscosity*?
6. Do you have any questions that came up while watching the movie/reading the notes?

Week 5 Computational exercise - Lorenz attractor

In a famous 1963 paper "Deterministic Nonperiodic Flow", Edward Lorenz wrote down a system of three coupled ODEs as a simplified model of thermal convection. These equations are **deterministic** in that given a set of initial values, the evolution of the system in time can be computed exactly. However, they show **sensitivity to initial conditions**: a tiny change in the initial conditions leads to a dramatically different evolution in time. This became known as the "butterfly effect" and launched the field of **Chaos Theory**. Chaotic systems show unpredictability and randomness despite being governed by a deterministic set of equations.

1. Rayleigh Benard convection

The equations that Lorenz wrote down were a very much simplified version of the fluid equations for thermal convection. In particular, a famous problem in fluids is **Rayleigh-Benard** convection in which heat is transported through a fluid that lies between two plates, the bottom plate hot and the top cold with a temperature difference ΔT. The response of the fluid depends on the dimensionless **Rayleigh number**

$$\mathrm{Ra} = \frac{\alpha \Delta T g H^3}{\kappa \nu},$$

where H is the thickness of the fluid layer, κ and ν are the thermal diffusivity and viscosity, α is the thermal expansion coefficient $(d\ln\rho/dT)$ and g is the acceleration due to gravity.

Question

- Take a look at this set of movies showing 2D Rayleigh-Benard convection at increasing values of Ra from 10^3 to 10^8. The movies show the temperature (red=hot and blue=cold) as a function of time as the simulation evolves[3]. What does the response of the fluid look like at different Rayleigh number? What are the similarities and differences in the flow? You should be able to see four different regimes.

2. The Lorenz equations

The simplified equations are

$$\begin{aligned}\dot{X} &= -\sigma X + \sigma Y \\ \dot{Y} &= -XZ + rX - Y \\ \dot{Z} &= XY - bZ\end{aligned}$$

[3]If you are interested in running your own convection simulations, take a look at the Dedalus code which I used for these simulations.

where the three variables are

$$\begin{aligned} X &\propto \text{intensity of convective motions} \\ Y &\propto \text{temperature differences between ascending and descending currents} \\ Z &\propto \text{deviation of temperature from initial linear profile} \end{aligned}$$

There are three parameters: r, which plays the role of Ra, σ which represents the Prandtl number (ratio of viscosity to thermal diffusivity ν/κ; this is ≈ 7 for water), and b which is related to the horizontal scale of the motion. Lorenz took $\sigma = 10$ and $b = 8/3$ (related to the most unstable wavelength in the linear instability theory for convection).

The Lorenz equations have a steady state solution $X = Y = Z = 0$ for $r < 1$ and an additional two steady-state solutions $Z = r - 1$, $X = Y$, $X^2 = b(r - 1)$ for $r > 1$.

Questions

- Have a look at the different terms in the Lorenz equations. Given your knowledge of the fluid equations, can you see where any of these terms might come from?
- Write a code to integrate the Lorenz equations in time. Your code should plot (1) X, Y and Z against time, and (2) the "phase space trajectory" by plotting X, Y and Z against each other (either in 3D or in 2D, e.g. plot X against Y). Mark the steady state solutions on your plots.
- Start with the same choices for σ and b as Lorenz — $\sigma = 10$ and $b = 8/3$ — and study the dependence of the solutions on r. Do you see analogous behavior to the Rayleigh-Benard convection simulations as r increases? What role if any do the steady-state solutions ("fixed points") play? Do you see sensitivity to initial conditions? Expanding to other values of σ and b, how many different types of behavior can you find?

Week 6: Energy in fluids

This week we'll discuss energy in fluids and the different forms of the energy equation.

Kinetic energy in inviscid fluids

The kinetic energy density is $\frac{1}{2}\rho u^2$. To derive an equation for the kinetic energy, take

$$\mathbf{u}\cdot(\text{momentum equation})$$

or

$$\mathbf{u}\cdot\left[\rho\frac{\partial\mathbf{u}}{\partial t}+\rho\left(\mathbf{u}\cdot\nabla\right)\mathbf{u}=-\nabla P\right]$$

We'll ignore the viscous term for now. The three terms are:

$$\mathbf{u}\cdot\rho\frac{\partial\mathbf{u}}{\partial t}=\rho\frac{\partial}{\partial t}\left(\frac{1}{2}u^2\right)$$

$$\rho\mathbf{u}\cdot\left(\mathbf{u}\cdot\nabla\right)\mathbf{u}=\rho\mathbf{u}\cdot\left(-\mathbf{u}\times\left(\nabla\times\mathbf{u}\right)+\nabla\frac{1}{2}u^2\right)=\rho\mathbf{u}\cdot\nabla\frac{1}{2}u^2$$

$$-\mathbf{u}\cdot\nabla P=-\nabla\cdot\left(\mathbf{u}P\right)+P\nabla\cdot\mathbf{u}$$

so we get

$$\rho\frac{\partial}{\partial t}\left(\frac{1}{2}u^2\right)+\rho\mathbf{u}\cdot\nabla\frac{1}{2}u^2=-\nabla\cdot\left(\mathbf{u}P\right)+P\nabla\cdot\mathbf{u}.$$

Now add to this

$$\left(\frac{1}{2}u^2\right)\times(\text{continuity equation})$$

or

$$\frac{1}{2}u^2\frac{\partial\rho}{\partial t}+\frac{1}{2}u^2\nabla\cdot\left(\rho\mathbf{u}\right)=0$$

which gives the **kinetic energy equation**

$$\boxed{\frac{\partial}{\partial t}\left(\frac{1}{2}\rho u^2\right)+\nabla\cdot\left(\mathbf{u}\left[\frac{1}{2}\rho u^2+P\right]\right)=P\nabla\cdot\mathbf{u}}$$

Internal energy and total energy

For the internal energy of the fluid, we can use the first law of thermodynamics

$$TdS=dE-\frac{P}{\rho^2}d\rho$$

where

$$E=\text{internal energy per unit mass}$$

$$S = \text{entropy per unit mass}$$

The last term is the "PdV" term but again written per unit mass (volume per unit mass $1/\rho$).

Consider first an **adiabatic flow** which has

$$T\frac{DS}{Dt} = 0.$$

The first law of thermodynamics then gives

$$\begin{aligned}
&\Rightarrow \quad \frac{DE}{Dt} = \frac{P}{\rho^2}\frac{D\rho}{Dt} = -\frac{P}{\rho}\nabla\cdot\mathbf{u} \\
&\Rightarrow \quad \frac{D}{Dt}(\rho E) = \rho\frac{DE}{Dt} + E\frac{D\rho}{Dt} = -P\nabla\cdot\mathbf{u} - \rho E\nabla\cdot\mathbf{u} \\
&\Rightarrow \quad \frac{\partial}{\partial t}(\rho E) + \mathbf{u}\cdot\nabla(\rho E) + \rho E\nabla\cdot\mathbf{u} = -P\nabla\cdot\mathbf{u}.
\end{aligned}$$

We arrive at the **internal energy equation**

$$\frac{\partial}{\partial t}(\rho E) + \nabla\cdot(\mathbf{u}\rho E) = -P\nabla\cdot\mathbf{u}.$$

For a **non-adiabatic flow**, we can add back in the DS/Dt term

$$\boxed{\frac{\partial}{\partial t}(\rho E) + \nabla\cdot(\mathbf{u}\rho E) = -P\nabla\cdot\mathbf{u} + \rho T\frac{DS}{Dt}}$$

The term

$$P\nabla\cdot\mathbf{u} = -\frac{P}{\rho}\frac{D\rho}{Dt}$$

represents $P\,dV$ work. It appears in both the kinetic energy and internal energy equations but with opposite sign: $P\,dV$ work transfers energy from the bulk kinetic energy to internal energy and vice versa.

The **total energy equation** is given by adding the kinetic energy and internal energy equations

$$\boxed{\frac{\partial}{\partial t}\left(\rho E + \frac{1}{2}\rho u^2\right) + \nabla\cdot\left(\mathbf{u}\left[\rho E + P + \frac{1}{2}\rho u^2\right]\right) = \rho T\frac{DS}{Dt}}$$

The left hand side is in flux-conservative form but notice that the energy flux has a pressure term in it. The energy flux is actually kinetic energy flux plus enthalpy flux where the **enthalpy** is the quantity $E + P/\rho$ (per unit mass). This takes into account the $P\,dV$ work as fluid moves around. We saw the enthalpy appear earlier in Bernoulli's constant.

Thermal diffusion and volumetric heating. An example of a DS/Dt term is heat flow from thermal diffusion. The heat flux in that case is

$$\mathbf{F} = -K\nabla T$$

were K is the thermal conductivity. Then

$$\rho T \frac{DS}{Dt} = -\nabla \cdot \mathbf{F}$$

since the heat deposited in a local volume is the integral of the heat flux over the surface. We can also add a volumetric heating or cooling term ε (J/kg/s) e.g. from chemical reactions, giving

$$\boxed{\rho T \frac{DS}{Dt} = -\nabla \cdot \mathbf{F} + \varepsilon}$$

Adiabatic flows. If the flow is adiabatic, we can write

$$\frac{DS}{Dt} = 0 = \frac{D}{Dt}\left(\frac{P}{\rho^\gamma}\right)$$

where γ is the ratio of specific heats C_P/C_V. This gives

$$\frac{1}{P}\frac{DP}{Dt} = \frac{\gamma}{\rho}\frac{D\rho}{Dt}$$

A good approximation if the flow time is much shorter than the time for heat generation/diffusion in the fluid.

Viscous dissipation

With viscosity included, there is an extra term in the kinetic energy equation

$$u_i \frac{\partial}{\partial x_j}\sigma_{ij} = u_i \frac{\partial}{\partial x_j}\left(2\mu e_{ij}\right)$$

(you might need to go back and look again at the notes on viscosity to get a reminder of the viscous stress tensor σ_{ij}). We're using index notation because we're now dealing with tensors. It is helpful to rewrite this term as

$$u_i \frac{\partial}{\partial x_j}\left(2\mu e_{ij}\right) = \frac{\partial}{\partial x_j}\left(2\mu u_i e_{ij}\right) - 2\mu e_{ij}\frac{\partial}{\partial x_j}u_i.$$

The first term is the divergence of a flux and adds to the divergence term on the LHS of the kinetic energy equation. The second term represents viscous dissipation: loss of energy to heat when fluid elements are deformed. If you look back at the kinetic energy equation for inviscid fluid, a similar thing happened to the pressure term when we wrote

$$-\mathbf{u}\cdot\nabla P = -\nabla\cdot(\mathbf{u}P) + P\nabla\cdot\mathbf{u}.$$

Just as $P\nabla \cdot \mathbf{u}$ represents $P\,dV$ work that changes the internal energy of fluid elements, the same thing happens with shearing distortions of fluid elements that involve viscous dissipation.

The viscous dissipation term can be written in a simpler form using the fact that e_{ij} is symmetric:

$$\begin{aligned}
-2\mu e_{ij}\frac{\partial u_i}{\partial x_j} &= -2\mu \times \frac{1}{2}\left[e_{ij}\frac{\partial u_i}{\partial x_j} + e_{ji}\frac{\partial u_j}{\partial x_i}\right] \\
&= -2\mu e_{ij} \times \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \\
&= -2\mu e_{ij}e_{ij} \\
&= -2\mu (e_{ij})^2
\end{aligned}$$

Note that this term is always negative, so viscous dissipation always acts to decrease the kinetic energy. This energy goes into internal energy.

The **viscous dissipation rate** is defined as the positive quantity

$$\Phi_V = \sigma_{ij}\frac{\partial u_i}{\partial x_j} = 2\mu (e_{ij})^2.$$

Note that there is a double sum implied here over i and j, so that Φ_V is a scalar quantity. It gives the energy per unit volume per second dissipated as internal energy and removed from the kinetic energy by viscous effects.

Questions for this week

1. Derive equations for the kinetic energy, internal energy, and total energy for an adiabatic flow in a gravitational field. How does your answer relate to Bernoulli's theorem?

2. A fluid has a constant kinematic viscosity ν. Derive an expression for the viscous dissipation rate in terms of ν and the velocity gradients.

Week 6 Computational exercise - Advection-diffusion

The problem we want to look at this week is the following:

> A fluid flows at constant velocity through a pipe that is heated at the far end. Calculate the temperature distribution in the fluid along the pipe.

We'll use this to look at two numerical techniques: implicit methods and operator splitting.

Equation and boundary conditions. Treating this as a 1D problem, the equation to solve is

$$\frac{\partial T}{\partial t} = -v\frac{\partial T}{\partial x} + D\frac{\partial^2 T}{\partial x^2},$$

with constant thermal diffusivity D and fluid velocity v. The coordinate x measures the distance along the pipe. Assume that the temperatures at each end of the pipe are fixed: $T = 0$ at $x = 0$ and $T = T_0$ at $x = L$. As in earlier exercises, we can solve this equation with finite differencing, where we follow the temperatures on a grid and calculate the spatial and time derivatives with finite differences. The notation T_i^n denotes the temperature at grid point i at time n.

Operator splitting. A straightforward way to advance the solution in time is to use operator splitting, in which we apply each operator (diffusion and advection) in turn. We start with the temperatures on the grid at time n. We update them using the diffusion term with time-step Δt to obtain intermediate values which we then use as input to an advection step for time Δt. This has the advantage that we can develop our numerical techniques for advection and diffusion separately rather than having to come up with a scheme to solve both at once.

Diffusion step. We looked at diffusion previously when we calculated the flow of viscous fluid. The finite-differenced version of the diffusion equation is

$$\frac{T_i^{n+1} - T_i^n}{\Delta t} = D\frac{T_{i+1}^n - 2T_i^n + T_{i-1}^n}{(\Delta x)^2}$$

giving

$$T_i^{n+1} = T_i^n + \alpha(T_{i+1}^n - 2T_i^n + T_{i-1}^n) = \alpha T_{i+1}^n + (1-2\alpha)T_i^n + \alpha T_{i-1}^n$$

where $\alpha = D\Delta t/(\Delta x)^2$. This is an *explicit* method where we write the new values of temperature explicitly in terms of the old values. Note that we can write this as a matrix equation

$$\mathbf{T}^{n+1} = \mathbf{A}\cdot\mathbf{T}^n$$

where $\mathbf{T}^n$ is the vector of temperature values at time n and the tri-diagonal matrix $\mathbf{A}$ is

$$\mathbf{A} = \begin{pmatrix} \ddots & & & & \\ & 1-2\alpha & \alpha & & \\ & \alpha & 1-2\alpha & \alpha & \\ & & \alpha & 1-2\alpha & \\ & & & & \ddots \end{pmatrix}$$

The endpoints of the matrices are where we implement the boundary conditions. For example, the first grid point $i = 1$ has an update equation $T_1^{n+1} = T_1^n$ since it doesn't change with time. Therefore we need the top left of the matrix to implement this:

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ \alpha & 1-2\alpha & \alpha & 0 & \cdots \\ 0 & \alpha & 1-2\alpha & \alpha & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Similarly for the bottom right. Then we can set the boundary values of temperature at the start of the calculation and they will not change over time which is the boundary condition we have for this problem[4].

The disadvantage of the explicit scheme is that it has a limit on the timestep – we need $\alpha \leq 1/2$ to avoid numerical instabilities. In an *implicit* scheme, we instead write

$$T_i^{n+1} = T_i^n + \alpha(T_{i+1}^{n+1} - 2T_i^{n+1} + T_{i-1}^{n+1})$$

so that the diffusion term is evaluated using the future values of T rather than their current values. This method is stable for large timesteps $\alpha > 1$. The solution for $\alpha \gg 1$ is the steady state $\partial^2 T/\partial x^2 = 0$, so although you lose accuracy on small scales by taking large timesteps, the solution evolves to steady-state which is usually the correct outcome.

In matrix form the implicit update is

$$\mathbf{B} \cdot \mathbf{T}^{n+1} = \mathbf{T}^n$$

with

$$\mathbf{B} = \begin{pmatrix} \ddots & & & & \\ & 1+2\alpha & -\alpha & & \\ & -\alpha & 1+2\alpha & -\alpha & \\ & & -\alpha & 1+2\alpha & \\ & & & & \ddots \end{pmatrix}.$$

Once again you need to make sure that the edges of the matrix are adjusted so that the boundary conditions are satisfied.

We can take the timestep using the inverse matrix:

$$\mathbf{T}^{n+1} = \mathbf{B}^{-1} \cdot \mathbf{T}^n.$$

In python, it's even easier because we can call `np.linalg.solve` which will do the matrix inversion for us. The routine `np.eye` is also really useful to construct tridiagonal matrices.

[4]Other boundary conditions, such as insulating boundaries $T_1 = T_2$ can be implemented in a similar way by adjusting the matrix entries so that when you multiply out the matrix equation you get the correct equations for the boundary points.

Advection step. Although it seems like the advection term should be straightforward, in fact advection is extremely hard to do accurately. The simplest differencing you could imagine

$$\frac{T_i^{n+1}-T_i^n}{\Delta t} = v\frac{T_{i+1}^n - T_{i-1}^n}{2\Delta x}$$

(where we use a second-order accurate derivative on the right hand side) turns out to be numerically unstable for any choice of timestep! One method that is stable is the Lax method:

$$T_i^{n+1} = \frac{1}{2}(T_{i+1}^n + T_{i-1}^n) - \frac{\beta}{2}(T_{i+1}^n - T_{i-1}^n)$$

where $\beta = v\Delta t/\Delta x$. This is stable as long as $\beta \leq 1$.

Questions

- Write a code to solve the advection-diffusion problem using operator splitting, the Lax method for the advection step and implement both explicit and implicit methods for the diffusion step so that you can compare them.

- Try running first with $v = 0$, ie. diffusion only. Compare the explicit and implicit methods. You should find that you can take much bigger timesteps with the implicit method without running into stability problems.

- Now turn on advection only ($D = 0$). What happens?

- With both advection and diffusion turned on, does the solution behave the way you expect? How does the temperature profile depend on the ratio v/D? Compare with the analytic solution for the steady-state.

- If you have time, try to implement a different boundary condition (e.g. insulating boundary or fixed heat flux rather than temperature) and see what happens.

Week 9: Compressible flows

Part I: Traffic flow simulation

(a) A simple way to model traffic flow is a "car following" model. For a 1D traffic flow, the acceleration of car i depends on its velocity difference and separation from the car in front of it $i+1$ according to

$$\frac{dv_i}{dt} = C\frac{v_{i+1} - v_i}{x_{i+1} - x_i} \tag{8}$$

where C is a constant (e.g. a classic paper is Gipps 1981 http://www.sciencedirect.com/science/article/pii/0191261581900370).

Implement this method in python for a line of cars, equally-spaced and moving with the same speed initially. Calculate what happens when the lead car suddenly changes its speed (up or down). You should find the cars evolve towards a new steady state. (For plotting purposes, it may be helpful to set the $x = 0$ point to be the location of the first car after every timestep, also a plot of v against x is helpful to look at.)

(b) As a function of the speed of the lead car v, find the steady-state separation Δx, and the flux of cars J (number of cars per second, or the velocity divided by the average spacing). Plot J and Δx against v.

(c) Discuss the shape of the $J(v)$ curve. Does it make sense? How does it depend on the constant C? By inspecting the plots can you come up with an analytic form for Δx and J as a function of v?

Part II: Compressible fluids

(a) Write down the continuity and momentum equations for a 1D isentropic gas. In the momentum equation, the only force you need to consider is the pressure gradient, which you can then rewrite in terms of the density gradient ($dP = c_s^2\, d\rho$ for adiabatic changes).

(b) For a steady flow, show that

$$\frac{dJ}{dv} = \rho\left(1 - \frac{v^2}{c_s^2}\right), \tag{9}$$

where $J = \rho v$ is the mass flux. Solve for and plot $J(v)$ and $\rho(v)$. For what v is the flux J maximum, and what is the value of the maximum flux?

(c) Discuss the connection between Parts I and II. Do you see how you could write down a continuum model for a traffic flow? Is the analogy with a compressible fluid exact? Is there an equivalent to a sound speed for the model in Part I?

Week 10: Internal Gravity Waves

The week, we'll discuss internal gravity waves, which arise in stratified fluids such as the atmosphere or the ocean. As you will see, they have some interesting properties and it will also give us a chance to tackle an eigenvalue problem numerically.

Eulerian and Lagrangian Perturbations

An important concept that we need to investigate internal gravity waves is the idea of **Lagrangian perturbations**. We already used the idea of Eulerian perturbations when we discussed sound waves. An Eulerian perturbation describes a perturbation to a fluid quantity at a fixed point in space, e.g. density

$$\delta\rho(\mathbf{r}) = \rho(\mathbf{r}) - \rho_0(\mathbf{r}),$$

where ρ_0 is the density in the unperturbed flow, and ρ the density in the perturbed flow.

Now extend this idea to perturbations of particular fluid elements. If the fluid element has a location $\mathbf{r_0}$ in the unperturbed flow and location $\mathbf{r}$ in the perturbed flow, the displacement of the fluid element between the two flows is

$$\xi = \mathbf{r} - \mathbf{r}_0.$$

The Lagrangian perturbation is

$$\Delta\rho(\mathbf{r}_0) = \rho(\mathbf{r}) - \rho_0(\mathbf{r}_0).$$

Using $\mathbf{r} = \mathbf{r}_0 + \xi$ and expanding assuming small displacements gives

$$\begin{aligned}\Delta\rho(\mathbf{r}_0) &= \rho(\mathbf{r}_0 + \xi) - \rho_0(\mathbf{r}_0) \\ &= \rho(\mathbf{r}_0) + \xi \cdot \nabla\rho_0 - \rho_0(\mathbf{r}_0) \\ &= \delta\rho(\mathbf{r}_0) + \xi \cdot \nabla\rho_0,\end{aligned}$$

which relates the Eulerian perturbation $\delta\rho$ to the Lagrangian perturbation $\Delta\rho$. To clean up the notation, we can drop the subscripts and write simply

$$\boxed{\Delta\rho = \delta\rho + \xi \cdot \nabla\rho}$$

As an example, consider an incompressible fluid, which must have $\Delta\rho = 0$. If there is a density gradient in the fluid, the Eulerian perturbation $\delta\rho$ can be non-zero because a fluid element displaced in the direction of the density gradient will have a different density than the background fluid once it gets to its new location.

Dispersion relation for internal gravity waves

Perturbation equations. Now consider a fluid in which the density varies with height $\rho(z)$. For example, this could arise in the ocean because of variations in temperature or in salt concentration. If the timescale associated with perturbations is short compared to the time for the density to adjust to its surroundings, e.g. by thermal diffusion or salt diffusion, then the fluid elements obey

$$\boxed{\Delta\rho = 0}$$

The perturbed continuity equation is

$$\frac{\partial\delta\rho}{\partial t} + \delta\mathbf{v}\cdot\nabla\rho = -\rho\nabla\cdot\delta\mathbf{v}.$$

In a Lagrangian picture, the velocity perturbation is related to the displacement of fluid elements by $\delta\mathbf{v} = \partial\xi/\partial t$, and so

$$\delta\rho + \xi\cdot\nabla\rho = -\rho\nabla\cdot\delta\xi.$$

The left hand side is $\Delta\rho = 0$, and so

$$\boxed{\nabla\cdot\xi = 0}$$

which is another statement that the perturbations are incompressible.

The perturbed momentum equation is

$$\boxed{\rho\frac{\partial\delta\mathbf{v}}{\partial t} = -\nabla\delta P - \delta\rho\,\mathbf{g}.}$$

The boxed equations describe the evolution of the perturbations.

WKB approximation. The perturbation equations have wave-like solutions of the form

$$\delta\rho \propto e^{-i\omega t}e^{ik_x x}f(z),$$

and similarly for δP and $\delta\mathbf{v}$. Here k_x is the horizontal wavevector (from the symmetry we are free to chose the alignment of the horizontal axes so that $k_y = 0$).

The vertical part of the solution $f(z)$ in general depends on the background density profile $\rho(z)$. However, if the lengthscale on which the background density is changing (i.e. the density scale height $dz/d\ln\rho$) is much larger than the vertical wavelength we are considering, then we can approximate the solution as a plane wave in the vertical direction also

$$f(z) \propto e^{ik_z z}.$$

This is known as the WKB approximation (you may have seen this for example in a quantum class when solving the Schrödinger equation).

Questions

Question 1. (a) Making the WKB approximation, use the perturbation equations to calculate the dispersion relation of the waves, i.e. the relation between ω, k_x and k_z. A useful quantity which you can use to simplify your expression is the Brunt-Väisälä frequency or buoyancy frequency, defined as

$$N^2 = -\frac{g}{\rho}\frac{d\rho}{dz}.$$

(b) Calculate the phase velocity $\mathbf{v}_p$ and the group velocity $\mathbf{v}_g$. Calculate $\mathbf{v}_p \cdot \mathbf{v}_g$ and interpret your answer.

Question 2. (a) Now consider a layer of fluid of thickness H with a constant value of N^2. The layer is in hydrostatic balance so $dP/dr = -\rho g$. Show that the vertical dependence of the perturbations $\delta\rho(z)$, $\delta P(z)$, $\xi_z(z)$, and $\xi_x(z)$ are given by the equations

$$\frac{d\xi_z}{dz} = \frac{k_x^2}{\rho\omega^2}\delta P \tag{10}$$

$$\frac{d\delta P}{dz} = -\rho(N^2 - \omega^2)\xi_z \tag{11}$$

with $\xi_x = ik_x\delta P/\rho\omega^2$ and $\delta\rho = \rho N^2\xi_z/g$.

(b) Integrate the two coupled ODEs equations (10) and (11) numerically to determine the spectrum of oscillation frequencies and corresponding eigenfunctions. You need to look for the values of ω for which the boundary conditions are satisfied. One way to do that is to start at the bottom of the layer, integrate across the layer assuming a value of ω, and then check at the top whether the solution satisfies the boundary condition. By varying ω, you should find solutions where boundary conditions are satisfied on both sides. This is known as a *shooting method*.

The boundary condition at the base of the layer assuming a rigid lower boundary is $\xi_z = 0$. At the top, assuming there is a free surface, we can write $\Delta P = 0$, or

$$\Delta P = \delta P + \xi_z\frac{dP}{dz} = \delta P - \rho g\xi_z = 0 \Rightarrow \delta P = \rho g\xi_z.$$

Calculate the oscillation frequencies ω and inspect the eigenfunctions $\delta P(z)$, and $\xi_z(z)$. Do they approach the WKB form for short wavelengths? How does the frequency depend on the number of nodes?

Week 11: Waves and Instabilities

We've seen two examples of waves in the course so far: sound waves in week 4 and internal gravity waves last week. The approach we took in each case was to consider small perturbations to the fluid and look for solutions $\propto e^{-i\omega t}$. The relation between the frequency ω and the wavevector $\mathbf{k}$ is the **dispersion relation** for the wave. For sound waves this took the simple form $\omega^2 = c_s^2 k^2$ with both phase and group velocities equal to the sound speed c_s; for internal gravity waves, the behavior is more complicated and we saw that the group and phase velocities are orthogonal in that case.

In some problems, the dispersion relation is such that ω^2 is negative, implying that ω is imaginary. If we write $\omega = i\sigma$, this means that the time-dependence is $\propto e^{\sigma t}$, and small perturbations grow exponentially – there is an **instability**.

Some examples of waves and instabilities in fluids

1. Internal gravity waves

Background state being perturbed: A stratified fluid in hydrostatic balance. It can be constant density like the ocean or compressible like the atmosphere. The important thing is that entropy should increase upwards so that perturbations are stable and create waves rather than convection (see convective instability, number 4).
Form of the fluid equations: These waves are not driven by compression (as are sound waves for example), so the perturbations can be taken as incompressible ($\nabla \cdot \xi = 0$). Otherwise, in the momentum equations we need the pressure gradient and gravity terms. In the gravity term, $\delta\rho$ can be found from the relation between density and pressure perturbations, $\Delta\rho/\rho = (1/\gamma)(\Delta P/P)$ (for adiabatic perturbations).
Dispersion relation:

$$\omega^2 = N^2 \frac{k_\perp^2}{k^2}$$

where $k_\perp$ is the component of the wavevector in the horizontal direction, $k^2 = k_\perp^2 + k_z^2$, and the Brunt-Väisälä frequency or buoyancy frequency N is given by

$$N^2 = g\left(\frac{1}{\gamma}\frac{d\ln P}{dz} - \frac{d\ln\rho}{dz}\right).$$

For incompressible fluid (ocean case) $\gamma \to \infty$ and N^2 only has a density term. Note that this dispersion relation has the interesting property that the phase velocity and group velocity are perpendicular to each other.
Basic physics The restoring force for the wave comes from the fact that when a fluid element is displaced upwards, it is denser than the surroundings and falls back due to gravity. The key assumption is that the fluid element maintains constant entropy (adiabatic perturbations) because the time for heat transport is long compared to the oscillation period. The condition for the fluid element to be denser than its surrounding at the new location and so want to fall back is $N^2 > 0$.

2. Capillary waves

Background state being perturbed: An interface between two fluids for which there is a surface energy and an associated *surface tension*. The classic case is capillary waves on the surface of water.
Form of the fluid equations: The simplest case is a layer of constant density fluid with air (vacuum) above it. With gravity waves last week, we had the surface boundary condition $\Delta P = 0$; here we instead must have

$$\Delta P = -T\frac{\partial^2 \xi_z}{\partial x^2}$$

so that when the surface is deformed, the pressure of the fluid element at the surface increases or decreases as needed to balance the surface tension.
Dispersion relation:

$$\omega^2 = k_\perp^3 \frac{T}{\rho}$$

(assuming short wavelengths compared to the height of the fluid layer). Note that these waves are dispersive: $v_p \propto k_\perp^{1/2}$.
Basic physics: Basically what we are doing is squeezing (in the crests, or "un-squeezing" in the troughs of the wave) columns of fluid in the layer. A column that is squeezed gets slightly taller, pushing the surface upwards, whereas the column next to it gets slightly shorter and less tall. The surface tension acts as a restoring force because it resists the deformation of the surface (because the lowest energy state is minimum surface area, or a flat surface).

3. Rossby waves

Background state being perturbed: A thin shell on a rotating sphere supports Rossby waves.
Form of the fluid equations: Often, a local region is considered in a plane-parallel approximation using the *shallow water* approximation in which the layer is assumed to be vertically thin. Rossby waves come from the latitudinal variation of the vertical component of rotation. The background has a vertical component of Ω which is written as $2\Omega \sin\theta = f + \beta y$ (because of the β term, this is called the β-plane). The shallow water momentum equations are then:

$$\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}\right) u - (f + \beta y)v = -g\frac{\partial h}{\partial x}$$
$$\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}\right) v + (f + \beta y)u = -g\frac{\partial h}{\partial y}.$$

A term proportional to β then appears as a source term in the vorticity equation.
Dispersion relation:

$$\omega = -\frac{\beta k_x}{k_x^2 + k_y^2}.$$

An interesting property is that the phase velocity of the wave is always negative (backwards compared to the rotation) (the group velocity can be in either direction depending on the relative size of k_x and k_y).

Basic physics: On a rotating sphere absolute vorticity $2\Omega + \omega$ is the conserved quantity, not just the relative vorticity $\omega = \nabla \times \mathbf{u}$. When a fluid element is perturbed northwards for example, towards the pole, it has more vertical component of Ω and so its relative vorticity must decrease. This changes the velocity field in such a way that neighbouring fluid elements are displaced. They change the flow in such a way that the original fluid element is advected back towards its original location, giving a restoring "force" for the wave.

4. Convective instability

Background state being perturbed: A stratified fluid. This is a very close cousin of number 1. internal gravity waves, except now the fluid has entropy *decreasing* with height. This situation naturally arises when you heat a fluid from below, so you get hot high entropy material underneath cold low entropy material.

Form of the fluid equations: Pressure and gravity are all that is needed in the momentum equations to get the basic instability; viscosity may be important, acting to stabilize the convection. There are two versions of this for constant density (e.g. water) or compressible (e.g. air) fluids. For compressible fluid in the atmosphere or a star for example, we assume the perturbations are adiabatic ($\Delta P/P = \gamma \Delta\rho/\rho$) and heat transport and viscosity can be neglected. For water, viscosity and thermal diffusion may be important. Although water is basically incompressible, convection relies on the slight decrease in density when the fluid is heated. Usually the density change is written in terms of the coefficient of thermal expansion $\delta\rho/\rho = 1 - \alpha\delta T$ where $\alpha \sim 10^{-4}$ for water. In all cases, the density perturbation need only be put into the gravity term of the momentum equation; for continuity it is enough to assume incompressible (no sound waves).

Dispersion relation: For adiabatic perturbations in an atmosphere for example, we get the same as 1., but now $N^2 < 0$ so that ω is imaginary. For situations in which viscosity and thermal conductivity must be considered, the extra derivatives in the fluid equations give a more complicated dispersion relation (something like 6th order), but the instability criterion can be written in a simple way:

$$\mathrm{Ra} = \frac{\alpha g \Delta T H^3}{\nu\kappa} > \mathrm{Ra_c},$$

where Ra is the *Rayleigh number*, ΔT is the temperature difference across the layer of height H and κ is the thermal diffusivity.

Basic physics: When entropy decreases upwards, a fluid element that is adiabatically displaced will arrive at its new location to find itself lighter than its surroundings. The buoyancy force will then keep accelerating it upwards: the background profile is unstable to small perturbations. For water, the hotter water in a layer heated from below is less dense (by a tiny amount) and wants to rise. Thermal diffusion and viscosity can kill the instability by either equalizing the temperature and therefore density, or by slowing down the rising fluid elements.

5. Salt-fingering or doubly-diffusive convection

Background state being perturbed: Hot salty water on top of cold fresh (non-salty) water. This can happen for example when a river flows into the ocean. Note that there are two things going on here: cold water on top of hot water (a stable situation) and salty (heavier) water on top of fresh (lighter) water (an unstable situation). The background for this instability is one in which the stabilizing temperature gradient overcomes the destabilizing composition gradient, so that the background state is stable to convection in the sense of instability number 4.

Form of the fluid equations: As implied by the name "doubly-diffusive" it is crucial to include the diffusion of heat in the energy equation and the diffusion of chemical (here salt). For convection, we would usually assume adiabatic perturbations: here the key thing is that heat can diffuse out of or into a perturbed fluid element.

Dispersion relation: The dispersion relation is more complex that the case of convection; the growth rate depends on the diffusivities of both heat and salt. E.g. see Huppert and Turner 1981 J. Fluid Mech. 106, 299 for more details.

Basic physics: As mentioned before, the background state is stable to convection, but once the diffusion of heat is included instability can develop. A fluid element displaced downwards from the salty layer into the fresh layer can come into thermal equilibrium faster than chemical equilibrium. Heat diffuses until the temperature of the perturbed fluid element matches the background; at that point we have a salty fluid element at the same temperature as the background fresh water – it is therefore more dense (contains salt) and will sink further. The instability develops in the form of thin "fingers" that slowly penetrate the underlying layer, hence the name.

6. Kelvin Helmholtz instability

Background state being perturbed: Two layers of fluid moving with different velocities. Perturbations of the interface are unstable.

Form of the fluid equations: Incompressible; pressure gradient in the momentum equation. One aspect of the calculation is that because there is a non-zero velocity U in the background, the nonlinear term gives a contribution $Ud\delta u/dx = ikU\delta u$. Combined with the $\partial/\partial t$ term, you'll get acceleration terms that look like $i(\omega - kU)\delta u$ instead of just $i\omega\delta u$.

Dispersion relation:

$$\omega^2 + 2k_x\frac{\rho_1 U_1 + \rho_2 U_2}{\rho_1 + \rho_2}\omega + k_x^2\frac{\rho_1 U_1^2 + \rho_2 U_2^2}{\rho_1 + \rho_2} = 0.$$

The simplest case is when $\rho_1 = \rho_2$, then you can show that there is always instability, with a growth rate $\propto k_x(U_2 - U_1)$ so that short wavelengths grow fastest.

Basic physics: A good way to think about why this situation is unstable is in terms of energy. If you calculate the kinetic energy for the two layers and then mix the two layers conserving momentum, you should find that the mixed state has a smaller kinetic energy. So the instability releases energy.

Extensions: Surface tension will stabilize the shortest wavelengths, and if gravity is included and $\rho_1 \neq \rho_2$, long wavelengths are stabilized. For a background with a shear dU/dz rather than a sharp jump in velocity, the growth rate turns out to be $\sim (dU/dz)$, and for a shear in a stratified background, a famous criterion for shear instability is that $\mathrm{Ri} < 1/4$ where the *Richardson number* is $\mathrm{Ri} = N^2/(dU/dz)^2$. This just says that large stratification (as measured by N^2) stops the instability because the kinetic energy released is overcome by the gravitational energy used to mix the two fluids.

7. Stability of Couette flow

Background state being perturbed: Couette flow is the steady flow between two rotating concentric cylinders, with $u_\phi = r\Omega(r)$.
Form of the fluid equations: Inviscid flow (no viscosity) in cylindrical geometry. This is an axisymmetric instability, so consider perturbations with $\partial/\partial\phi = 0$.
Dispersion relation: The *Rayleigh criterion* is a famous condition for instability of this flow:

$$\frac{d}{dr}(r^2\Omega)^2 < 0,$$

so that the flow is unstable if angular momentum decreases outwards.
Basic physics: If angular momentum in the background flow decreases outwards, a fluid element that is displaced outwards conserving angular momentum will be rotating faster than its surroundings. The centrifugal force on it will therefore be larger that the pressure gradient in the background, which is balancing the background centrifugal force. So the fluid element has a net outwards force and will keep going.

Summary of topics covered

The fluid equations

Basic ideas. A fluid as a continuum. Mean free path much smaller than macroscopic lengthscales.

Lagrangian and Eulerian points of view. The advective derivative

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla.$$

Definition of a streamline.

Continuity equation (mass conservation).

$$\frac{D\rho}{Dt} = -\rho \nabla \cdot \mathbf{u} \qquad \frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho \mathbf{u})$$

Momentum equation. Body forces and surface stresses.

$$\rho \frac{D\mathbf{u}}{Dt} = \mathbf{f} + \nabla \cdot \mathbf{T}.$$

The stress tensor T_{ij}. Stress tensor for pressure $T_{ij} = -P\delta_{ij}$. With gravity and pressure

$$\rho \frac{D\mathbf{u}}{Dt} = \rho \mathbf{g} - \nabla P.$$

Hydrostatic balance. Isothermal atmosphere with scale height $H = k_B T/\mu m_p g$. Ocean $P = P_0 + \rho g z$.

Bernoulli's principle. The Bernoulli constant

$$H = \frac{P}{\rho} + \frac{1}{2}u^2 + \chi$$

is constant along streamlines in a steady flow, where $\mathbf{g} = -\nabla \chi$. For irrotational flow, H is the same constant on all streamlines. Examples: water flowing out of a hole in a vessel; Venturi tube; lift force.

Vorticity and circulation

Vorticity. $\omega = \nabla \times \mathbf{v}$. Measures local rotation of the fluid element; value of vorticity is two times the local angular velocity. For a rigidly rotating fluid, $\omega = 2\Omega$. The circulation $\Gamma = \oint \mathbf{u} \cdot \mathbf{dl}$. Kelvin's theorem $D\Gamma/Dt = 0$ for a material loop.

The vorticity equation.

$$\frac{D\omega}{Dt} = (\omega \cdot \nabla)\, \mathbf{u}.$$

The same equation is obeyed by the separation **dl** between two fluid elements, which implies that vortex lines are "frozen" into the fluid. The term on the right hand side represents "vortex tilting" and "vortex stretching".

Line vortex flow $\mathbf{u} = (k/r)\hat{\theta}$ (which is irrotational except at the origin). The Rankine vortex as a simple model for a vortex.

Generation of vorticity by a force with a non-zero curl. Viscosity causes diffusion of vorticity. The baroclinic vector $(\nabla\rho \times \nabla P)/\rho^2$ and why baroclinicity changes vorticity.

Viscosity and viscous flow; energy equation

Viscosity. The microscopic origin of viscosity. Kinematic viscosity ν and dynamical viscosity $\mu = \rho\nu$. Viscosity of water 10^{-2} in cgs units. A Newtonian fluid has stress proportional to velocity gradient.

Viscous timescale $t_{\rm visc} \sim L^2/\nu$ where L is the lengthscale on which the velocity changes

Stress tensor with viscous stress $T_{ij} = -P\delta_{ij} + \sigma_{ij}$. For an incompressible fluid, $\sigma_{ij} = 2\mu e_{ij}$ where the deformation tensor is

$$e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right).$$

For constant viscosity, the momentum equation is

$$\rho\frac{D\mathbf{u}}{Dt} = -\nabla P + \mu\nabla^2\mathbf{u}.$$

Reynolds number ${\rm Re} = UL/\nu$. Dimensionless numbers and dynamical similarity.

No slip boundary condition for viscous flow. Free slip boundary condition for irrotational flow. Boundary layer width is $\sim L/{\rm Re}^{1/2}$.

Similarity solutions when a problem has no intrinsic lengthscale. For example, viscous diffusion of momentum/vorticity from an impulsively moved boundary has a solution in terms of $\eta = x/\sqrt{\nu t}$.

Energy equation. Kinetic energy

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho u^2\right) + \nabla\cdot\left(\mathbf{u}\cdot\left(\frac{1}{2}\rho u^2 + P\right)\right) = P\nabla\cdot\mathbf{u} - \Phi_V$$

Internal energy

$$\frac{\partial}{\partial t}(\rho E) + \nabla\cdot(\mathbf{u}\rho E) = -P\nabla\cdot\mathbf{u} + \Phi_V$$

Viscous dissipation $\Phi_V = 2\mu(e_{ij})^2$. In a steady flow, the total viscous dissipation in the flow matches the work done to keep the flow moving.

Cylindrical flow, $u_\theta(r) = r\Omega(r)$. Steady state solution $\Omega = A + B/r^2$. Viscous stress $\tau = \mu r d\Omega/dr$. Viscous dissipation $\Phi_V = \mu r^2 (d\Omega/dr)^2$.

Sound waves and steepening; other types of waves and instabilities

Linear sound waves. The general idea of linear perturbation theory. Eulerian and Lagrangian perturbations and the relation between them. The dispersion relation $\omega(k)$ and how to find the phase velocity $v_p = \omega/k$ and group velocity $v_g = \partial\omega/\partial k$ (including in >1D). Waves in a compressible constant density fluid. The wave equation

$$\frac{\partial^2 \delta\mathbf{u}}{\partial t^2} = c_s^2 \nabla^2 \delta\mathbf{u}.$$

Sound speed $c_s^2 = \partial P/\partial\rho$. Adiabatic sound speed $c_s^2 = \gamma P/\rho$. Dispersion relation $\omega^2 = c_s^2 k^2$. The basic physics driving the wave.

Steepening and nonlinear waves. The nonlinear advection term causes steepening. This can be balanced by in some way to create a non-linear structure that propagates without change of shape, a nonlinear wave. Shocks are an example, described by Burger's equation which includes steepening from the nonlinear term and diffusion

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = -\nu\frac{\partial^2 u}{\partial x^2}.$$

Shock thickness $\delta^2/\nu \sim \delta/v$.

Compressible flow. The different behavior of incompressible vs. compressible flows (e.g. rivers vs. traffic),

$$\frac{dJ}{dv} = \rho\left(1 - \frac{v^2}{c_s^2}\right)$$

(you should be able to derive this). When a flow can be considered incompressible or not (ie. very subsonic flow is incompressible).

Internal gravity waves. The dispersion relation $\omega^2 = N^2 k_z^2/k^2$ and its implications, i.e. frequency decreases with decreasing wavelength, phase and group velocities are perpendicular.

WKB approximation. The WKB approximation as the approximate solution for short-wavelength waves.

Instabilities. The idea of a complex ω leading to instability. Different examples of waves and instabilities, their dispersion relations, the physics driving the wave/instability.

Turbulence and nonlinear dynamics

Turbulence Characteristics of a turbulent flow: irregular, highly diffusive, large Re number, three-dimensional in nature, dissipative. Turbulence is a property of the flow not the fluid.

The idea of an **energy cascade.** The scaling arguments that lead to the Kolmogorov result $E(k) \propto k^{-5/3}$. The stirring or outer scale L, inertial range with $E(k) \propto k^{-5/3}$ and inner scale $l_d = L/\mathrm{Re}^{3/4}$. Typical Eddy velocity $v \sim (\epsilon l)^{1/3}$ on scale l.

Turbulent transport. The Reynolds decomposition into a mean flow and fluctuating flow. Transport of momentum or heat comes from correlated fluctuations, e.g. Reynolds stress is $T_{ij} = -\rho\overline{u_i' u_j'}$, heat flux is $F_j = \rho c_P \overline{T' u_j'}$.

Transition to turbulence. The Lorenz equations for convection as a simple model. The idea that chaotic behavior and unpredictability can arise in a deterministic system.

Numerical techniques

Finite differencing. First and second order derivatives.

$$f_j' = \frac{f_{j+1} - f_j}{\Delta x} = \frac{f_j - f_{j-1}}{\Delta x} + \mathcal{O}(\Delta x) \qquad f_j' = \frac{f_{j+1} - f_{j-1}}{2\Delta x} + \mathcal{O}\left(\Delta x^2\right)$$

$$f_j'' = \frac{f_{j+1} - 2f_j + f_{j-1}}{2\Delta x} + \mathcal{O}\left(\Delta x^2\right)$$

The difference between an explicit and implicit scheme.

Boundary conditions. How to use boundary conditions to obtain quantities just off the grid (e.g. grid cell $N + 1$) which you need to update the boundary grid point.

Advection. The Lax method

$$f_j^{n+1} = \frac{1}{2}\left(f_{j+1}^n + f_{j-1}^n\right) - \frac{v\Delta t}{2\Delta x}\left(f_{j+1}^n - f_{j-1}^n\right).$$

The Courant condition $\Delta t \leq \Delta x/v$. The idea of numerical dissipation and numerical viscosity ($\sim \Delta x^2/\Delta t$). Upwind differencing.

Diffusion. Explicit schemes and limitations on the timestep $\Delta t \leq \Delta x^2/2D$,

$$\frac{f_j^{n+1} - f_j^n}{\Delta t} = \frac{D}{(\Delta x)^2}\left(f_{j+1}^n - 2f_j^n + f_{j-1}^n\right).$$

Why this is slow for a large number of grid cells. Implicit methods: how to solve them and perform a timestep using matrix inversion $\mathbf{A} \cdot \mathbf{f}^{n+1} = \mathbf{f}^n \rightarrow \mathbf{f}^{n+1} = \mathbf{A}^{-1}\mathbf{f}^n$

Operator splitting. How to combine, for example, advection and diffusion, e.g. to solve the advection-diffusion equation

$$\frac{\partial f}{\partial t} = -v\frac{\partial f}{\partial x} + D\frac{\partial^2 f}{\partial x^2}.$$

Finite volume methods. Flux conserving formulation

$$\frac{f_j^{n+1} - f_j^n}{\Delta t} = \frac{J_{j+1/2}^{n+1/2} - J_{j-1/2}^{n+1/2}}{\Delta x}$$

and why this conserves the quantity f. Different ways to choose the fluxes. Donor cell advection, e.g.

$$J_{j\pm1/2} = v_{j\pm1/2} f_j^n \quad (v_{j\pm1/2} > 0) \qquad J_{j\pm1/2} = v_{j\pm1/2} f_{j+1}^n \quad (v_{j\pm1/2} < 0).$$

Integration of ODEs. The example of integrating hydrostatic balance for some equation of state $P(\rho)$. How to solve a two-point boundary value problem with a shooting method, e.g. finding the eigenvalues and eigenfunctions for internal gravity waves.

Practice problems

Week 1 Problems

1. **Hydrostatic balance: Isothermal atmosphere and ocean**

(a) Write down the momentum equation for a plane-parallel atmosphere with constant gravity g where the gas is at rest $v = 0$ and there is no time-dependence $\partial/\partial t = 0$. Assume an ideal, isothermal gas (same temperature T everywhere). Show that the density of the gas as a function of height z from the surface at $z = 0$ is

$$\rho = \rho_0 e^{-Mgz/k_B T}$$

where M is the mass of an air molecule. Give a physical interpretation of this result.

(b) In the ocean the density of water ρ is very close to a constant. What is the pressure as a function of depth in the ocean in this case?

(c) Plug in some numbers: I've included a figure on the next page showing the pressure against height for the Earth's atmosphere (this from the book "An Introduction to Earth's Atmosphere" by Liou). Does your formula roughly match the figure? How far up in the atmosphere do you have to go for pressure to drop by $1/e$? How far down in the ocean do you have to go for pressure to increase by the same factor?

2. **Streamlines etc.**

(Acheson question 1.8) Consider the unsteady flow $u = u_0$, $v = kt$, $w = 0$, where u, v and w are the Cartesian components of velocity, and u_0 and k are positive constants. Show that (a) the streamlines are straight lines, and sketch them at two different times, and (b) that a given fluid element follows a parabolic path as time proceeds.

3. **The coffee pot**

Watch the video I made when I went to get coffee in the Trottier building. You can find it at this link: http://www.physics.mcgill.ca/~cumming/teaching/432/coffee.mov. Make some numerical estimates and see whether you can understand what is happening.

4. **A rotating bucket of water**

(This one is Acheson question 1.2.) An ideal fluid is rotating under gravity g with constant angular velocity Ω so that relative to fixed Cartesian axes $\mathbf{u} = (-\Omega y, \Omega x, 0)$. We wish to find the surfaces of constant pressure, and hence the surface of a uniformly rotating bucket of water (which will be at atmospheric pressure).

(a) Consider the following argument: "By Bernoulli" $p/\rho + (1/2)u^2 + gz$ is constant, so the constant pressure surfaces are

$$z = \text{constant} - \frac{\Omega^2}{2g}(x^2 + y^2).$$

But this means that the surface of a rotating bucket of water is at its highest in the middle, whereas we know from experience that it is lowest in the middle. What is wrong with the argument?

(b) Write down the momentum equations in component form, integrate them directly to find the pressure p, and hence obtain the correct shape of the free surface.

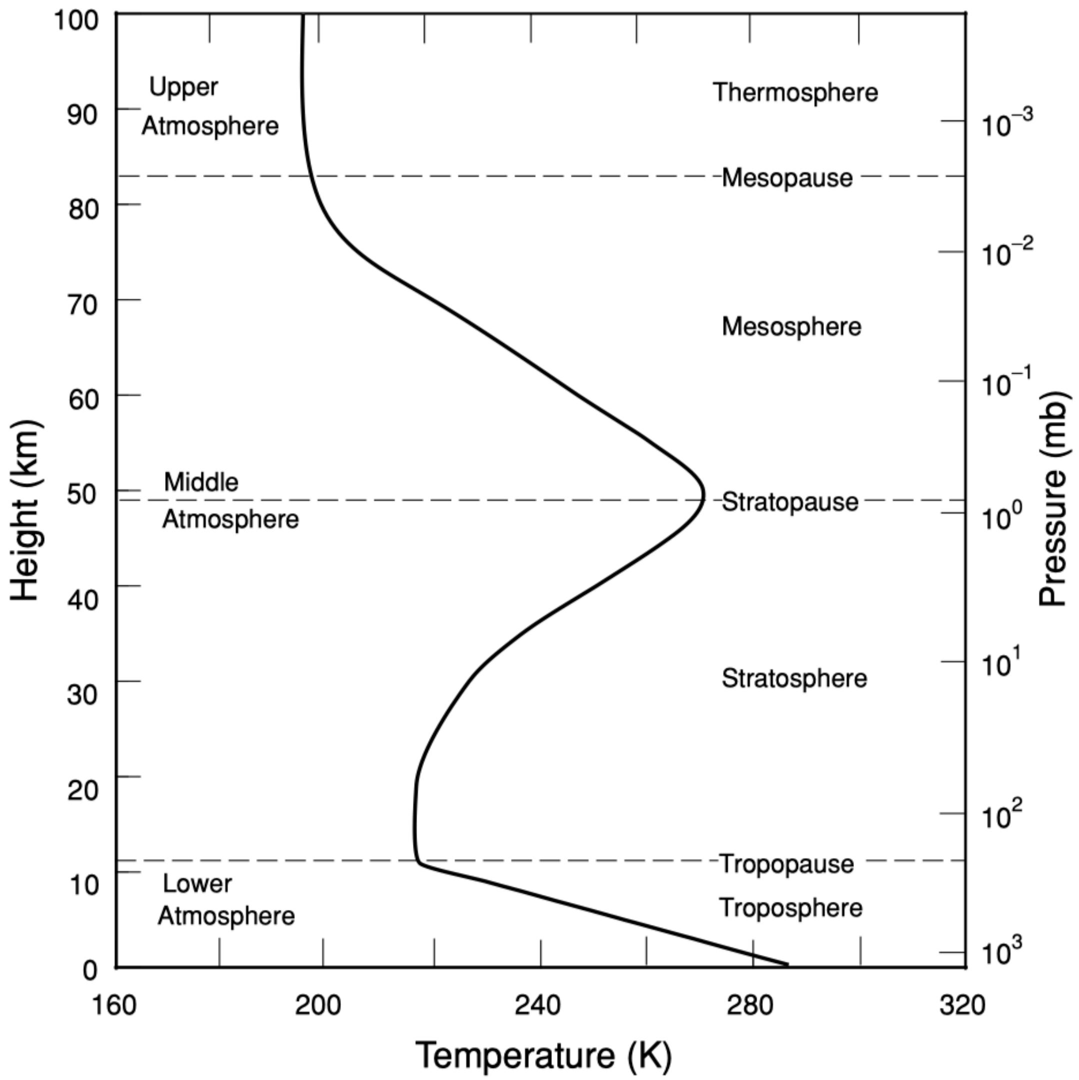


Figure 3.1 Vertical temperature profile after the U.S. Standard Atmosphere and definitions of atmospheric nomenclature.

Additional questions:

5. **Polytropic atmospheres**

(a) A useful equation of state in many problems is a polytropic equation of state which has $P \propto \rho^\gamma$. What values of the constant γ should be chosen to model an incompressible fluid or an isothermal fluid?

(b) Calculate the density and pressure as a function of height for a polytropic atmosphere with arbtrary γ. Assuming that the pressure vanishes $P = 0$ at the top of the atmosphere, derive an expression for the thickness Δz of the atmosphere.

(c) Are your answers to part (b) what you expect if you choose γ to be the right values for an isothermal gas or for an incompressible fluid?

6. **Bernoulli for compressible flows**

In the notes it is assumed that the density ρ is constant when deriving Bernoulli's principle. Generalize that to compressible flows as follows:

(a) For a flow which is adiabatic, $P \propto \rho^\gamma$, where the constant γ is the adiabatic index. Derive the form of Bernoulli's constant in that case.

(b) An adiabatic flow is one in which the entropy is constant for any given fluid element. Use the first law of thermodynamics $TdS = dU + PdV$ and the definition of enthalpy (per unit mass $h = u + P/\rho$, where u is the internal energy per unit mass) to show that the Bernoulli constant for an adiabatic flow can be written $H = h + \frac{1}{2}u^2 + \chi$, where h is the enthalpy per unit mass.

(c) For a *barotropic* flow, P is a function of ρ only. Derive the form of Bernoulli's principle for this case.

Week 2 Problems

The vorticity equation for a compressible fluid

(Acheson question 1.5) Use the momentum and continuity equations to show that

$$\frac{D}{Dt}\left(\frac{\omega}{\rho}\right) = \left(\frac{\omega}{\rho}\cdot\nabla\right)\mathbf{u} - \frac{1}{\rho}\nabla\left(\frac{1}{\rho}\right)\times\nabla p.$$

(In the momentum equation, assume that pressure gradient and gravity are the only forces.)

How is this result different from the vorticity equation for an incompressible fluid? Interpret this difference physically.

Vortex dynamics

(a) A pair of line vortices with the same circulation Γ are placed next to each other. What happens?

(b) What happens if the two vortices have equal and opposite circulations?

(c) Consider a set of line vortices spaced out equally along a line. What happens?

(d) Now imagine one of the vortices in the line is displaced upwards slightly. What happens?

Vorticity on a rotating sphere

On the surface of a rotating sphere, the momentum equation can be written

$$\frac{D\mathbf{u}}{Dt} + 2\boldsymbol{\Omega}\times\mathbf{u} + \boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\mathbf{r}) = -\frac{\nabla P}{\rho} + \mathbf{g}.$$

Show that the fluid obeys a vorticity equation with vorticity $\omega = \nabla\times\mathbf{u}$ replaced by the *absolute vorticity* $\omega_{\mathbf{a}} = \omega + 2\Omega$.

A vortex near the equator on a rotating sphere moves upward towards the pole. What happens to it?

Tornado explosions

(This is question 2 in chapter 5 of Kundu's book). A tornado can be idealized as a Rankine vortex with a core of diameter 30m. The gauge pressure at a radius of 15m is $-2000\mathrm{N/m^2}$ (atmospheric pressure would correspond to zero pressure on this scale).

(a) Show that the circulation around any circuit surrounding the core is 5485 $\mathrm{m^2/s}$. [Hint: apply Bernoulli between infinity and the edge of the core]

(b) Such a tornado is moving at a linear speed of 25 m/s relative to the ground. Find the time required for the gauge pressure to drop from -500 to $-2000\mathrm{N/m^2}$. Neglect compressibility effects and assume an air temperature of 25°C.

[Note that the tornado causes a sudden decrease of the local atmospheric pressure. The damage to structures is often caused by the resulting excess pressure on the interiors of walls, which can cause a house to explode].

Image vortices

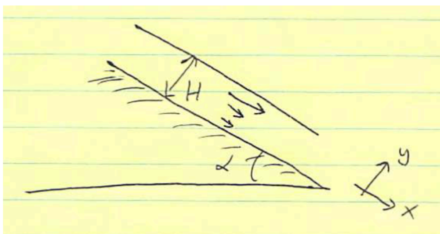
(Another one from Kundu's book, question 4 in chapter 5, I've rewritten it to try to make it easier to follow). Consider fluid in a 90 degree angle in the x-y plane. A vortex is initially at location (x, y). Show that the vortex will move and follow a trajectory given by

$$\frac{1}{x^2} + \frac{1}{y^2} = \text{constant.}$$

Week 3 Problems

Flow down an inclined plane

A layer of viscous fluid of thickness H flows down an incline that is at angle α to the horizontal. Assume that the flow is steady, gravity acts vertically downwards, and that the flow variables are functions of y only, where the x and y axes are defined in the diagram below:



(a) Does it make sense to assume that there is no dependence on x?

(b) Use the continuity equation to argue that the flow velocity is only in the x-direction $\mathbf{u} = u(y)\hat{\mathbf{x}}$.

(c) Use the y-component of the momentum equation to show that the pressure decreases linearly with y.

(d) Use the x-component of the momentum equation to calculate the velocity profile in the layer $u(y)$.

(e) What is the velocity at the top of the layer of fluid? Interpret your expression physically.

(f) What is the viscous stress at the base of the layer? Interpret your expression physically.

Bonus: (h) Now consider two layers of fluid on top of each other with thicknesses h_1 and h_2 and viscosities μ_1 and μ_2. You can assume they have the same density ρ. Redo the problem, and in particular show that the velocity of the lower fluid is dependent on the depth h_2 but not the viscosity of the upper fluid. Why is this?

Flow near an impulsively moved plane

A semi-infinite layer of viscous fluid lies stationary on a flat plate. At time $t = 0$ the plate starts moving to the right with speed U_0.

(a) Show that the subsequent motion of the fluid is governed by the diffusion equation,

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2}.$$

(b) In this problem, you will find a so-called *similarity solution* in which the velocity is a function of the combination

$$\eta = \frac{x}{\sqrt{\nu t}},$$

i.e. $u = f(\eta)$ where f is a function that is to be determined. Can you explain why this might be a reasonable guess for the solution?

(c) Show that $f(\eta)$ satisfies

$$f'' + \frac{1}{2} f' \eta = 0$$

(d) Solve the equation to arrive at an expression for $u(x, t)$. Sketch the velocity profile at different times to show how it evolves.

(e) What does the vorticity look like as a function of time?

(f) Now consider a layer with finite thickness H. What is $\omega(x, t)$ now? Hint: you can use the method of images from electrostatics to immediately write down the solution using your previous answer.

(g) Use your answer from (f) to calculate the velocity profile $u(y)$, and plot the velocity profile as a function of time, and the velocity at the surface of the fluid as a function of time.

Bonus: (h) What would the solution be if the plate started oscillating

$$U_0 \propto \cos(\omega t)$$

at time $t = 0$ instead of moving at constant velocity.

Week 4 Problems

1. **Damped sound waves**

(a) Consider adiabatic perturbations to a uniform density gas initially at rest. Show that the velocity perturbations $\delta\mathbf{u}$ obey the same wave equation as the density perturbations $\delta\rho$.

(b) Why is it a good assumption to assume that the perturbations are adiabatic? Another assumption we could make would be isothermal perturbations $\delta T = 0$. When would this be appropriate? Show that this gives

$$\frac{\delta P}{P} = \frac{\delta\rho}{\rho},$$

and derive an expression for the sound speed. Which is a more appropriate choice for sound waves in air, adiabatic or isothermal perturbations?

(c) A possible source of damping of sound waves in air is viscosity. Show that when the viscous term is included in the momentum equation, the dispersion relation for adiabatic sound waves in a uniform medium is

$$\omega^2 - i\nu k^2\omega - c_s^2 k^2 = 0,$$

where $c_s = \gamma P/\rho$ is the adiabatic sound speed and ν is the kinematic viscosity. Use this to calculate the distance over which a 400 Hz sound wave will damp in air.

2. **Steepening and shocks**
We saw in the notes that a velocity field evolving according to

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0 \tag{12}$$

steepens over time in regions where $du/dx < 0$, eventually forming a discontinuity, or shock. In reality, viscosity will cause diffusion that acts against the steepening from the non-linear term, giving the shock a width.

(a) To model the combination of steepening and viscosity, we can add a viscous term to equation (1),

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \nu\frac{\partial^2 u}{\partial x^2},$$

and look for a solution that has $u = U_0$ at $x = -\infty$, and $u = 0$ for $x = \infty$. Show that the solution $u = f(x - Vt)$ where

$$f(\xi) = U_0\left(1 + \exp(\xi/\Delta)\right)^{-1}$$

satisfies the equation and these boundary conditions, and derive expressions for the shock speed V and width Δ in terms of U_0 and ν.

(b) A sonic boom is a shock created by a supersonic aircraft. Estimate the width of such a shock assuming it is limited by the viscosity of air. Comment on whether it is appropriate to treat air as a fluid on these lengthscales.

Week 5 Problems

1. **Scalings in turbulence**

(a) In the movie on turbulence, they show two jets with Reynolds numbers different by a factor of 50 (you can also see this in Figure 7 in the film notes). They look the same on the large scale, but one has finer structures than the other on small scales. Explain this observation in terms of the energy spectrum of the turbulence (sketch it for each case on the same plot). How much finer is the small scale structure in the larger Re number jet?

(b) Also discussed in the film was the use of models in simulating exploding ships or erupting volcanoes for movies. Consider a real volcano eruption and a model volcano eruption. Assuming the microscopic properties of the fluid is the same in each, sketch the energy spectrum of the turbulence on the same plot. Estimate the outer and inner scales and the number of order of magnitudes in scale covered by the inertial range.

(c) A fluid is stirred, generating a turbulent cascade. At time $t = 0$ the stirring is stopped, and the turbulence begins to decay. Sketch the energy spectrum $E(k)$ against k that you expect at different times. In particular, how does the smallest lengthscale scale with time?

2. **Turbulence in a cloud**

Estimate the energy dissipation in a cumulus cloud, both per unit mass and for the entire cloud. You'll need to estimate or look up typical lengthscales and velocities for the fluid motion. Also estimate the inner scale of the turbulence.

Week 6 Problems

1. **Viscous flow down an inclined plane**

For the problem of a viscous fluid flowing down an inclined plane that we looked at in week 3, calculate the viscous energy dissipation rate as a function of position. What is the total energy dissipation rate? Interpret it physically.

2. **Rotating cylinder**

Consider a solid cylinder of radius R steadily rotating at angular speed Ω in an infinite viscous fluid. Show that the work that must be done to keep the cylinder rotating is equal to the energy dissipated in the fluid by viscous dissipation.

Some useful formulae in cylindrical coordinates are included on the next page (taken from Acheson's book).

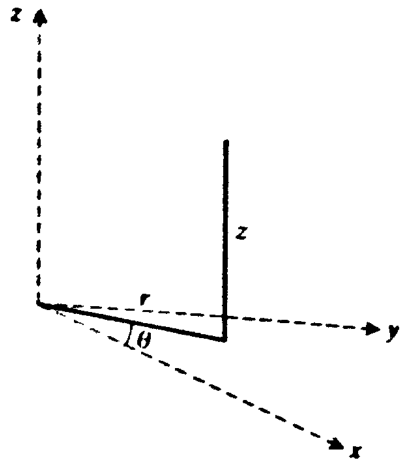


Fig. A.2 Cylindrical polar coordinates.

Also,

$$\nabla\phi = \frac{\partial\phi}{\partial r}\boldsymbol{e}_r + \frac{1}{r}\frac{\partial\phi}{\partial\theta}\boldsymbol{e}_\theta + \frac{\partial\phi}{\partial z}\boldsymbol{e}_z, \tag{A.30}$$

$$\nabla\cdot\boldsymbol{F} = \frac{1}{r}\frac{\partial}{\partial r}(rF_r) + \frac{1}{r}\frac{\partial F_\theta}{\partial\theta} + \frac{\partial F_z}{\partial z}, \tag{A.31}$$

$$\nabla\wedge\boldsymbol{F} = \frac{1}{r}\begin{vmatrix} \boldsymbol{e}_r & r\boldsymbol{e}_\theta & \boldsymbol{e}_z \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial\theta} & \dfrac{\partial}{\partial z} \\ F_r & rF_\theta & F_z \end{vmatrix}, \tag{A.32}$$

$$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} + \frac{\partial^2}{\partial z^2}, \tag{A.33}$$

$$\boldsymbol{u}\cdot\nabla = u_r\frac{\partial}{\partial r} + \frac{u_\theta}{r}\frac{\partial}{\partial\theta} + u_z\frac{\partial}{\partial z}. \tag{A.34}$$

The Navier–Stokes equations in cylindrical polar coordinates are:

$$\frac{\partial u_r}{\partial t} + (\boldsymbol{u} \cdot \nabla)u_r - \frac{u_\theta^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu\left(\nabla^2 u_r - \frac{u_r}{r^2} - \frac{2}{r^2}\frac{\partial u_\theta}{\partial \theta}\right),$$

$$\frac{\partial u_\theta}{\partial t} + (\boldsymbol{u} \cdot \nabla)u_\theta + \frac{u_r u_\theta}{r} = -\frac{1}{\rho r}\frac{\partial p}{\partial \theta} + \nu\left(\nabla^2 u_\theta + \frac{2}{r^2}\frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r^2}\right),$$

$$\frac{\partial u_z}{\partial t} + (\boldsymbol{u} \cdot \nabla)u_z = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\,\nabla^2 u_z,$$

$$\frac{1}{r}\frac{\partial}{\partial r}(r u_r) + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z} = 0. \tag{A.35}$$

The components of the rate-of-strain tensor are given by:

$$e_{rr} = \frac{\partial u_r}{\partial r}, \qquad e_{\theta\theta} = \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r}, \qquad e_{zz} = \frac{\partial u_z}{\partial z},$$

$$2e_{\theta z} = \frac{1}{r}\frac{\partial u_z}{\partial \theta} + \frac{\partial u_\theta}{\partial z}, \qquad 2e_{zr} = \frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}, \tag{A.36}$$

$$2e_{r\theta} = r\frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right) + \frac{1}{r}\frac{\partial u_r}{\partial \theta}.$$

Solutions to practice problems

Week 1 Problem solutions

Hydrostatic balance: Isothermal atmosphere and ocean

(a) $dP/dr = -\rho g$ with the ideal gas equation of state $P = \rho k_B T/M$ gives the solution.

(b) $P = \rho g z$

(c) For the scale height (distance over which pressure varies by $1/e$), you should get about 10 km for the atmosphere but only 10m for the ocean.

Streamlines etc.

(a) To find an equation for the streamlines, write the streamline as $\mathbf{x}(s)$ (where s is a coordinate measuring distance along the streamline) where

$$\frac{d\mathbf{x}}{ds} = \mathbf{u}.$$

Integrating gives an equation for a straight line

$$\frac{kt}{u_0}(x - x_0) = y - y_0.$$

(b) The coordinates of a fluid element as a function of time are given by $\mathbf{x}(t) = \int \mathbf{u}(t)dt$, which gives a parabola.

The coffee pot

The exit speed of the coffee can be estimated from the time taken to fill the cup and an estimate of the area of the stream. Bernoulli then gives the pressure drop, which can be compared to the drop in the level of the indicator ($\Delta P \approx \rho g \Delta z$).

A rotating bucket of water

(a) Bernoulli's constant is constant along streamlines. In general each streamline has a different value of the constant (unless $\nabla \times \mathbf{u} = 0$ which does not apply here).

(b) Constant pressure surface has a height $z = (\Omega^2/2g)(x^2 + y^2)$.

Polytropic atmospheres

(a) Isothermal $\gamma = 1$; incompressible $\gamma = \infty$.

(b)

$$P = P_b \left[1 - \left(\frac{\gamma - 1}{\gamma}\right)\frac{z}{H_b}\right]^{\gamma/(\gamma-1)},$$

$\Delta z = H_b\gamma/(\gamma - 1)$, $H_b = P_b/\rho_b g$

(c) For $\gamma = 1$ you will need the result $(1 + \epsilon x)^{1/\epsilon} \rightarrow e^x$ as $\epsilon \rightarrow 0$.

Bernoulli for compressible flows

(a)

$$\mathcal{H} = \frac{\gamma}{\gamma - 1}\frac{P}{\rho} + \frac{1}{2}u^2 + \chi$$

(b) The specific enthalpy h satisfies $dh = Tds + dP/\rho$, or $dh = dP/\rho$ for an isentropic flow.

(c) If $P(\rho)$ then $\nabla \times (\nabla P/\rho) = 0$ therefore there must be a function f such that $\nabla f = \nabla P/\rho$ and then

$$\mathcal{H} = f + \frac{1}{2}u^2 + \chi$$

.

Week 2 Problem solutions

The vorticity equation for a compressible fluid

The vorticity equation is derived by taking the curl of the momentum equation, so start there. The key steps are (1) write $(\mathbf{u} \cdot \nabla)\mathbf{u}$ in terms of $\nabla u^2/2$ and $\omega \times \mathbf{u}$ using a vector identity, (2) take the curl of the momentum equation, (3) use the continuity equation to replace $\nabla \cdot \mathbf{u}$ with $D\rho/Dt$, and (4) use the chain rule to combine $D\omega/Dt$ and $D\rho/Dt$.

For the physical interpretation, think about a constant density fluid first, and why vortex stretching leads to spin up (see notes if you need a reminder). Now think about a fixed volume cylinder of fluid and change the density of the fluid without stretching or tilting the vortex. How would the rotation respond if angular momentum is conserved? You should be able to argue that $\omega \propto \rho$, and that therefore the incompressible vorticity equation is modified to the equation shown.

Vortex dynamics

Use the fact that any given vortex is advected by the flow induced by the other vortex/vortices to try to predict the vortex trajectories.

(a) They orbit around each other.

(b) They move parallel to one another (e.g. vortices created by canoe paddle).

(c) This is a stationary state for an infinitely long line of vortices.

(d) The vortex line is unstable to small perturbations.

You look at these different situations in the computational exercise for this week.

Vorticity on a rotating sphere

As in question 1, take the curl of the momentum equation.

As a vortex moves towards the pole, it has a larger contribution from $\mathbf{\Omega}$ to ω_a. Therefore ω must decrease and the vortex rotates more slowly.

Tornado explosions

Bernoulli's principle can be used here because the $u \propto 1/r$ part of the flow is curl free, and so Bernoulli's constant is the same on all streamlines. At large distance from the vortex, Bernoulli's constant vanishes (using gauge pressure as the zero-point of pressure). Use that to determine the velocity at the edge of the vortex.

Image vortices

First convince yourself that you can replace the 90 degree boundary with a set of image vortices, just like an image problem in electrostatics. Evaluate the velocity at the vortex location and show that $dy/dx = -y^3/x^3$ which gives the result when integrated.

Week 3 Problem solutions

Flow down an inclined plane

(a) As long as the layer is thin, so the horizontal extent is much greater than the vertical extent, then we can treat the layer as infinite in the horizontal direction, in which case there should be no dependence on the particular x location we are working at.

(b) Continuity is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

but since there is no x-dependence, then $\partial v/\partial y = 0 \Rightarrow v$ must be independent of y. But there can be no perpendicular flow at the solid boundary $\Rightarrow v = 0$ everywhere.

(c) The y-component of the momentum equation is

$$0 = -\rho g \cos\alpha - \frac{\partial P}{\partial y}$$

where α is the angle of the slope. Therefore we must have a linear dependence of P on y. Integrating gives

$$P = P_0 + \rho g (H - y) \cos\alpha$$

where P_0 is the atmospheric pressure (at $y = H$).

(d) The x-component of momentum is

$$\mu \frac{\partial^2 \mu}{\partial y^2} + \rho g \sin\alpha = 0$$

which we can solve with boundary conditions: $u = 0$ at $y = 0$ (no slip) and $\partial u/\partial y = 0$ at $y = H$ (free surface so there is no viscous stress). The answer is

$$u(y) = \frac{g}{\nu}\left(H - \frac{y}{2}\right) y \sin\alpha.$$

A quadratic dependence on height.

(e) The velocity at the top $y = H$ is

$$u = \frac{1}{2} g \sin\alpha \frac{H^2}{\nu}.$$

Writing it this way gives a physical interpretation: the velocity is $(1/2)at^2$ where a is the acceleration and the time is the viscous time across the layer. The velocity reached by the top of the layer is the velocity reached after accelerating due to gravity for a viscous time across the layer. This is very similar to the terminal velocity of a sky diver.

(f) Viscous stress at the base is

$$\mu \left. \frac{\partial u}{\partial y} \right|_{y=0} = \rho g H \sin \alpha$$

which is the stress required to balance the component of the weight of the fluid along the surface.

(g) This part is optional because it's quite a lot of slightly painful algebra, but I recommend you try it to make sure you understand these kind of problems. The boundary condition at the interface between the two layers is that the velocity must be continuous there and the viscous stress must be continuous there. The solution is

$$u_1 = \frac{\rho g \sin \alpha}{\mu_1} \left(h_1 + h_2 - \frac{y}{2} \right) y$$

$$u_2 = \rho g \sin \alpha h_1 \left(\frac{h_1}{2} + h_2 \right) \left(\frac{1}{\mu_1} - \frac{1}{\mu_2} \right) + \frac{\rho g \sin \alpha}{\mu_2} \left(h_1 + h_2 - \frac{y}{2} \right) y$$

where 1 refers to the lower layer and 2 the upper layer. (You can check that these solutions satisfy the boundary conditions and become equal in the limit $\mu_1 \to \mu_2$). The velocity of the lower layer does not depend on the viscosity of the upper layer because the stress the lower layer has to apply to the upper layer depends only on the mass (weight) of the upper layer. It doesn't matter how quickly the stress is communicated through the upper layer.

Flow near an impulsively-moved plane

(a) We have the same symmetry and same equation as the first problem, but without the gravity term.

(b) The layer is semi-infinite, meaning that there is no intrinsic lengthscale set by the height of the layer. The ony lengthscale in the problem is therefore the diffusion length $\sqrt{\nu t}$ so it makes sense that the solution would be in terms of this lengthscale. We could also apply this to the early-time behaviour of a finite thickness layer (i.e. for times $t \ll H^2/\nu$).

(c) Change variables using

$$\frac{\partial u}{\partial t} = f' \frac{\partial \eta}{\partial t} = -f' \frac{y}{2\nu^{1/2} t^{3/2}}$$

$$\frac{\partial u}{\partial y} = f' \frac{\partial \eta}{\partial y} = f' \frac{1}{\nu^{1/2} t^{1/2}}$$

$$\Rightarrow \frac{\partial^2 u}{\partial y^2} = \frac{f''}{\nu t}$$

$$\Rightarrow \frac{f''}{\nu t} = -\frac{1}{\nu} f' \frac{y}{2\nu^{1/2} t^{3/2}}$$

which reduces to

$$f'' + \frac{1}{2} f' \eta = 0$$

(d) Integrating once gives $f' = Be^{-\eta^2/4}$ for some constant B. Integrating again gives

$$f = A + B \int_0^\eta e^{-s^2/4} ds$$

The boundary conditions determine the constants A and B: since the velocity vanishes either at early times or large distances from the plate, we need $f(\infty) = 0$. The lower boundary is $f(0) = U_0$. Implementing these gives

$$u = U_0 \left[1 - \frac{1}{\sqrt{\pi}} \int_0^\eta e^{-s^2/4} ds \right]$$

or

$$u = U_0 \left[1 - \text{erf}\left(\frac{\eta}{2} \right) \right]$$

in terms of the error function erf. This is a fixed function when plotted in terms of η, but when you plot in terms of y you get a similar profile at different times, it is just stretched along the y axis at larger and larger times.

(e) Vorticity here is

$$\omega = -\frac{\partial u}{\partial y} = \frac{U_0}{(\pi \nu t)^{1/2}} e^{-y^2/4\nu t}$$

which shows that vorticity diffuses away from the wall.

(f) For a finite thickness layer, we need to have $du/dy = 0$ at $y = H$. We can arrange this by adding another boundary at $y = 2H$ with opposite vorticity (like an image charge in electrostatics):

$$\omega = \frac{U_0}{(\pi\nu t)^{1/2}}\left[e^{-y^2/4\nu t} - e^{-(y-2H)^2/4\nu t}\right]$$

which satisfies $\omega = 0$ at $y = H$ for all time.

(g) Again, a bit of algebra for this one. You have to integrate the expression for $\omega = -du/dy$ from $y = 0$ where $u = U_0$ to some value y where $u = u(y)$. The answer is

$$u(y) = U_0\left[1 - \mathrm{erf}\left(\frac{y}{2\sqrt{\nu t}}\right) + \mathrm{erf}\left(\frac{y-2H}{2\sqrt{\nu t}}\right) + \mathrm{erf}\left(\frac{H}{\sqrt{\nu t}}\right)\right]$$

You can see that $u = U_0$ for $y = 0$ (note erf is an odd function $\mathrm{erf}(-x) = -\mathrm{erf}(x)$), $u = 0$ for $t = 0$.

(h) To solve this one, you can look for an oscillatory solution $u = e^{i\omega t}f(y) \Rightarrow i\omega f = \nu f''$. The solution is

$$f(y) = Ae^{ky} + Be^{-ky}$$

where $k = (i\omega/\nu)^{1/2} = (1+i)(\omega/2\nu)^{1/2}$. Then the boundary conditions give $A = 0$ (otherwise the solution blows up at infinity) and $B = U_0$ since $u = U_0 e^{i\omega t}$ at $y = 0$. The solution is therefore

$$u = U_0 e^{i\omega t}e^{-iy/\ell}e^{-y/\ell}$$

where $\ell = (2\nu/\omega)^{1/2}$. The velocity is given by the real part

$$u = U_0\cos\left(\omega t - \frac{y}{\ell}\right)e^{-y/\ell}$$

You should compare this with your numerical solution from the numerical exercise this week!

Week 4 Problem solutions

Damped sound waves

(a) We can do the opposite to what we did in the notes: take the time-derivative of equation (2) in the notes and combine with equation (1). It's a bit tricky because whereas before we ended up with $\nabla \cdot (\nabla \delta\rho) = \nabla^2 \delta\rho$, we now end up with

$$\nabla(\nabla \cdot \delta\mathbf{u})$$

which is not obviously the same as $\nabla^2 \delta\mathbf{u}$. However, we can use the vector identity

$$\nabla(\nabla \cdot \mathbf{A}) - \nabla \times (\nabla \times \mathbf{A}) = \nabla^2 \mathbf{A}$$

which gives the result we want because $\nabla \times \delta\mathbf{u} \propto \nabla \times (\nabla \delta\rho) = 0$.

(b) Whether adiabatic or isothermal is a good assumption depends on how the timescale for heat loss compares with the oscillation period. An isothermal gas has $P \propto \rho$ and so the result in the question follows. The sound speed this time is $c_s^2 = P/\rho$ (the same expression as adiabatic sound waves but with $\gamma = 1$).

To assess whether sound waves in air are adiabatic or not, choose a frequency, e.g. $\nu_s =$ 400 Hz. The wavelength of the wave is then $\lambda = c_s/\nu_s = 0.825$ m for $c_s = 330$ m s^{-1}. The thermal diffusion timescale on that lengthscale is λ^2/κ where $\kappa \approx 2 \times 10^{-5}$ m^2 s^{-1} is the thermal diffusivity of air (I looked up the value on wikipedia). This gives a thermal diffusion time of ≈ 9.5 hours, obviously much longer than the oscillation period! The sound waves are adiabatic.

(c) With the viscous term included in the momentum equation, the perturbation equations are

$$\frac{\partial}{\partial t}\delta\rho = -\rho_0 \nabla \cdot \delta\mathbf{u}$$

$$\frac{\partial}{\partial t}\delta\mathbf{u} = -\frac{\nabla \delta P}{\rho_0} + \nu \nabla^2 \delta\mathbf{u}.$$

Putting in a plane wave dependence $e^{i\mathbf{k}\cdot\mathbf{r}} e^{-i\omega t}$ for the perturbations will give you the dispersion relation in the question [**except for a typo! The viscous term should have a plus sign not minus sign..**]

One way to approach the next part is to write the frequency as $\omega = \omega_R + i\sigma$ with real and imaginary parts ω_R and σ respectively. Because the viscous term is a small correction, we have $\sigma \ll \omega_R$. Inserting this form for ω into the dispersion relation and taking the real and imaginary parts of the equation gives

$$\omega_R^2 \approx c_s^2 k^2$$

and

$$\sigma \approx -\frac{\nu k^2}{2}.$$

This shows that the wave is $\propto e^{-\nu k^2 t/2}$, so decays by one e-folding on a timescale $\sim \lambda^2/2\pi^2\nu$ or over a distance $\sim c_s\lambda^2/2\pi^2\nu$. For the 400 Hz wave we looked at above, we had $\lambda = 0.825$ m, $c_s = 330$ m s^{-1}, and with viscosity $\nu = 1.5 \times 10^{-5}$ m^2 s^{-1}, the decay distance is ~ 800 km.

Steepening and shocks

(a) First change variables: $\xi = x - Vt$ gives

$$\frac{\partial}{\partial t} = -V\frac{d}{d\xi}$$

$$\frac{\partial}{\partial x} = \frac{d}{d\xi}$$

$$\Rightarrow -Vu' + uu' = \nu u''.$$

Then check the solution works:

$$f = \frac{U_0}{1 + e^{\xi/\Delta}}$$

$$f' = \frac{-f}{1 + e^{\xi/\Delta}}\frac{e^{\xi/\Delta}}{\Delta} = -\frac{f}{\Delta} + \frac{f^2}{U_0\Delta}$$

(the strategy here is to write things in terms of f as much as possible which avoids lots of exponential factors in the equations and makes it a bit easier..)

$$f'' = -\frac{f'}{\Delta} + \frac{2ff'}{U_0\Delta}$$

By comparing with the equation, we see that this solution does indeed work as long as

$$\frac{\nu}{\Delta} = V, \qquad \frac{2\nu}{U_0\Delta} = 1$$

or

$$\Delta = \frac{2\nu}{U_0}, \qquad V = \frac{U_0}{2}$$

which gives the shock speed and width.

(b) With the viscosity of air $\sim 10^{-5}$ m^2 s^{-1} and $c_s \approx 300$ m s^{-1}, $\Delta \sim 10^{-5}/300 \sim 3\times10^{-8}$ m. This is actually comparable to the mean free path in air! Another way to see this is that for a gas, the viscosity is $\nu \approx c_s\lambda_{\rm mfp}$ and so the shock thickness is of order the mean free path by necessity. Of course, at these scales the fluid approximation breaks down so we should be careful if we really want to look at what's happening on the scale of the shock itself. In many applications, we can treat the shock as a discontinuity and use conservation laws to map density and velocity etc. from one side to the other (you can look up "shock jump conditions" if you want to learn more about this).

Week 5 Problem solutions

Scalings in turbulence

For (a) and (b) you need the idea that the energy spectrum extends to a smaller cutoff scale for larger Reynolds number; $\ell_d = L/\mathrm{Re}^{3/4}$.

For (c), the dissipation scale moves to larger scales over time as the small scale energy dissipates. At time t, the scale that is dissipating is the one with $\ell/v \sim t$, and since $v \propto \ell^{1/3}$, the shortest lengthscale therefore is $\ell \propto t^{3/2}$. Note that this is faster than viscous decay alone which would give $\ell \propto t^{1/2}$!

Turbulence in a cloud

Your answer will obviously depend a lot on the values you assume for wind speed and size of the cloud. With speed 1 m s^{-1} and size 300 m, the energy dissipation rate per unit volume is

$$\frac{\rho u^3}{\ell} \sim 0.003 \text{ W m}^{-3}.$$

With a volume (300 m)^3, the total energy dissipation rate is $\sim$ 100 kW. (For scale, we can look up the amount of energy per unit area from the solar irradiation, it's about 1 kW m^{-2}).

With viscosity $\nu \sim 2 \times 10^{-5}$ m^2 s^{-1}, the Reynolds number is $\mathrm{Re} \sim 10^7$, giving an inner scale $\sim 300 \text{ m}/\mathrm{Re}^{3/4} \sim 1$ mm.

Week 6 Problem solutions

Viscous flow down an inclined plane

(a) From the solutions to Week 3, the velocity is

$$u(y) = \frac{g}{\nu}\left(H - \frac{y}{2}\right) y \sin\alpha.$$

The viscous dissipation rate per unit volume is

$$\Phi_V = \mu\left(\frac{\partial u}{\partial y}\right)^2 = \frac{(\rho g \sin\alpha)^2}{\mu}\left(H - y\right)^2.$$

Integrate over height to get the viscous dissipation per unit area:

$$\frac{(\rho g \sin\alpha)^2}{\mu}\int_0^H \left(H - y\right)^2 dy = (\rho g \sin\alpha)^2\frac{H^3}{3\mu}.$$

This energy is supplied by the work done by gravity on the fluid

$$\int_0^H dy\ u(y)\ \rho g \sin\alpha = \frac{(\rho g \sin\alpha)^2}{\mu}\int_0^H dy\, y\left(H - \frac{y}{2}\right) = \frac{(\rho g \sin\alpha)^2}{\mu}\frac{H^3}{3}.$$

Rotating cylinder

The flow in the fluid is that of a vortex with circulation $\Gamma = 2\pi R \times R\Omega$, ie.

$$u_\phi(r) = \frac{\Gamma}{2\pi r} = \frac{R^2\Omega}{r} \qquad r \geq R$$

You can argue this either (1) based on the symmetry of the problem and then evaluating the circulation at the surface of the cylinder and at some radius r and setting them equal or (2) by writing down the momentum equation for a steady flow

$$\frac{\partial u_\phi}{\partial t} = 0 = \nu\left[\frac{\partial^2 u_\phi}{\partial r^2} + \frac{1}{r}\frac{\partial u_\phi}{\partial r} - \frac{u_\phi}{r^2}\right]$$

which has a general solution $u_\phi = Ar + B/r$, the constants A and B can be obtained from the boundary conditions.

The viscous dissipation rate per unit volume is

$$\mu\, r^2\left(\frac{d}{dr}\left(\frac{u_\phi}{r}\right)\right)^2.$$

Integrating to get the dissipation per unit length along the cylinder gives $4\pi\mu R^4\Omega^2$.

To compare this to the work done on the cylinder, we need the viscous stress at the surface of the cylinder, which is

$$\tau = \mu\, r \frac{d}{dr}\left(\frac{u_\phi}{r}\right).$$

To get this, you can use $\sigma_{ij} = 2\mu e_{ij}$ and take the $r\theta$ component of e_{ij} from the cylindrical geometry formulae included with the question.

Evaluate the viscous stress at $r = R$ to get the force per unit area acting on the surface of the cylinder (trying to slow it down). You can then calculate the rate of work done by this force on the cylinder (again per unit length along the cylinder) which should give you the same answer as the viscous dissipation. To keep the cylinder spinning at a constant angular velocity requires work to replace the energy lost to viscous dissipation.