PHYS 642
Radiative Processes in Astrophysics
Winter term 2009

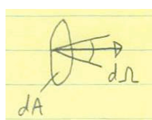
Contents

1. Radiative Transfer

These are notes for the first part of PHYS 642 Radiative Processes in Astrophysics. The idea is to get as far as we can without worrying about the microphysics by which radiation is emitted, absorbed, or scattered. We will develop a formalism to follow the radiation from its source to the observer through intervening material, taking into account absorption, emission, and scattering, and discuss the properties of thermal radiation. Examples covered are radiative diffusion in stellar interiors and the Rosseland mean opacity, the grey atmosphere as the simplest example of a stellar atmosphere, the spectrum of an atmosphere and limb darkening, and the origin of emission and absorption lines.

1.1. The specific intensity and its moments

We describe the radiation propagating in a particular direction in terms of the specific intensity I_ν. The energy crossing per second per unit area perpendicular to the beam is



$$dE = I_\nu \, dA \, dt \, d\Omega \, d\nu \tag{1.1}$$

in the frequency interval ν to $\nu + d\nu$ and in the cone of solid angle $d\Omega$ about the propagation direction.

The specific intensity has a simple relation to the phase space density of the photons. Defining the single particle occupation number f_α such that the number of particles with momentum $\vec{p}$ and position $\vec{x}$ is

$$dn = \sum_{\alpha=1}^{2} f_\alpha \, d^3\vec{x} \, d^3\vec{p}. \tag{1.2}$$

We sum over the two polarizations of the photons, labelled by α. We make the connection with I_ν by rewriting the volume element in momentum space in terms of the magnitude and direction of the momentum,

$$d^3\vec{p} = p^2 dp \, d\Omega = \frac{h^3 \nu^2 d\nu \, d\Omega}{c^3} \tag{1.3}$$

and the spatial volume element in terms of the area element perpendicular to the photon propagation direction, and a length cdt, which gives the distance travelled by photons in time dt,

$$d^3\vec{x} = dA \; c \, dt. \tag{1.4}$$

This gives

$$dn = \sum_\alpha f_\alpha h^3 \frac{h\nu^3}{c^2}\ dA\ dt\ d\Omega\ d\nu \tag{1.5}$$

and therefore

$$I_\nu \equiv \sum_\alpha f_\alpha h^3 \frac{h\nu^3}{c^2}. \tag{1.6}$$

The energy density of the radiation is

$$U = \int\ d^3\vec{p}\ \sum_\alpha f_\alpha h\nu = \frac{1}{c}\int I_v\ d\nu\, d\Omega \tag{1.7}$$

from which we see that

$$U_\nu = \frac{1}{c}\int I_\nu\ d\Omega. \tag{1.8}$$

We can also write an expression for the energy flux $\vec{F}$. In the x-direction, for example,

$$F_x = \int d^3\vec{p} \sum_\alpha f_\alpha\ v_x\ h\nu \tag{1.9}$$

where we construct the flux by multiplying the number density of particles by the velocity in the x-direction and the quantity being carried, here energy. If θ is the angle with respect to the photon propagation direction, then

$$F_x = \int d^3\vec{p} \sum_\alpha f_\alpha\ h\nu c\ \cos\theta = \int I_\nu\ d\nu\ d\Omega \cos\theta \tag{1.10}$$

Similarly, we can derive the pressure of the radiation by calculating the momentum flux across a unit area. The flux of x-momentum in the x-direction is

$$P_{xx} = \int d^3\vec{p} \sum_\alpha f_\alpha\ v_x\ p_x = \int d^3\vec{p} \sum_\alpha f_\alpha\ h\nu\ \cos^2\theta = \frac{1}{c}\int I_\nu\ d\nu\ d\Omega \cos^2\theta \tag{1.11}$$

We see that the energy density, pressure, and flux can be expressed in terms of the zeroth, first, and second moments of the radiation field,

$$\frac{cU_\nu}{4\pi} = J_\nu = \frac{1}{4\pi}\int I_\nu d\Omega = \frac{1}{2}\int_{-1}^{1} d\mu\ I_\nu(\mu) \tag{1.12}$$

$$\frac{F_\nu}{4\pi} = H_\nu = \frac{1}{4\pi}\int I_\nu \cos\theta\ d\Omega = \frac{1}{2}\int_{-1}^{1} d\mu\ \mu I_\nu(\mu) \tag{1.13}$$

$$\frac{cP_\nu}{4\pi} = K_\nu = \frac{1}{4\pi}\int I_\nu \cos^2\theta\ d\Omega = \frac{1}{2}\int_{-1}^{1} d\mu\ \mu^2 I_\nu(\mu) \tag{1.14}$$

where the integrals over μ are for an axially symmetric radiation field, where $d\Omega = 2\pi\ d\mu$ with $\mu = \cos\theta$. The quantity J_ν is known as the *mean intensity*.

Let's do some simple examples. An *isotropic* radiation field has I_ν constant for photons propagating in all directions. Then

$$\frac{cU_\nu}{4\pi} = J_\nu = I_\nu = \frac{3cP_\nu}{4\pi} \tag{1.15}$$

or

$$P_\nu = \frac{1}{3}U_\nu. \tag{1.16}$$

The flux vanishes for integration over all solid angles. The flux from a surface is given by integrating over a hemisphere in solid angle,

$$F_\nu = \pi I_\nu. \tag{1.17}$$

Another example is a unidirectional radiation field, e.g. $I_\nu = I_0\delta(\mu)$. This gives $P_\nu = U_\nu$ in contrast to the result for an isotropic radiation field. In an atmosphere as we move towards the surface, the radiation field becomes more and more outwards directed, and P_ν/U_ν goes from $1/3 \rightarrow 1$. Keeping track of this variation is important in modeling stellar atmospheres.

1.2. Thermal radiation

An important case is when the radiation is in thermal equilibrium at temperature T. Then the photon distribution function is given by the Bose-Einstein distribution with zero chemical potential ($\mu = 0$)

$$h^3 f_\alpha = \frac{1}{e^{h\nu/k_BT} - 1} \tag{1.18}$$

the same for both spin states α, giving

$$I_v \equiv B_\nu(T) = \frac{2h\nu^3}{c^2}\frac{1}{e^{h\nu/k_BT} - 1} \tag{1.19}$$

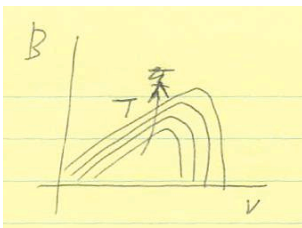
which defines the Planck distribution $B_\nu(T)$ or blackbody spectrum. The two limits $h\nu \gg k_BT$ and $h\nu \ll k_BT$ both have names. The Rayleigh-Jeans part of the spectrum at low frequency is given by

$$B_\nu = \frac{2\nu^2}{c^2}k_BT \propto \nu^2. \tag{1.20}$$

This formula can be obtained by counting the photon modes and assuming each has k_BT of energy in thermal equilibrium. Taken to large frequency, this predicts infinite energy in the photon field, the so-called ultraviolet catastrophe. The resolution is in the quantization of the photon energy spectrum. At high frequencies, the photon energy becomes much greater than the thermal energy $h\nu/k_BT$, and the occupation number is exponentially suppressed, giving the Wien tail

$$B_\nu = \frac{2h\nu^3}{c^2}\exp\left(\frac{-h\nu}{k_BT}\right). \tag{1.21}$$

The other point to note about the blackbody spectrum is that $\partial B_\nu(T)/\partial T|_\nu > 0$, that is the emissivity increases at every frequency when the temperature increases. In other words, the blackbody curves all fit inside each other in a plot.



The peak of B_ν is at $h\nu_{\rm max} = 2.28k_BT$ (the Wien displacement law), or $\nu_{\rm max}/T = 5.88\times 10^{10}$ Hz K^{-1}. The peak of B_λ is given by $\lambda_{\rm max}T = 0.290$ cm K.

Integrating over frequency[1] gives $B = \int B_\nu d\nu = acT^4/4\pi$ where the radiation constant

$$a = \frac{8\pi^5}{15}\frac{k_B^4}{(hc)^3} = 7.5657\times 10^{-15}\ \text{cgs}. \tag{1.22}$$

Since B_ν is isotropic, we can use our earlier results for the energy density and pressure, which are

$$U = aT^4 \qquad P = \frac{1}{3}aT^4. \tag{1.23}$$

The flux from a surface is $F_\nu = \pi B_\nu$, or integrated over frequency,

$$F = \pi B = \frac{1}{4}acT^4 = \sigma_{SB}T^4 \tag{1.24}$$

where the Stefan-Boltzmann constant is $\sigma_{SB} = 5.67\times 10^{-5}$ cgs.

1.3. The transfer equation for emission and absorption

Having defined I_ν and looked at some examples, we now ask how I_ν changes as photons propagate through space. First, consider propagation through vacuum so that photons

[1] Use the result $\int_0^\infty dx\, x^3/(e^x-1) = \pi^4/15$.

are neither created or destroyed. The single particle distribution function f_α satisfies the collisionless Boltzmann equation

$$\frac{1}{c}\frac{\partial f}{\partial t} + \vec{\hat{k}} \cdot \vec{\nabla} f_\alpha = 0 \tag{1.25}$$

where $\vec{k}$ is a unit vector giving the photon propagation direction. This is straightforward to derive. The idea is that photons initially at position $(\vec{x}, \vec{p})$ in phase space will be at position $(\vec{x} + \vec{k}cdt, \vec{p})$ a time dt later. The photons conserve phase space volume as they propagate. Setting the number of photons in a phase space volume $d^3\vec{x}d^3\vec{p}$ constant implies that $f_\alpha(\vec{x}, \vec{p}, t) = f_\alpha(\vec{x} + \vec{k}cdt, \vec{p}, t + dt)$. Equation (1.25) follows by a Taylor expansion.

Since I_ν and f_α are related by a constant factor, then

$$\frac{1}{c}\frac{\partial I_\nu}{\partial t} + \vec{k} \cdot \vec{\nabla} I_\nu = 0. \tag{1.26}$$

There are two points to make about this equation. First, the first term is often much smaller than the second term if the timescale for evolution of the system we're interested in is much longer than the light crossing time for that system. Second, in general photons are not conserved but scattered, absorbed, and emitted and we account for these processes by adding source and sink terms to the RHS. Define coordinate s along the photon path, we then have

$$\frac{dI_\nu}{ds} = (\text{sources}) - (\text{sinks}). \tag{1.27}$$

In general, we must solve a number of equations for I_ν at different photon frequencies and propagation directions. Emission and absorption of photons by matter are obvious sources and sinks that we must include. Also, scattering moves photons from one direction to another and perhaps from one frequency to another if it is inelastic.

The spontaneous emission coefficient j_ν is defined as the energy emitted per unit time, volume, in a given direction and frequency, so that

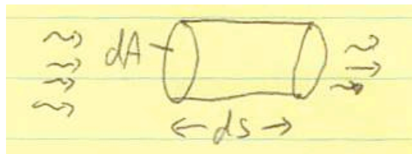
$$\frac{dI_\nu}{ds} = j_\nu \tag{1.28}$$

with units erg cm^{-3} s^{-1} Hz^{-1} sterad^{-1}. Often the emission process is isotropic, and it's useful to define an emissivity ϵ_ν, where

$$j_\nu = \frac{\rho \epsilon_\nu}{4\pi} \tag{1.29}$$

(units of ϵ_ν are erg g^{-1} s^{-1} Hz^{-1}). We'll calculate j_ν due to various physical processes later in the course. Notice that locally where we can treat j_ν as constant, the specific intensity increases linearly due to emission.

Now consider absorption of photons. Draw a cylinder around the direction of photon propagation, with length ds and cross-section dA.



If the absorbers have number density n, and each has a cross-section for photon absorption of σ, the absorption cross-section looking along the cylinder is $\sigma n\, dA\, ds$. The probability that a ray is absorbed on traversing the cylinder is therefore $\sigma n\, ds = ds/l$, where $l = 1/n\sigma$ is the photon mean free path[2].

Therefore, as the beam passes through the material,

$$\frac{dI_\nu}{I_\nu} = -n\sigma ds = -\frac{ds}{l}. \tag{1.30}$$

Rather than writing l in terms of n and σ, in astrophysics it is usual to write

$$l = \frac{1}{n\sigma} = \frac{1}{\kappa\rho} = \frac{1}{\alpha} \tag{1.31}$$

where ρ is the mass density, κ is the opacity (units cm^2 g^{-1}) and α is the opacity coefficient (units cm^{-1}), giving

$$\frac{dI_\nu}{ds} = -\rho\kappa I_\nu = -\alpha_\nu I_\nu. \tag{1.32}$$

The absorption results in exponential decrease in I_ν as the photons propagate (one e-folding per photon mean free path).

The final result is

$$\frac{dI_\nu}{ds} = j_\nu - \alpha_\nu I_\nu \tag{1.33}$$

which describes radiative transfer with emission and absorption.

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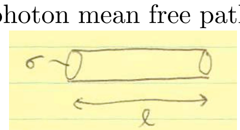
[2]To see that this is the photon mean free path, construct a cylinder of cross-section σ and length l along the direction of the photon. 

We are guaranteed an absorption on average if there is one absorber in the cylinder $n\sigma l = 1$. This defines the mean free path l.

1.4. Optical depth, source function, and Kirchoff's theorem

We define the optical depth τ by $d\tau = \alpha ds = \kappa\rho ds = ds/l$, or

$$\tau(s) - \tau(s_0) = \int_{s_0}^{s} \alpha(s) ds \tag{1.34}$$

If $\tau \gg 1$ when integrated along a typical path in a medium, the medium is said to be *optically thick* (most photons absorbed), whereas if $\tau \ll 1$ along a typical path, the medium is said to be *optically thin* (most photons escape).

To get a sense of the size of the mfp as a function of density, we can estimate $\sigma \approx \sigma_T$ where $\sigma_T = 8\pi r_0^2/3 = 6.63 \times 10^{-25}$ cm^2 is the Thomson cross-section and $r_0 = e^2/m_e c^2$ is the classical electron radius[3]. This cross-section is for Thomson *scattering* rather than absorption, but gives a starting point for an estimate of an interaction cross-section. For a gas of protons, the cross-section per gram is then

$$\kappa = \frac{\sigma_T}{m_p} = 0.40 \text{ cm}^2 \text{ g}^{-1}. \tag{1.35}$$

The photon mfp is

$$l = \frac{1}{n\sigma} = \frac{0.5 \text{ Mpc}}{(n/\text{cm}^{-3})} \tag{1.36}$$

or

$$l = \frac{1}{\rho\kappa} = \frac{2.5 \text{ cm}}{(\rho/\text{g cm}^{-2})}. \tag{1.37}$$

The first case is a typical interstellar medium (ISM) density, the second case is for the mean density of the Sun. For the Sun, the mean free path is a factor $\sim 10^{11}$ smaller than the radius of the star, so that the solar interior is extremely optically thick. For the ISM, the mean free path we estimate is larger than the size of the Galaxy. Of course, we haven't included the correct opacity sources for optical photons travelling through the ISM, but still this estimate makes the point that at ISM densities the mfp can be large.

In terms of optical depth, the transfer equation is

$$\frac{dI_\nu}{d\tau_\nu} = \frac{j_\nu}{\alpha_\nu} - I_\nu = S_\nu - I_\nu \tag{1.38}$$

where we define the *source function* $S_\nu \equiv j_\nu/\alpha_\nu$.

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[3]Note that in cgs, the electron charge is $e = 4.8032 \times 10^{-10}$ cgs.

The general solution of the transfer equation is[4]

$$I_\nu(\tau_\nu) = I_\nu(0)e^{-\tau_\nu} + \int_0^{\tau_\nu} e^{-(\tau_\nu - \tau'_\nu)} S_\nu(\tau'_\nu)\, d\tau'_\nu. \tag{1.39}$$

Each term has a simple physical interpretation. The first term describes absorption of the incident radiation $I_\nu(0)$. The second term is an integral over the emitted photons given by the source function, and a factor to include absorption of those emitted photons as they propagate to optical depth τ_ν.

As a simplified case, consider $S =$constant. Then the solution is

$$I_\nu(\tau_\nu) = I_\nu(0)e^{-\tau_\nu} + S_\nu\left(1 - e^{-\tau_\nu}\right) = S_\nu + e^{-\tau_\nu}\left(I_\nu(0) - S_\nu\right) \tag{1.40}$$

which shows that for large optical depths, $I_\nu \to S_\nu$. If initially $I_\nu > S_\nu$, then photons are absorbed from the beam until $I_\nu = S_\nu$. Similarly, if $I_\nu < S_\nu$ initially, then photons are added to the beam until $I_\nu = S_\nu$. For small optical depth, $I_\nu(\tau_\nu) \approx I_\nu(0)(1 - \tau_\nu) + \tau_\nu S_\nu$.

An extremely important result is *Kirchoff's law*, which states that a material in thermodynamic equilibrium at temperature T has

$$j_\nu = \alpha_\nu B_\nu(T) \tag{1.41}$$

or

$$S_\nu = B_\nu(T). \tag{1.42}$$

One way to see that this must be the case is to consider an object placed inside a thermal cavity and allowed to come into equilibrium with it. It must replace any radiation it absorbs, frequency by frequency.

A true blackbody has α_ν constant, independent of frequency (a "perfect absorber" absorbs all frequencies equally), and so has $j_\nu \propto B_\nu$. But this is not true for real materials, which have an emissivity weighted by a non-constant absorption coefficient. At frequencies which are readily absorbed, the emissivity is high, and vice versa. An example of this is emission lines from thermal optically thin gas (e.g. in the chromosphere of the Sun). The absorption coefficient κ_ν is larger at the frequencies of line transitions, and therefore so is the emissivity.

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[4]To see this, first take out the expected $e^{-\tau}$ behavior by defining $f = I_\nu e^\tau$, and $g = Se^\tau$. Then $df/d\tau = g$ can be integrated to give eq. [1.39].

1.5. Examples

1.5.1. Stellar interiors

Let's consider optically thick regions such as stellar interiors. A good assumption is often *local thermodynamic equilibrium* (LTE), in which the degrees of freedom associated with the particles (e.g. atomic energy levels) are characterized by their values in thermodynamic equilibrium (TE) at temperature T. In this case, $S_\nu = B_\nu(T)$. The difference from full TE is that the radiation field in general does not have a Planck distribution, $I_\nu = B_\nu(T)$. However, our solution for the radiative transfer equation tells us that $I_\nu \rightarrow B_\nu$ for optically thick LTE material.

We first write ds in terms of the radial coordinate r, as $ds = dr\cos\theta = \mu dr$ for a photon propagating at angle θ to the radial direction. Then

$$\mu\frac{dI_\nu}{dr} = j_\nu - \alpha_\nu I_\nu. \tag{1.43}$$

Since the optical depth increases inwards, it makes sense to define the radial optical depth as $d\tau_\nu = -\alpha_\nu dr$, and so

$$\mu\frac{dI_\nu}{d\tau_\nu} = -S_\nu + I_\nu \tag{1.44}$$

where the factor of μ on the LHS accounts for the fact that τ_ν is the *radial* optical depth.

Next, we take the moments of equation (1.44). Integrating over solid angle gives

$$\frac{1}{4\pi}\frac{dF_\nu}{d\tau_\nu} = -S_\nu + J_\nu \tag{1.45}$$

and multiplying by μ and integrating gives

$$c\frac{dP_\nu}{d\tau_\nu} = F_\nu. \tag{1.46}$$

We have assumed S_ν is isotropic so that $\int d\Omega \mu S_\nu = 0$.

In a stellar interior, we already mentioned the fact that $\tau \gg 1$, and therefore we expect $I_\nu \approx B_\nu$. However, there must be some anisotropy in the radiation field since the photons transport heat outwards. Therefore

$$I_\nu = B_\nu(T) + (\text{small anisotropic part}) \tag{1.47}$$

Now look at equation (1.46). The flux F_ν must come from the anisotropic part of I_ν, but the pressure is mostly set by the isotropic part, $P_\nu \approx 4\pi B_\nu/3c$. Therefore

$$F_\nu = -\frac{4\pi}{3\rho\kappa_\nu}\frac{dB_\nu}{dT}\frac{dT}{dr}. \tag{1.48}$$

Integrating over frequency gives the total flux

$$F = -\frac{4\pi}{3\rho}\frac{dT}{dr}\int d\nu \frac{1}{\kappa_\nu}\frac{dB_\nu}{dT}. \tag{1.49}$$

Next, we define the *Rosseland mean opacity*

$$\left[\int d\nu \frac{dB_\nu}{dT}\right]\frac{1}{\kappa_R} = \left[\int d\nu\ \frac{1}{\kappa_\nu}\ \frac{dB_\nu}{dT}\right]. \tag{1.50}$$

The factor on the LHS is

$$\int d_\nu \frac{dB_\nu}{dT} = \frac{d}{dT}\int d\nu B_\nu = \frac{d}{dT}\left(\frac{acT^4}{4\pi}\right) \tag{1.51}$$

and therefore we arrive at

$$F = -\frac{4acT^3}{3\kappa_R\rho}\frac{dT}{dr} \tag{1.52}$$

the *radiative diffusion equation.*

We see that radiation diffuses down the temperature gradient, as would be expected. We can rewrite equation (1.52) as

$$F = -\frac{1}{3}c\ l\ \frac{d}{dr}\left(aT^4\right) \tag{1.53}$$

exactly what we would have guessed from a kinetic theory approach. The 1/3 factor is the usual factor from integration over angles, and the transported quantity is the photon energy density aT^4. At a given location, the photons coming from deeper in the star are hotter (by an amount $\approx l dT/dr$) than those coming from cooler regions above.

We can use equation (1.52) to understand the solar luminosity. We expect

$$\begin{aligned} L_\odot &\approx 4\pi R^2 F \\ &\approx 4\pi R^2 \frac{1}{3} c l \frac{aT_c^4}{R} \\ &\approx \left(\frac{4\pi R^3}{3} aT_c^4\right)\left(\frac{cl}{R^2}\right). \end{aligned} \tag{1.54}$$

In the second line, we approximate $d(aT^4)/dr \approx aT_c^4/R$, where T_c is the central temperature. In the last line, the first term is the total energy content in radiation in the solar interior. In the second term, we are dividing by the time for a photon to random walk out of the Sun. Recall that for a random walk, the total distance travelled is $R = \sqrt{N}l$ where N is the number of steps. Therefore the time to escape is $Nl/c = R^2/lc = t_{\rm esc}$, and

$$L_\odot \approx \frac{E_\gamma}{t_{\rm esc}}. \tag{1.55}$$

Let's plug in some numbers: the central density of the Sun is $\rho \approx 150$ g cm^{-3} which gives $l/R \approx 10^{-13}$ (see eq. [1.37]). The light travel time is $R/c \approx 2$ s, and therefore $t_{\rm esc} \approx 10^6$ years. The luminosity of the Sun $L_\odot = 4\times10^{33}$ erg s^{-1}. Putting this together gives $T_c \approx 9\times10^6$ K. Not bad, the actual value is 1.5×10^7 K.

1.5.2. Grey atmosphere: temperature profile, limb darkening

Next, we consider the solution of the radiative transfer equation in the stellar atmosphere, in which the optical depth drops from $\tau \gg 1$ to $\tau \ll 1$. The simplest case is a *grey atmosphere*, in which "grey" refers to a frequency-independent opacity $\kappa_\nu = \kappa$. Equation (1.45) integrated over frequency is

$$\frac{1}{4\pi}\frac{dF}{d\tau} = -S + J \tag{1.56}$$

which for a constant flux F implies that we must have $S = J$. Similarly, the frequency-integrated equation (1.46),

$$c\frac{dP}{d\tau} = F \tag{1.57}$$

gives the simple result $P = (F/c)(\tau+\tau_0)$. To close these equations, we make the *Eddington approximation* that $U = 3P$ or $3P = 4\pi J/c$. Then,

$$S = J = \frac{3cP}{4\pi} = \frac{3F}{4\pi}(\tau+\tau_0). \tag{1.58}$$

To find the constant τ_0, we solve the radiative transfer equation for $I(\tau)$ and then use it to find the flux F at the surface. Only for the correct choice of τ_0 is the solution self-consistent in this way. The specific intensity is

$$I(\tau,\mu) = \int_\tau^\infty e^{-(\tau-\tau')/\mu} S(\tau')\,\frac{d\tau'}{\mu}. \tag{1.59}$$

Substituting our expression for S gives $I(0) = (3F/4\pi)(\tau_0+\mu)$ for $\mu > 0$ and $I(0) = 0$ for $\mu < 0$. The flux at the surface can be found from

$$\int_0^1 2\pi\mu\, d\mu\, I(0), \tag{1.60}$$

which is equal to F only if $\tau_0 = 2/3$.

If we assume LTE, then $S = B$, and therefore

$$B = S = \frac{3F}{4\pi}\left(\tau + \frac{2}{3}\right) \tag{1.61}$$

but $F = \sigma T_{\rm eff}^4$ and $B = \sigma T^4/\pi$, giving

$$T^4 = \frac{3}{4} T_{\rm eff}^4 \left(\tau + \frac{2}{3}\right), \tag{1.62}$$

the temperature profile of the grey atmosphere in the Eddington approximation. Note that $T = T_{\rm eff}$ at $\tau = 2/3$. This optical depth is often taken as the photosphere.

The specific intensity for arbitrary depth is (for outgoing rays, $\mu > 0$)

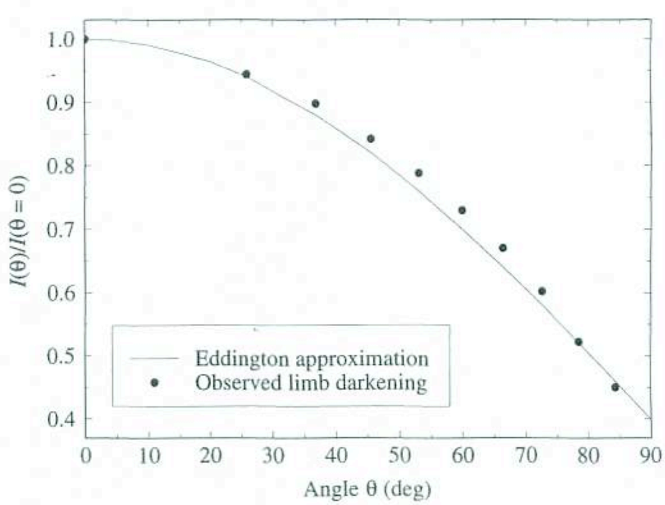
$$I(\mu, \tau) = \frac{3F}{4\pi}\left(\mu + \frac{2}{3} + \tau\right) = B + \frac{3F\mu}{4\pi}. \tag{1.63}$$

This shows that at large optical depth, the anisotropic part of the specific intensity is $\approx 1/\tau$ of the isotropic part.

We can also estimate the amount of *limb darkening* we expect. This is the effect that when we look at the edge of the Sun, we see to cooler layers for a given optical depth. The limb therefore appears darker than the face of the Sun. In our solution,

$$\frac{I(\tau = 0, \mu = 0)}{I(\tau = 0, \mu = 1)} = \frac{2}{5}. \tag{1.64}$$

A full solution to the grey atmosphere (without the Eddington approximation) gives 0.35 (see Appendix which summarizes the exact solution to the grey atmosphere from Chandrasekhar's book). Here is a comparison of the observed limb-darkening of the Sun compared to the Eddington approximation result, taken from Carroll and Ostlie.



It may seem surprising that the Eddington approximation $P = U/3$ gives such good results given that it holds for an isotropic photon distribution, whereas the photon distribution is anisotropic in the stellar atmosphere. In fact, the Eddington approximation holds for more general angular distributions of the photons. For example, if the photons are isotropic in the outgoing and ingoing hemispheres, but with different intensities, the Eddington approximation holds. Similarly, the Eddington approximation holds for the anisotropic intensity $I = a + b\mu$ (or with additional terms as long as only odd powers of μ are included).

1.5.3. Spectrum of a grey atmosphere

Having calculated the temperature profile of the grey atmosphere, we can now go back and calculate its spectrum if we assume LTE so that the source function is $B_\nu(T)$ at each depth. Then the outgoing specific intensity at the surface is

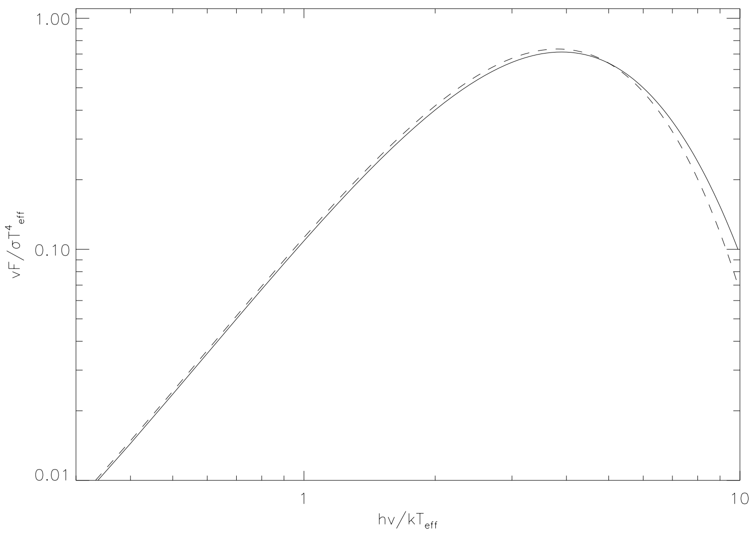
$$I_\nu(0,\mu) = \int_0^\infty e^{-\tau/\mu} B_\nu\left(T(\tau)\right) \frac{d\tau}{\mu} \tag{1.65}$$

where we can take $T(\tau)$ as previously calculated using the Eddington approximation. The emergent flux is

$$\begin{aligned} F_\nu(0) &= \int_0^1 d\mu\ 2\pi\mu\ I_\nu(0,\mu) && (1.66)\\ &= \int_0^1 d\mu\ 2\pi\mu \int_0^\infty e^{-\tau/\mu} B_\nu(T)\frac{d\tau}{\mu} && (1.67)\\ &= 2\pi \int_0^\infty d\tau B_\nu(T) \int_0^1 d\mu\ e^{-\tau/\mu} && (1.68)\\ &= 2\pi \int_0^\infty d\tau B_\nu(T) \int_1^\infty \frac{dx}{x^2} e^{-\tau x} && (1.69)\\ &= 2\pi \int_0^\infty d\tau B_\nu(T) E_2(\tau) && (1.70) \end{aligned}$$

where we have made the substitution $x = 1/\mu$, and $E_2(\tau)$ is an exponential integral[5].

[5] Defined as $E_n(\tau) = \int_1^\infty x^{-n} e^{-\tau x}\, dx$. These functions occur often in analytic solutions to the radiative transfer problem. Some properties (which are straightforward to prove) are: $E_n(x) \to e^{-x}/x$ for $x \to \infty$, $E_1(x) \to \ln(1/x)$ as $x \to 0$, $E_n(x) \to 1/(n-1)$ as $x \to 0$ ($n > 1$), $(n-1)E_n(x) = e^{-x} - xE_{n-1}(x)$, $dE_n/dx = -E_{n-1}(x)$, and $\int_0^\infty dx E_n(x) = 1/n$. This last result can be used to show that eq. [1.70] gives the correct result for an isothermal atmosphere, $F_\nu = \pi B_\nu(T)$.



The first question in the homework 1 is about calculating the spectrum using equation (1.70). The plot above compares the grey atmosphere spectrum with a blackbody spectrum. The grey atmosphere spectrum is *harder* than a blackbody at the effective temperature, i.e. it has enhanced emission at higher photon energies. This is due to the increasing temperature profile with depth.

1.5.4. Emission and absorption lines

Of course, the spectrum of the Sun is not smooth and featureless like a blackbody or grey atmosphere spectrum, but has many absorption lines. The figure below is taken from Carroll and Ostlie. We can use the general solution to the equation of radiative transfer to get a feeling for when to expect absorption and emission lines.

Before we do this, it is interesting to note that the smooth part of the Sun's spectrum differs from a blackbody in the opposite way to a grey atmosphere – i.e. higher frequencies are suppressed slightly and lower frequencies enhanced. As discussed in chapter 4 of Shu's book, this difference was used to infer that the opacity in the solar atmosphere must increase with frequency (so that lower frequency photons come from deeper in the atmosphere, where the temperature is greater). This frequency variation was explained in the 1930s/40s when it was realized that H^- dominates the opacity in the atmosphere.

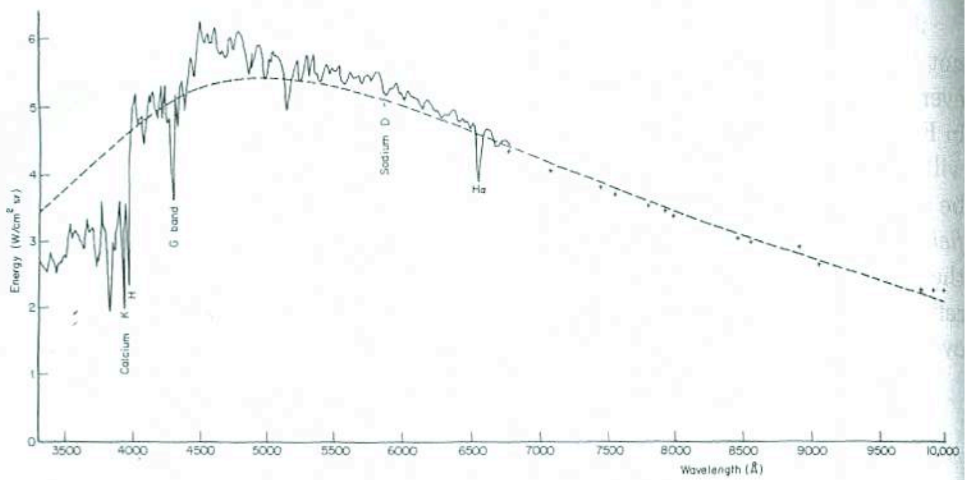
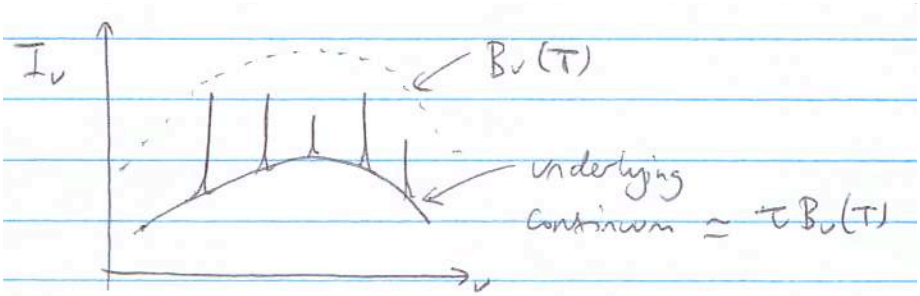


Figure 9.5 The spectrum of the Sun. The dashed line is the curve of an ideal blackbody having the Sun's effective temperature. (Figure from Aller, *Atoms, Stars, and Nebulae*, Revised Edition, Harvard University Press, Cambridge, MA, 1971.)

We already briefly mentioned optically thin, thermal gas, for which if we assume that the source function does not depend on position,

$$I_\nu = \int_0^\tau S_\nu(\tau')e^{-\tau'_\nu}d\tau'_\nu = S_\nu(1 - e^{-\tau_\nu}) \approx \tau_\nu S_\nu = \tau_\nu B_\nu. \tag{1.71}$$

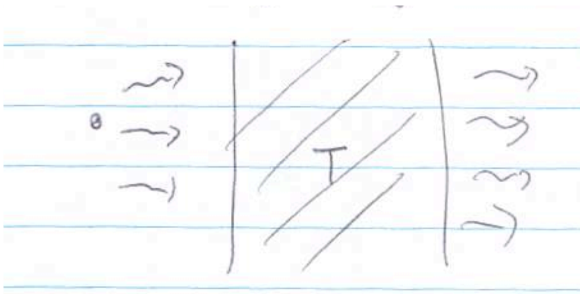
At the frequencies associated with line transitions, the absorption coefficient is large, and therefore so is the emissivity, $j_\nu = \alpha_\nu B_\nu$. The spectrum looks like this:



Note that since $1 - e^{-\tau} < 1$ the brightness of the emission lines can never exceed the Planck spectrum at the temperature of the gas (dashed line in the Figure).

A stellar atmosphere produces an absorption line spectrum. In this case, the object we're looking at is optically thick. At the frequencies of the absorption lines, a given optical depth corresponds to a much smaller physical depth, and therefore the photons come from a lower temperature region, giving a lower intensity than the continuum.

To get some intuition for this, a simple problem is a background source viewed through a layer of gas in LTE at temperature T.



The solution is

$$I_\nu = I_\nu(0)e^{-\tau_\nu} + B_\nu(T)(1 - e^{-\tau_\nu}). \tag{1.72}$$

Now, if $I_\nu(0) > B_\nu(T)$ then

$$I_\nu = [I_\nu(0) - B_\nu(T)]\, e^{-\tau_\nu} + B_\nu(T) \tag{1.73}$$

is smaller at larger τ, and so would give absorption lines at frequencies where α_ν is larger than the continuum absorption coefficient. (See also problem 1.9 in Rybicki and Lightman). If $I_\nu(0) < B_\nu(T)$ on the other hand (e.g. the source is cooler than the layer of gas), then the brightness will be larger at frequencies with a higher optical depth, and an emission line spectrum would be seen.

1.6. Scattering

So far, we have not included scattering processes. The reason is that in general including scattering introduces significant complexity in solving the equation of radiative transfer. Scattering abruptly changes the direction and possibly frequency of the scattered photon, resulting in an integro-differential equation as we will see below.

1.6.1. Scattering only

Start by considering a medium with no emission or absorption, scattering only. We will assume that the scattering is *monochromatic*, *coherent*, *elastic* – i.e. no change in photon frequency occurs on scattering, only direction. In general, we can write

$$\frac{dI_\nu(\vec{k})}{ds} = -\sigma_\nu I_\nu(\vec{k}) + \sigma_\nu \int \phi_\nu\left(\vec{k}, \vec{k}'\right) I_\nu\left(\vec{k}'\right) d\Omega' \tag{1.74}$$

where σ_ν is the *scattering coefficient*, the first term on the RHS describes photons removed from the beam by scattering, and the second term describes photons added to the beam by scattering from other directions. The function ϕ gives the probability of scattering from initial direction $\vec{k}'$ into direction $\vec{k}$, and is normalized such that $\int d\Omega \phi(\vec{k}, \vec{k}') = \int d\Omega' \phi(\vec{k}, \vec{k}') = 1$.

For simplicity, we will assume isotropic scattering, for which $\phi = 1/4\pi$ is a constant, i.e. all scattering angles are equally likely. In that case,

$$\frac{dI_\nu}{ds} = -\sigma_\nu I_\nu + \sigma_\nu J_\nu. \tag{1.75}$$

If we define an optical depth $d\tau_\nu = \sigma_\nu ds$, then

$$\frac{dI_\nu}{d\tau_\nu} = -I_\nu + S_\nu \tag{1.76}$$

with the source function for scattering $S_\nu = J_\nu$.

The number of scatterings required to escape a medium depends on whether it is optically thick or optically thin. In the optically thick case, $\tau \gg 1$, the scattering photon executes a random walk, and the number of scatterings is given by $L = \sqrt{N} l$, or $N = (L/l)^2 = \tau^2$. In the optically thin case, $\tau \ll 1$, the chance of scattering is l/L which gives $N \approx \tau$.

1.6.2. Scattering and absorption

Putting scattering and absorption terms into the radiative transfer equation gives

$$\frac{dI_\nu}{ds} = -\sigma_\nu I_\nu + \sigma_\nu J_\nu + j_\nu - \alpha_\nu I_\nu. \tag{1.77}$$

If the gas is in LTE, then $j_\nu = \alpha_\nu B_\nu$, and

$$\frac{dI_\nu}{ds} = \alpha_\nu (B_\nu - I_\nu) + \sigma_\nu (J_\nu - I_\nu) \tag{1.78}$$

$$= -(\alpha_\nu + \sigma_\nu)(I_\nu - S_\nu) \tag{1.79}$$

where we have defined a *source function for absorption and scattering*,

$$S_\nu = \frac{\alpha_\nu B_\nu + \sigma_\nu J_\nu}{\alpha_\nu + \sigma_\nu}. \tag{1.80}$$

If we define the total optical depth $d\tau_\nu = (\alpha_\nu + \sigma_\nu) ds$, then we recover the same form of the transfer equation as earlier

$$\frac{dI_\nu}{d\tau_\nu} = -I_\nu + S_\nu. \tag{1.81}$$

We can check the limits of this expression: if $J_\nu \approx B_\nu$ then $S_\nu \approx B_\nu$; if $J_\nu \approx 0$ then $S_\nu \approx B_\nu \alpha_\nu/(\alpha_\nu + \sigma_\nu) < B_\nu$.

Another way to write the source function is to define the absorption probability $\epsilon_\nu = \alpha_\nu/(\alpha_\nu + \sigma_\nu)$. The source function is then $S_\nu = \epsilon_\nu B_\nu + (1 - \epsilon_\nu) J_\nu$.

Now think about the random walk of a photon in a gas with scattering and absorption. The number of steps before being absorbed is $1/\epsilon_\nu$, giving the mean free path to absorption $l_\star = \sqrt{N} l = l/\sqrt{\epsilon_\nu}$. But $l = 1/(\alpha_\nu + \sigma_\nu)$, and so

$$l_\star = \frac{1}{\sqrt{\alpha_\nu(\alpha_\nu + \sigma_\nu)}}. \tag{1.82}$$

This length is known as the *diffusion length*, *thermalization length*, or the *effective mean free path*. The *effective optical thickness* is $\tau_\star = L/l_\star = \sqrt{\epsilon}\tau$. When $\tau_\star \ll 1$, most photons escape without being absorbed (but they might scatter multiple times depending on the value of τ). This implies a luminosity $L = 4\pi\alpha_\nu B_\nu V$ where V is the volume. For $\tau_\star \gg 1$, we expect $I_\nu \to B_\nu$ and $S_\nu \to B_\nu$, giving a luminosity $L \approx 4\pi\alpha_\nu B_\nu (A l_\star)$ where $A l_\star$ is the volume from which photons can escape, or since $\alpha_\nu l_\star = \sqrt{\epsilon_\nu}$, we get $L \approx 4\pi\sqrt{\epsilon_\nu} B_\nu A$. For $\epsilon_\nu = 1$, we should get $L = \pi B_\nu A$, so this estimate is off by a factor of 4, but the important point is that we see that when scattering is included, the emissivity is reduced by a factor of $\sqrt{\epsilon_\nu}$. There are two competing effects. First, the emitting volume near the surface is increased by scattering, since the depth from which photons escape is $l/\sqrt{\epsilon}$. However, the mean free path l is shorter by a factor of ϵ, so the overall emitting volume is actually smaller by $\sqrt{\epsilon}$.

Summary and Further Reading

Here are the main ideas and results that we covered in this part of the course:

- Specific intensity I_ν and its moments F_ν, P_ν, $U_\nu = 4\pi J_\nu/c$. Source function, $S_\nu = j_\nu/\alpha_\nu$. Outwards flux $F_\nu = \pi I_\nu$ for isotropic I_ν. Closure relations: $P_\nu = U_\nu/3$, $P_\nu = U_\nu$.
- Mean free path $l = 1/\alpha = 1/n\sigma = 1/\rho\kappa$. Optical depth.
- Radiative transfer equation
$$\frac{dI_\nu}{ds} = j_\nu - \alpha_\nu I_\nu$$
- General solution. For constant source function,
$$I_\nu = I_\nu(0) e^{-\tau_\nu} + S_\nu \left(1 - e^{-\tau_\nu}\right),$$

giving $I_\nu \to S_\nu$ for $\tau \gg 1$ and $I_\nu \approx I_\nu(0) + \tau S_\nu$ for $\tau \ll 1$.

- Thermal radiation. $U = aT^4$, $P = (1/3)aT^4$. Properties of Planck spectrum. $\lambda_{\rm max} = 0.29\ {\rm cm}/T$. Kirchoff's Law $j_\nu = \alpha_\nu B_\nu$.
- Local thermodynamic equilibrium (LTE). Radiative diffusion equation. Rosseland mean opacity.
- Stellar atmospheres. The Eddington approximation and the source function and temperature profile of a grey atmosphere. $\tau = 2/3$ as the photosphere. Limb darkening.
- Emission lines from optically thin thermal gas. Conditions for forming absorption lines.
- Scattering as a random walk. Number of scatterings to escape $\max(\tau, \tau^2)$. Source function
$$S_\nu = \frac{\alpha_\nu B_\nu + \sigma_\nu J_\nu}{\alpha_\nu + \sigma_\nu} = \epsilon_\nu B_\nu + (1 - \epsilon_\nu) J_\nu.$$
Thermalization depth. Emissivity of a scattering atmosphere $F_\nu = \sqrt{\epsilon_\nu}\pi B_\nu$.

Reading

- Rybicki and Lightman, chapter 1.
- Chandrasekhar, S. "Radiative Transfer" Dover 1960. Classic treatise on radiative transfer. Analytic solution for grey atmosphere.
- Mihalas, D. "Stellar atmospheres" W.H. Freeman & Co. 1978. Now unfortunately out of print. Detailed treatment of the physics of atmospheres and also it tells you how to calculate a "real" stellar atmosphere.

Appendix: Chandra's exact solution for a grey atmosphere

In his book *Radiative Transfer*, Chandrasekhar presents a beautiful analytic solution to the grey atmosphere. We summarize it here, and compare the results with our solution derived using the Eddington approximation.

For a grey atmosphere, $S = J$ regardless of the degree of scattering or absorption. The transfer equation is
$$\mu \frac{dI}{d\tau}(\tau, \mu) = I(\tau, \mu) - \frac{1}{2}\int_{-1}^{1} d\mu' I(\tau, \mu') \tag{1.83}$$

an integral equation for $I(\tau, \mu)$.

If we follow only a finite set of μ values, the integral can be written as a sum,

$$\mu_i \frac{dI_i}{d\tau} = I_i - \frac{1}{2}\sum a_j I_j \tag{1.84}$$

for $i = \pm 1, \pm 2, ... \pm n$. Gaussian quadrature is used to choose the appropriate μ_i's and the corresponding a_i's. For

$$\int_{-1}^{1} f(\mu) d\mu = \sum_{j=1}^{m} a_j f(\mu_j) \tag{1.85}$$

the appropriate choice is to choose the μ_i to be the zeroes of $P_m(\mu)$, and

$$a_j = \frac{1}{P_m'(\mu_j)} \int_{-1}^{1} \frac{d\mu P_m(\mu)}{\mu - \mu_j} \tag{1.86}$$

(where $\sum_{j=1}^{m} a_j = 1$) (e.g. see Chandra's book or numerical recipes). This choice gives an exact solution for $f(\mu)$ a polynomial with order $< 2m$. For integrals of the form $\int e^{-x} f(x) dx$, the Laguerre polynomials are used instead. For any weight function (e^{-x} in this case), a set of μ_i and a_i can be constructed that solves the integral exactly for $f(x)$ a polynomial of degree $< 2m$. This is therefore a good technique for numerically integrating smooth functions. Note that Chandra uses the terminology "nth approximation" for $m = 2n$, i.e. we use the roots of $P_{2n}(\mu)$ for which $a_j = a_{-j}$ and $\mu_{-j} = -\mu_j$.

Chandra derived an analytic solution for equation (1.84). First, we look for a solution with exponential dependence on τ, $I_i = g_i e^{-k\tau}$, $i = \pm 1, ... \pm n$. Substituting this into equation (1.84) gives

$$g_i = \frac{\text{constant}}{1 + \mu_i k} \tag{1.87}$$

where k is determined by

$$1 = \sum_{j=1}^{n} \frac{a_j}{1 - \mu_j^2 k^2}. \tag{1.88}$$

There are $2n-2$ roots $\pm k_\alpha$, for $\alpha = 1, ... n-1$. We keep only the $k > 0$ roots, since we want I finite at large optical depth. There is also a solution linear in τ, $I_i = b(\tau + q_i)$, $i = \pm 1, ... \pm n$. Substituting into equation (1.84) gives $q_i = Q + \mu_i$ where Q is a constant. Therefore, the solution is

$$I_i = b\left[\sum_{\alpha=1}^{n-1} \frac{L_\alpha e^{-k_\alpha \tau}}{1 + \mu_i k_\alpha} + \tau + \mu_i + Q\right] \tag{1.89}$$

which has n constants Q and L_α to be determined. To fix their values, we set $I_{-i} = 0$ at $\tau = 0$ (no ingoing radiation at the surface) which gives

$$\sum_{\alpha=1}^{n-1} \frac{L_\alpha}{1 - \mu_i k_\alpha} - \mu_i + Q = 0 \qquad i = 1, ... n \tag{1.90}$$

Given the analytic expression for I_i, the flux, pressure and mean intensity can be calculated

$$\begin{aligned} F &= 2\pi \sum a_i \mu_i I_i & (1.91) \\ \frac{cP}{4\pi} = K &= \frac{1}{2} \sum a_i \mu_i^2 I_i = \frac{F}{4\pi} (\tau + Q) & (1.92) \\ J &= \frac{1}{2} \sum a_i I_i = \frac{3F}{4\pi} [\tau + q(\tau)] & (1.93) \end{aligned}$$

where

$$q(\tau) = Q + \sum_{\alpha=1}^{n-1} L_\alpha e^{-k_\alpha \tau}. \quad (1.94)$$

Putting $S = J$, we can then solve for $I(\mu)$ for all values of μ,

$$I(\tau, +\mu) = \frac{3F}{4\pi} \left[\sum_{\alpha=1}^{n-1} \frac{L_\alpha e^{-k_\alpha \tau}}{1 + k_\alpha \mu} + \tau + \mu + Q \right] \quad (1.95)$$

and at the surface

$$I(\mu) = \frac{3F}{4\pi} \left[\sum_{\alpha=1}^{n-1} \frac{L_\alpha}{1 + k_\alpha \mu} + \mu + Q \right] = \frac{3F}{4\pi} \frac{H(\mu)}{\sqrt{3}}. \quad (1.96)$$

The tables in Chandra's book give Q, L_α, and k_α for different n's. At all orders, the Hopf-Bronstein relation holds at $\tau = 0$, $J(0) = \sqrt{3}F/4\pi$.

To make a connection with our Eddington approximation solution, let's look at the first approximation ($n = 1$). Then looking at Chandra's table VIII we find $Q = 1/\sqrt{3}$, and we have two photon directions $\mu_\pm = \pm 1/\sqrt{3}$ which are the roots of $P_2(\mu)$ (there are no L_α's to consider for $n = 1$). This gives another way to understand the discussion in Rybicki and Lightman section 1.10, where they introduce a "two stream approximation" using these angles, motivating them by saying that this choice gives moments that satisfy the Eddington approximation. Another point to note is that this solution is very similar to the Eddington approximation solution, but with $Q = 1/\sqrt{3} = 0.58$ rather than $Q = 2/3 = 0.67$. The different approximations made in each case are that in Chandra's solution, the ratio $3P/U$ is not assumed to be constant, but only two photon directions are followed, whereas in the Eddington approximation the ratio $3P/U$ is fixed to be unity, but all photon angles are followed.

Let's check the limb darkening ratio. For the Eddington approximation, this was $I(\mu = 0)/I(\mu = 1) = 2/5$. In the first approximation $I(\mu = 0)/I(\mu = 1) = Q/(1+Q) = 0.37$. In the second approximation $n = 2$, we have $Q = 0.694$, $k_1 = 1.97$, and $L_1 = -0.117$. The angles are

$\mu_{\pm 1} = \pm 0.340$ and $\mu_{\pm 2} = \pm 0.861$. Then $I(\mu = 0)/I(\mu = 1) = (Q{+}L_1)/(1{+}Q{+}L_1/(1{+}k_1)) = 0.35$.

Next, look at the temperature profile which is given by equation (1.93) with $J = S = B$. Then

$$T^4 = \frac{3}{4} T_{\rm eff}^4 \left(\tau + q(\tau)\right). \tag{1.97}$$

In the Eddington approximation, $q(\tau) = 2/3$. In the first approximation $n = 1$, we have $q(\tau) = 1/\sqrt{3}$. In the second approximation,

$$q(\tau) = Q + L_1 e^{-k_1 \tau} \tag{1.98}$$

which changes from $Q + L_1 = 0.577$ at $\tau = 0$ to $Q = 0.694$ at large τ.

2. Radiation from Accelerating Charges

These are notes for part two of PHYS 642 Radiative Processes in Astrophysics. The basic physics underlying the radiation that we see is that *accelerating charges radiate*. The power radiated from a single charged particle q moving non-relativistically ($u \ll c$) is given by *Larmor's formula*

$$P = \frac{2q^2\dot{u}^2}{3c^3} \tag{2.1}$$

where $\dot{u}$ is the magnitude of the acceleration. In this section, we derive this equation, and use it to understand the emission from a thermal plasma due to scattering of electrons from ions, bremstrahlung radiation. Next , we consider emission from collections of particles using the multipole expansion. Applications include: X-ray emission from galaxy clusters, free-free opacity in stars, the spectra of compact HII regions, emission from spinning dust, and the spin down of radio pulsars.

2.1. Derivation of the radiation field of an accelerated charge

Here we go through the derivation of the radiation fields of an accelerating charge starting with Maxwell's equations. We'll skip a lot of the algebra and focus on the physical ideas.

In cgs units, Maxwell's equations are

$$\vec{\nabla} \cdot \vec{E} = 4\pi\rho \tag{2.2}$$

$$\vec{\nabla} \times \vec{B} = \frac{4\pi\vec{J}}{c} + \frac{1}{c}\frac{\partial\vec{E}}{\partial t} \tag{2.3}$$

$$\vec{\nabla} \times \vec{E} = -\frac{1}{c}\frac{\partial\vec{B}}{\partial t} \tag{2.4}$$

$$\vec{\nabla} \cdot \vec{B} = 0 \tag{2.5}$$

together with charge conservation

$$\frac{\partial\rho}{\partial t} + \vec{\nabla} \cdot \vec{J} = 0. \tag{2.6}$$

In these units, the Lorentz force is

$$\vec{F} = q\left(\vec{E} + \frac{\vec{v} \times \vec{B}}{c}\right). \tag{2.7}$$

It is convenient to work with potentials

$$\vec{\nabla} \times \vec{A} = \vec{B} \qquad \vec{E} = -\vec{\nabla}\phi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t} \tag{2.8}$$

(recall that the $\vec{B}$ field is always divergence free, but $\vec{E}$ can have both divergence and curl if $\partial\vec{B}/\partial t$ is non-zero).

Substituting these potentials into Maxwell's equations gives

$$\nabla^2 \vec{A} - \frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2} = -\frac{4\pi \vec{J}}{c} + \vec{\nabla}\left[\vec{\nabla}\cdot\vec{A} + \frac{1}{c}\frac{\partial\phi}{\partial t}\right] \tag{2.9}$$

$$\nabla^2 \phi - \frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} = 4\pi\rho + \frac{1}{c}\frac{\partial}{\partial t}\left[\vec{\nabla}\cdot\vec{A} + \frac{1}{c}\frac{\partial\phi}{\partial t}\right]. \tag{2.10}$$

I've written the equations this way to emphasize that ϕ and $\vec{A}$ satisfy the inhomogeneous wave equation

$$\nabla^2\left(\vec{A},\phi\right) - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\left(\vec{A},\phi\right) = 4\pi\left(-\frac{\vec{J}}{c},\rho\right) \tag{2.11}$$

if we choose

$$\vec{\nabla}\cdot\vec{A} + \frac{1}{c}\frac{\partial\phi}{\partial t} = 0. \tag{2.12}$$

In fact we are free to choose $\vec{\nabla}\cdot\vec{A}$ (only the curl of $\vec{A}$ which gives $\vec{B}$ is the physical quantity) in this way – this choice is the *Lorenz gauge*.

The solutions to equations (2.11) are the *retarded potentials*

$$\phi(\vec{r},t) = \int \frac{\rho\left(\vec{r'},t'\right) d^3\vec{r'}}{\left|\vec{r}-\vec{r'}\right|} \tag{2.13}$$

$$\vec{A}(\vec{r},t) = \frac{1}{c}\int \frac{\vec{J}\left(\vec{r'},t'\right) d^3\vec{r'}}{\left|\vec{r}-\vec{r'}\right|} \tag{2.14}$$

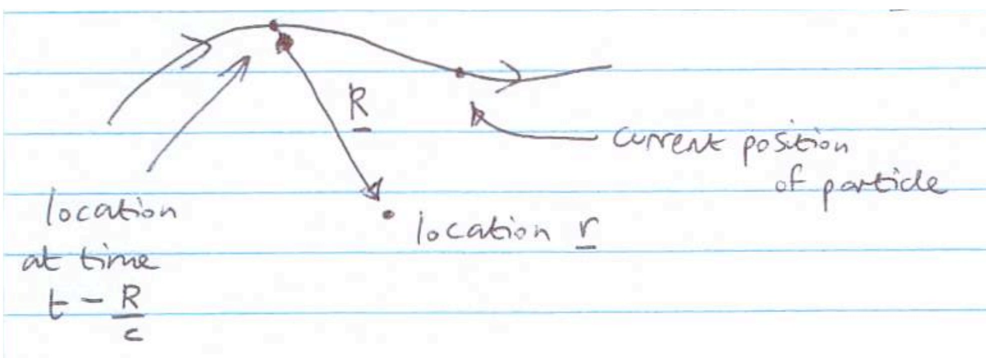
where the integrand is evaluated at the *retarded time*

$$t' = t - \frac{\left|\vec{r}-\vec{r'}\right|}{c}. \tag{2.15}$$

The physics of this makes sense – the relevant value of the source (ρ or $\vec{J}$) is the value a light travel time ago. Electromagnetic disturbances propagate at the speed of light. In electro- or magnetostatics, $t' \to t$, and the expressions for the potentials should be familiar.

For a point charge moving with velocity $\vec{u}$ along a path $\vec{r_0}(t)$, we can write

$$\rho(\vec{r},t) = q\,\delta\left(\vec{r}-\vec{r_0}(t)\right) \tag{2.16}$$
$$\vec{J}(\vec{r},t) = q\vec{u}\,\delta\left(\vec{r}-\vec{r_0}(t)\right). \tag{2.17}$$



Substituting these expressions into the potential integrals, and evaluating gives the *Lienard-Wiechart potentials* for a point charge,

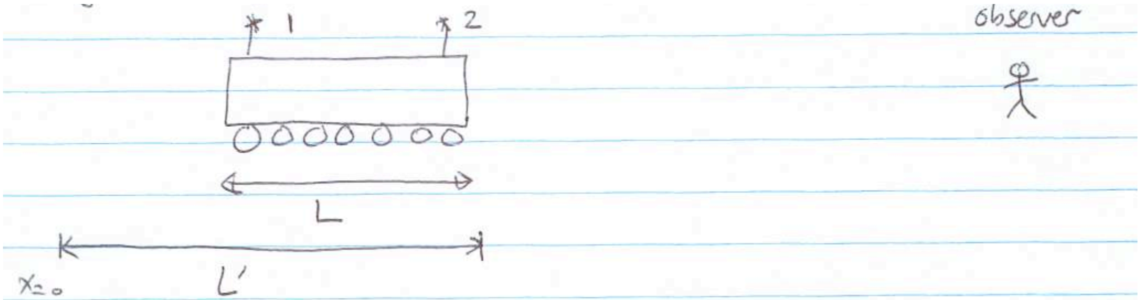
$$\phi = \left[\frac{q}{\kappa R}\right] \qquad \vec{A} = \left[\frac{q\vec{u}}{c\kappa R}\right] \tag{2.18}$$

where the notation [...] means that the quantity inside the brackets should be evaluated at the retarded time,

$$\kappa = 1 - \frac{\vec{n}\cdot\vec{u}}{c} \qquad R = |\vec{r}-\vec{r_0}(t)| \tag{2.19}$$

and $\vec{n} = \vec{e_R} = \vec{R}/\left|\vec{R}\right|$. This is the same idea as before – evaluate the source ρ or $\vec{J}$ at the retarded time, but now ρ and $\vec{J}$ are non-zero only on a particular track through space $\vec{r_0}(t)$. For $u \ll c$, we see that $\phi = [q/R]$ is the Coulomb potential.

The factor κ has a simple interpretation as the effect of the finite velocity on the apparent size of the volume element. The basic point is illustrated by the thought experiment where one measures the length of a moving train.



Photon 1 is emitted from the far side of the train at $t = 0$. It is straightforward to show that photon 2 from the front of the train should be emitted at position $x = L'$ if it is to arrive at the observer at the same time as photon 1, where

$$L' = \frac{L}{1 - v/c}. \tag{2.20}$$

The length L' is the apparent length of the train. The volume element $d^3\vec{r'}$ in the integral undergoes the same distortion

$$\int \rho d^3\vec{r'} = \frac{q}{1 - \vec{n} \cdot \vec{u}/c}. \tag{2.21}$$

We think the charge is spread out over a larger volume than it actually is.

Once we know the potentials, we can differentiate to find the fields. This is where we skip some algebra, and give the result,

$$\vec{E}(\vec{r}, t) = \left[\frac{q}{\kappa^3 R^2}\left(\vec{n} - \vec{\beta}\right)\left(1 - \beta^2\right)\right] + \left[\frac{q}{\kappa^3 Rc}\vec{n} \times \left\{\left(\vec{n} - \vec{\beta}\right) \times \frac{\partial \vec{\beta}}{\partial t}\right\}\right] \tag{2.22}$$

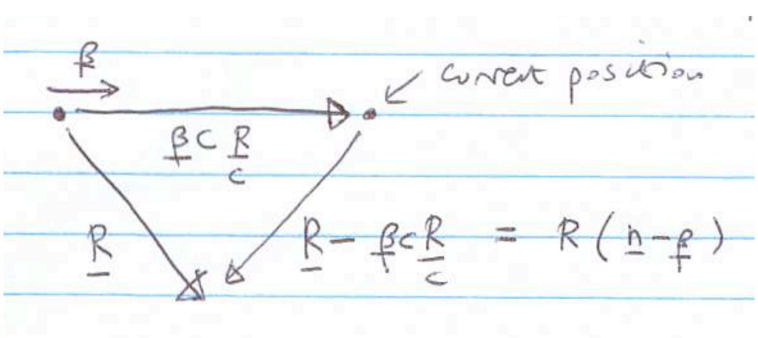
$$\vec{B}(\vec{r}, t) = \left[\vec{n} \times \vec{E}\right], \tag{2.23}$$

where $\vec{\beta} = \vec{u}/c$.

Let's take a closer look at each term and see if they make sense. The first term is known as the *velocity field*,

$$\vec{E_V} = \left[\frac{q}{\kappa^3 R^2}\left(\vec{n} - \vec{\beta}\right)\left(1 - \beta^2\right)\right]. \tag{2.24}$$

For $\beta \ll 1$, $\vec{E_V} = [q\vec{n}/R^2]$ which is Coulomb's law, and $\vec{B}$ is smaller by β than $\vec{E}$. The vector $\vec{n} - \vec{\beta}$ points to the current position of the particle:



Remarkably, for a particle moving at constant velocity, the electric field $\vec{E_V}$ points in the direction of the current position of the particle even though the electric field is determined by its position at the retarded time!

The second term, which involves $\partial\vec{\beta}/\partial t$, is called the *acceleration field* or *radiation field*,

$$\vec{E}_{\rm rad} = \left[\frac{q}{\kappa^3 Rc} \vec{n} \times \left\{ \left(\vec{n} - \vec{\beta} \right) \times \frac{\partial \vec{\beta}}{\partial t} \right\} \right]. \tag{2.25}$$

This is the term that gives rise to radiation. Since $E_{\rm rad} \propto 1/R$, the Poynting flux is $\propto 1/R^2$, giving a constant energy per unit area at large distance since the area of constant solid angle increases $\propto R^2$. The velocity field decreases as $1/R^2$ (as for the static Coulomb field), and does not therefore contribute at large R.

2.2. Radiation from non-relativistic particles: Larmor's formula

We will need the relativistic version of $\vec{E}_{\rm rad}$ later for emission by relativistic particles, but for now we assume $\beta \ll 1$, and therefore $\kappa = 1$, and we need not distinguish between the current time and the retarded time. Our aim is to calculate the Poynting flux

$$\vec{S} = \frac{c}{4\pi} \vec{E} \times \vec{B}. \tag{2.26}$$

Since $\vec{B} = \vec{n} \times \vec{E}$, then

$$\vec{S} = \frac{c}{4\pi} \left(\vec{E} \times \left(\vec{n} \times \vec{E} \right) \right) = \frac{c}{4\pi} \vec{n} \left| \vec{E} \right|^2 \tag{2.27}$$

where we use the fact that $\vec{n} \cdot \vec{E} = 0$.

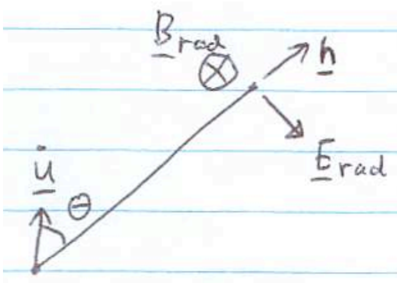
Now substitute

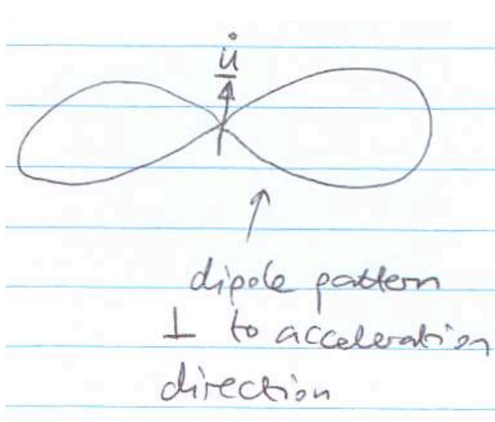
$$\vec{E}_{\rm rad} \approx \frac{q}{Rc} \vec{n} \times \left(\vec{n} \times \frac{\partial \vec{\beta}}{\partial t} \right) \tag{2.28}$$

gives

$$\vec{S} = \vec{n} \, \frac{q^2}{4\pi c^3 R^2} \left| \vec{n} \times \left(\vec{n} \times \dot{\vec{u}} \right) \right|^2. \tag{2.29}$$

The vector $\vec{n} \times (\vec{n} \times \dot{\vec{u}})$ is the piece of $\dot{\vec{u}}$ that is perpendicular to the direction of $\vec{n}$.





Therefore we see that the radiation electric field is perpendicular to the radial direction, which is why it leads to a radial Poynting flux.

Defining the angle Θ so that $\left|\vec{n} \times (\vec{n} \times \vec{\dot{u}})\right|^2 = \sin^2 \Theta$, we arrive at the final result

$$\vec{S} = \vec{n}\, \frac{q^2 \dot{u}^2}{4\pi R^2 c^3}\, \sin^2 \Theta, \tag{2.30}$$

which is the flux at distance R and angle Θ. Now since the area element is $dA = R^2 d\Omega$, then we can rewrite this as

$$\frac{dW}{dt d\Omega} = \frac{q^2 \dot{u}^2}{4\pi c^3}\, \sin^2 \Theta, \tag{2.31}$$

which is the power radiated per unit solid angle at angle Θ.

The total power radiated is given by integrating over all solid angles

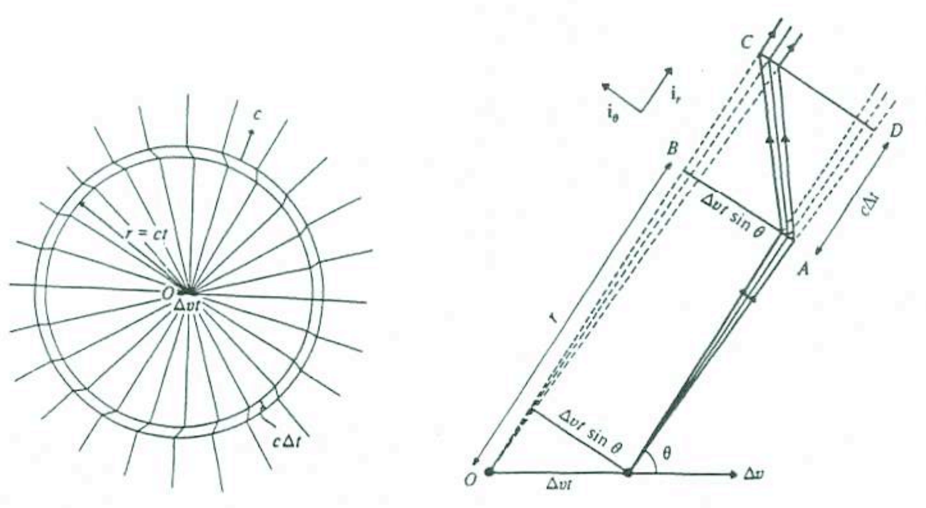
$$P = \frac{dW}{dt} = \frac{(q\dot{u})^2}{4\pi c^3} \int \sin \Theta^2 d\Omega \tag{2.32}$$

or

$$P = \frac{2 q^2 \dot{u}^2}{3c^3}, \tag{2.33}$$

which is Larmor's formula.

There is a nice graphical argument which can be used to obtain E_θ and therefore Larmor's formula, due to J. J. Thomson and presented in Longair's book *High Energy Astrophysics*. The idea is to consider accelerating a particle for time Δt by an amount Δv. In a frame moving at the original velocity of the particle, the particle now begins to move. The idea is that a time t later, there is a sphere at radius ct within which the electric field lines point to the current location of the particle, and outside which the electric field lines point back to the original location (the field doesn't "know" yet that the charge has moved).



The Figures you need are above, taken from Longair. The graphical argument gives

$$\frac{E_\theta}{E_r} = \frac{(\Delta v) t \sin\theta}{c\Delta t} \tag{2.34}$$

but $E_r = q/r^2 = q/(ct)^2$, and so $E_\theta = q\dot{u}\sin\theta/c^2 r$, exactly as we found earlier. This shows very nicely that E_θ/E_r grows with time $t \propto r$, and so $E_\theta \propto 1/r$.

2.3. The spectrum of the emitted radiation

Next, we consider the frequency spectrum of the radiation. As might be expected, the frequency spectrum is related to the time history of the acceleration of the particle. To see this, we write the power radiated per unit solid angle as

$$\frac{dW}{dt d\Omega} = \frac{c}{4\pi}\left|R\vec{E}\right|^2 = \left|\vec{A}(t)\right|^2 \tag{2.35}$$

(note that $\vec{A}$ is not the vector potential, but a temporary definition for this section; we are following the notation and argument of Jackson 14.5). The argument is to integrate over all time to get the total energy emitted per unit solid angle,

$$\frac{dW}{d\Omega} = \int_{-\infty}^{\infty} \left|\vec{A}(t)\right|^2 \, dt = \int_{-\infty}^{\infty} \left|\vec{A}(\omega)\right|^2 d\omega \tag{2.36}$$

where Parseval's theorem has been used to rewrite the integral in terms of the Fourier transform of $\vec{A}(t)$. We write the Fourier transforms as[6]

$$\vec{A}(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \vec{A}(\omega) e^{-i\omega t} d\omega \qquad \vec{A}(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \vec{A}(t) e^{i\omega t} dt. \tag{2.37}$$

If $\vec{A}(t)$ is real, then $\vec{A}(-\omega) = \vec{A}^*(\omega)$, and so we can integrate over positive frequencies over and multiply by two. The energy radiated per unit solid angle per unit frequency interval is therefore

$$\frac{dW}{d\omega d\Omega} = 2\left|\vec{A}(\omega)\right|^2. \tag{2.38}$$

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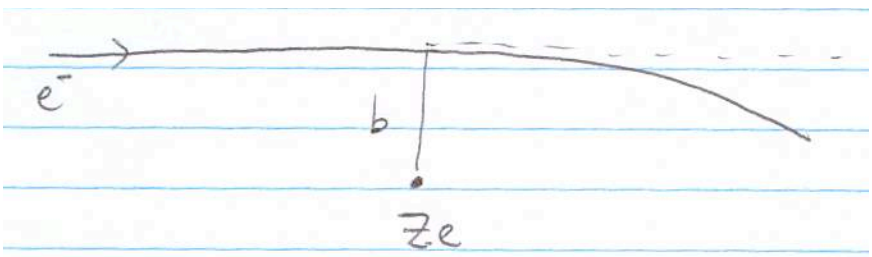
[6] As usual, beware of different normalizations used by different authors. I prefer the symmetry of putting $1/\sqrt{2\pi}$ in front of each integral in the pair; Rybicki and Lightman put the full $1/2\pi$ in front of the integral for $\vec{A}(\omega)$.

2.4. Thermal Bremsstrahlung

An important emission mechanism arises from Coulomb collisions between electrons and ions in a plasma. Consider an electron scattering from an ion, with impact parameter b. For the purposes of calculating the radiation, we assume that the path is undeviated (dashed line in the Figure), which is a good approximation since we are generally in the limit where small angle scattering dominates.



This gives us a chance to apply our formula for the frequency spectrum (eq. [2.38]) Since we assume the particle travels in a straight line, the angle to the observer Θ does not change with time. Therefore

$$\frac{dW}{d\omega d\Omega} = 2\left|\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{i\omega t}dt\frac{q\dot{u}}{(4\pi c^3)^{1/2}}\sin\Theta\right|^2 = \frac{q^2}{2\pi c^3}\left|\dot{u}(\omega)\right|^2\sin^2\Theta. \tag{2.39}$$

The angle of the incoming electron with respect to the line of sight is random, and so averaging over incoming angles is equivalent to integrating over the outgoing solid angle, giving the frequency spectrum of the radiation

$$\frac{dW}{d\omega} = \frac{4q^2}{3c^3}\left|\dot{u}(\omega)\right|^2 = \frac{4q^2}{3c^3}\left(\dot{u}_{\parallel}^2(\omega) + \dot{u}_{\perp}^2(\omega)\right) \tag{2.40}$$

where we write down the two components of the acceleration, perpendicular to the motion and parallel to the motion.

We parametrize the particle path so that at $t = 0$ the particle is at closest approach, distance b away from the ion. The perpendicular acceleration is then given by the perpendicular component of the Coulomb force,

$$\dot{u}_{\perp} = \frac{Ze^2}{m}\frac{1}{b^2 + u^2t^2}\frac{b}{\sqrt{b^2 + u^2t^2}} \tag{2.41}$$

where the last factor gives the perpendicular component of the force. The total change in velocity perpendicular to the path is given by

$$\Delta u_{\perp} = \frac{Ze^2}{m}\int_{-\infty}^{\infty}\frac{b\,dt}{(b^2 + u^2t^2)^{3/2}} = \frac{2Ze^2}{mbu} \tag{2.42}$$

It is straightforward to see that this is much larger than the acceleration parallel to the path. Energy conservation gives before and after scattering $u^2 = (u - \Delta u_\parallel)^2 + \Delta u_\perp^2$, or $\Delta u_\parallel / \Delta u_\perp \approx Ze^2/bmu^2$ which is small for small angle collisions.

Therefore,

$$\dot{u}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\omega t} dt \frac{Ze^2}{m} \frac{b}{(b^2 + u^2 t^2)^{3/2}} \tag{2.43}$$

$$= \frac{1}{\sqrt{2\pi}} \frac{Ze^2}{mub} \int_{-\infty}^{\infty} \frac{dx\, e^{ix\omega b/u}}{(1+x^2)^{3/2}} = \sqrt{\frac{2}{\pi}} \frac{Ze^2}{mub} y K_1(y) \tag{2.44}$$

where $y = \omega b/u$ and $K_1(y)$ is a modified Bessel function. The limits are $yK_1(y) \approx 1$ for low frequencies $\omega \ll v/b$, and $yK_1(y) \approx (y\pi/2)^{1/2} e^{-y}$ for $\omega \gg v/b$. That is the frequency spectrum is constant below $\omega = u/b$, and falls to zero for higher frequencies. This makes sense because since b is the distance of closest approach, b/v is the shortest timescale in the problem, and we might expect no higher frequency components. On the other hand, at low frequencies, the $e^{i\omega t}$ term is approximately unity, and $\dot{u}(\omega) \approx \Delta u/\sqrt{2\pi}$. Another way to look at it is that the interaction is strongly peaked around $t = 0$ when the particle is at closest approach, and therefore the frequency spectrum is very broad.

Substituting $\dot{u}_\perp(\omega)$ into equation (2.40) gives the spectrum for a single particle collision averaged over angles for a particle value of impact parameter b. The total emission rate per unit volume is

$$\frac{dW}{dt\ d\omega\ dV} = n_i n_e u \int_{b_{\rm min}}^{b_{\rm max}} 2\pi b\, db\, \frac{dW}{d\omega}(b) \tag{2.45}$$

$$= \frac{16 Z^2 e^6}{3 m^2 c^3 u} n_e n_i \int_{b_{\rm min}}^{b_{\rm max}} \frac{db}{b} \left[\left(\frac{\omega b}{u} \right) K_1 \left(\frac{\omega b}{u} \right) \right]^2 \tag{2.46}$$

An approximate way to write this is

$$\frac{dW}{dt\ d\omega\ dV} = \frac{16 Z^2 e^6}{3 m^2 c^3 u} n_e n_i \int_{b_{\rm min}}^{b_{\rm max}} \frac{db}{b} \tag{2.47}$$

for $\omega b/u < 1$, and zero for $\omega b/u > 1$. The integral over impact parameters gives the *Coulomb logarithm*

$$\Lambda = \ln \left(\frac{b_{\rm max}}{b_{\rm min}} \right), \tag{2.48}$$

which indicates a logarithmic divergence for large $b_{\rm max}$ that arises because the Coulomb force is a long range force.

How do we choose $b_{\rm max}$? To be consistent with our approximation for the integral, we should choose $b_{\rm max} = u/\omega$, that is only consider values of b that give a contribution to the spectrum at frequency ω. For $b_{\rm min}$ there are two possibilities. The "classical" approach is to choose $b_{\rm min}$ as the impact parameter where $\Delta u_\perp = u$, that is the impact parameter where a large angle scatter occurs, giving $b_{\rm min} = 2Ze^2/mu^2$. (This is also the distance of closest approach for a repulsive interaction.) However, if $b_{\rm min} = \hbar/m_e u$ is larger, then we should choose that instead. Then quantum mechanics sets $b_{\rm min}$. You can think of this as the impact parameter of a particle with one quantum of angular momentum. The changeover occurs when $2Ze^2/mu^2 = \hbar/mu$ or

$$\frac{1}{2}mu^2 = Z^2\left(\frac{e^2}{\hbar c}\right)^2 mc^2 = Z^2\alpha^2 mc^2 = Z^2(13.6\ {\rm eV}) \tag{2.49}$$

(where $\alpha = e^2/\hbar c$ is the fine structure constant). We see that the classical calculation is no longer appropriate when the electron energy exceeds $Z^2{\rm Ry}$.

In general, the emissivity is written as

$$\frac{dW}{dt\ d\omega\ dV} = \frac{16e^6}{3m^2c^3u}n_e n_i Z^2\left[\frac{\pi}{\sqrt{3}}g_{ff}\right] \tag{2.50}$$

where $g_{ff}(\omega, u)$ is the *Gaunt factor*. As our classical calculation indicated, the Gaunt factor is typically a slowly varying function of ω and u, so that the prefactor gives the major dependence. A classic paper which presents calculations of g_{ff} is Karzas and Latter (1961). The Gaunt factor is also plotted in Rybicki and Lightman's book. You'll see the parameter $\gamma = Z^2{\rm Ry}/k_BT$ which measures the transition between the classical and quantum regimes.

For a gas with a thermal distribution of velocities,

$$f(u)du = \left(\frac{m}{2\pi k_BT}\right)^{3/2}\exp\left(-\frac{mu^2}{2k_BT}\right)4\pi u^2 du \tag{2.51}$$

we can average over the velocity distribution to obtain the total emissivity due to *thermal bremsstrahlung*. However, we must be careful to cut off the velocity distribution at a minimum velocity $u_{\rm min}$ where $\hbar\omega = mu_{\rm min}^2/2$. This accounts for *photon discreteness*, that is the incoming electron must have enough energy to produce the photon of frequency ω. Combining equations (2.50) and (2.51), we can see that the answer will look like

$$\frac{dW}{dtdVd\omega} \propto \int_{u_{\rm min}}^{\infty} u^2du\ \exp\left(-\frac{mu^2}{2k_BT}\right)\frac{g_{ff}(u)}{u} \tag{2.52}$$

$$\propto \bar{g}_{ff}\int_{u_{\rm min}^2}^{\infty} d(u^2)\ \exp\left(-\frac{mu^2}{2k_BT}\right) \tag{2.53}$$

$$\propto \bar{g}_{ff}\exp\left(-\frac{\hbar\omega}{k_BT}\right) \tag{2.54}$$

for a suitably averaged Gaunt factor $\bar{g}_{ff}$. Keeping the prefactors, the result is

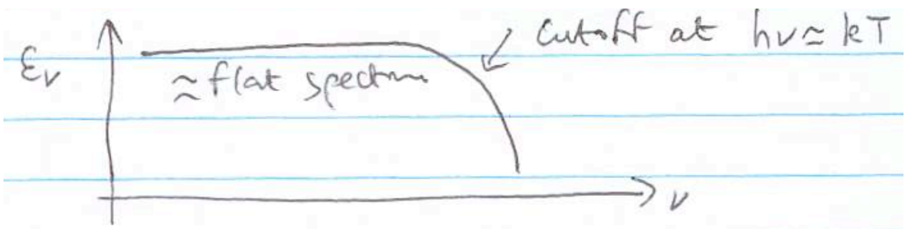
$$\frac{dW}{dtdVd\omega}=\frac{2^5\pi e^6}{3mc^3}\left(\frac{2\pi}{3k_Bm}\right)^{1/2}Z^2T^{-1/2}n_en_ie^{-\hbar\omega/k_BT}\bar{g}_{ff} \tag{2.55}$$

or

$$\epsilon_\nu^{ff}=6.8\times10^{-38}\ \mathrm{erg\ s^{-1}\ cm^{-3}\ Hz^{-1}}\ Z^2n_en_iT^{-1/2}e^{-h\nu/k_BT}\bar{g}_{ff} \tag{2.56}$$

where $\bar{g}_{ff}$ is the *thermally-averaged Gaunt factor*, ϵ_ν is the emissivity (where $j_\nu=\epsilon_\nu/4\pi$). We write "ff" for "free-free" which refers to the fact that we can think of the electron as making a transition between states in the continuum.

We see from equation (2.56) that the thermal bremsstrahlung spectrum is approximately flat at low frequencies, with a cutoff at $h\nu\approx k_BT$.



The spectrum is not completely flat at low frequencies, there is a small slope set by the frequency dependence of the Gaunt factor.

The total power per unit volume is $\int d\nu\,\epsilon_\nu^{ff}$, which gives

$$\frac{dW}{dtdV}=\left(\frac{2\pi k_BT}{3m}\right)^{1/2}\frac{2^5\pi e^6}{2hmc^3}Z^2n_en_i\bar{g}_B \tag{2.57}$$

or

$$\epsilon^{ff}=1.4\times10^{-27}\ \mathrm{erg\ s^{-1}\ cm^{-3}}\ T^{1/2}n_en_iZ^2\bar{g}_B, \tag{2.58}$$

where $\bar{g}_B(T)$ is the thermally-averaged and frequency-averaged Gaunt factor.

The classical example of gas emitting thermal bremsstrahlung is hot gas in Galaxy cliusters. There is a question on this in HW2 in which you can work out the details. The turnover in the spectrum gives a measure of the temperature of the gas, as a function of position in the cluster, and the total luminosity tells you about the gas mass.

2.5. Free-free absorption opacity

By Kirchoff's law, we know that there must be an absorption process corresponding to bremsstrahlung or free-free emission, which is known as free-free absorption. The absorption coefficient is given by

$$\alpha_\nu^{ff} = \frac{j_\nu^{ff}}{B_\nu(T)} = \frac{1}{4\pi B_\nu(T)}\frac{dW}{dtdVd\nu} \quad (2.59)$$

$$= \frac{4e^6}{3mch}\left(\frac{2\pi}{3k_Bm}\right)^{1/2} T^{-1/2}Z^2n_en_i\nu^{-3}\left(1-e^{-h\nu/k_BT}\right)\bar{g}_{ff} \quad (2.60)$$

$$= 3.7\times10^8\ \mathrm{cm^{-1}}\ T^{-1/2}Z^2n_en_i\nu^{-3}\left(1-e^{-h\nu/k_BT}\right)\bar{g}_{ff}. \quad (2.61)$$

In the Rayleigh-Jeans limit, $h\nu \ll k_BT$, $\alpha_\nu^{ff} = 0.018\ \mathrm{cm^{-1}}\ T^{-3/2}Z^2n_en_i\nu^{-2}\bar{g}_{ff}$.

In optically thick regions, the Rosseland mean opacity is the relevant quantity. Recall that the Rosseland mean is defined by

$$\left[\int d\nu\frac{dB_\nu}{dT}\right]\frac{1}{\alpha_R} = \left[\int d\nu\ \frac{1}{\alpha_\nu}\ \frac{dB_\nu}{dT}\right]. \quad (2.62)$$

The result is

$$\alpha_R^{ff} = 1.7\times10^{-25}\ T^{-7/2}Z^2n_en_i\bar{g}_R, \quad (2.63)$$

where the prefactor comes from Rybicki and Lightman. The $T^{-7/2}$ factor comes from the $T^{-1/2}\nu^{-3}$ dependence of the frequency-dependent opacity, since the averaging replaces $h\nu$ with a multiple of k_BT. For use in stellar interiors, it is more convenient to write down an expression for the opacity $\kappa_R = \alpha_R/\rho$. To do so, we write $n_e = \rho Y_e/m_p$, where Y_e is the number fraction of electrons, and $n_i = \rho Y_i/m_p$ where Y_i is the number fraction of nuclei. The opacity is then

$$\kappa_R^{ff} = 6.1\times10^{22}\ \mathrm{cm^2\ g^{-1}}\ \frac{\rho Y_e}{T^{7/2}}\,\bar{g}_R\sum_i\frac{X_iZ_i^2}{A_i}, \quad (2.64)$$

where the sum is over the charges Z_i, masses A_i, and mass fractions X_i of nuclei. In terms of the nuclear charges and masses, $Y_e = \sum_i X_iZ_i/A_i$ and $Y_i = \sum_i X_i/A_i$. **The prefactor here doesn't agree with Clayton or Itoh who have 7.53×10^{22}**. The result $\kappa \propto \rho T^{-7/2}$ is known as *Kramer's law*. A rough rule is that free-free absorption is important in stars less massive than the Sun, and Thomson scattering in stars more massive than the Sun. This is shown in HW2, where you will see that it results in a change in the slope of the luminosity-mass relation for main sequence stars at around 1 $M_\odot$.

Another application to mention is to compact HII regions, which can be optically thick to free-free absorption at low frequencies ($\alpha_\nu^{ff} \propto \nu^{-2}$ at low frequencies). They are said to be *self-absorbed* and this gives a falling spectrum at low frequencies, as you will see in HW 2.

2.6. Multipole radiation

So far we have dicussed radiation resulting from acceleration of a single particle. We now turn to a collection of particles, and use the multipole expansion to evaluate the radiated power.

First, a reminder of the multipole expansion in electrostatics or magnetostatics. The electrostatic potential at a large distance from a charge distribution can be expanded as

$$\phi(\vec{r}) = \int \frac{\rho(\vec{r'})d^3\vec{r'}}{\left|\vec{r}-\vec{r'}\right|} = \frac{Q}{r} + \frac{\vec{e}_r\cdot\vec{p}}{r^2} + \frac{\vec{e}_r\cdot\mathbf{Q_2}\cdot\vec{e}_r}{r^3} + ... \tag{2.65}$$

where

$$Q = \int \rho(\vec{r'})d^3\vec{r'} \tag{2.66}$$

is the total charge,

$$\vec{p} = \int \rho(\vec{r'})\vec{r'}d^3\vec{r'} \tag{2.67}$$

is the electric dipole moment, and

$$(\mathbf{Q_2})_{ij} = \int \rho(\vec{r'})\left[3r'_ir'_j - r'^2\delta_{ij}\right]d^3\vec{r'} \tag{2.68}$$

is the electric quadrupole moment tensor. Similarly, for a current distribution, the vector potential can be expanded

$$\vec{A}(\vec{r}) = \frac{1}{c}\int \frac{\vec{J}(\vec{r'})d^3\vec{r'}}{\left|\vec{r}-\vec{r'}\right|} = \frac{\vec{m}\times\vec{e}_r}{r^2} + ... \tag{2.69}$$

where

$$\vec{m} = \frac{1}{2c}\int \vec{r'}\times\vec{J}(\vec{r'})d^3\vec{r'} \tag{2.70}$$

is the magnetic dipole moment. To derive these results, expand

$$\frac{1}{\left|\vec{r}-\vec{r'}\right|} \approx \frac{1}{r}\left(1 + \frac{\vec{e}_r\cdot\vec{r'}}{r} + \frac{3(\vec{e}_r\cdot\vec{r'})^2 - r'^2}{r^2} + ...\right). \tag{2.71}$$

The idea is to now do something similar for the time-dependent case, in particular to expand the retarded potentials (eqs. [2.13] and [2.14]) and therefore radiation fields as a sum of multipole components. In the time-dependent case, there is a new lengthscale in the problem, which is the wavelength of the emitted radiation λ. We will assume that the size of the emitting region $d \ll \lambda \ll r$, and that the particles are non-relativistic. (In other words, the light-crossing time d/c is much smaller than the wave period $2\pi/\omega = 2\pi/ck = \lambda/c$.)

We start by looking at an individual Fourier components $\vec{J}(\vec{r}, t) = \vec{J}(\vec{r})e^{-i\omega t}$ etc. The spatial part of the vector potential is then, from equation (2.14),

$$\vec{A}(\vec{r}) = \frac{1}{c}\int \vec{J}(\vec{r'})d^3\vec{r'}\frac{e^{ik\left|\vec{r}-\vec{r'}\right|}}{\left|\vec{r}-\vec{r'}\right|} \tag{2.72}$$

since the integrand is evaluated at the retarded time ($t' = t - \left|\vec{r} - \vec{r'}\right|/c$). Our approach will be to calculate $\vec{A}(\vec{r})$ and then obtain the fields from

$$\vec{B} = \vec{\nabla} \times \vec{A} \qquad \vec{E} = \frac{i}{k}\vec{\nabla} \times \vec{B} \tag{2.73}$$

(this is simpler than expanding ϕ and using that to obtain $\vec{E}$). The relation between $\vec{E}$ and $\vec{B}$ holds since outside the source there are no currents and $\partial\vec{E}/\partial t = c\vec{\nabla} \times \vec{B}$.

2.6.1. Electric dipole

We start with the *electric dipole* term by writing $\left|\vec{r} - \vec{r'}\right| \approx r$. This means that we ignore variations in the retarded time across the source. Then

$$\vec{A}(\vec{r}) \approx \frac{1}{rc}e^{ikr}\int \vec{J}(\vec{r'})d^3\vec{r'}. \tag{2.74}$$

To simplify this term, integrate by parts using $\vec{\nabla} \cdot (r_i\vec{J}) = r_i\vec{\nabla} \cdot \vec{J} + \vec{J} \cdot \vec{\nabla}r_i = i\omega\rho r_i + J_i$. The surface term vanishes, giving

$$\vec{A}(\vec{r}) = -\frac{ik}{r}e^{ikr}\int \rho(\vec{r'})\vec{r'}d^3\vec{r'} = -\frac{ik\vec{p}e^{ikr}}{r}. \tag{2.75}$$

The radiation fields are therefore

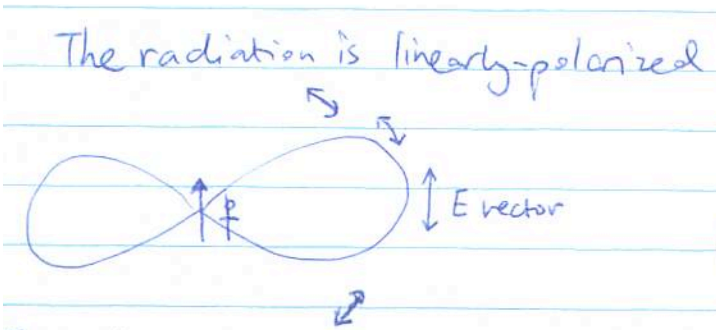
$$\vec{B} = \frac{k^2e^{ikr}}{r}\vec{n} \times \vec{p} \tag{2.76}$$

$$\vec{E} = \vec{B} \times \vec{n} = \frac{k^2e^{ikr}}{r}\vec{n} \times (\vec{p} \times \vec{n}). \tag{2.77}$$

The power radiated is

$$\frac{dP}{d\Omega} = \frac{c}{8\pi}\mathrm{Re}\left[r^2\vec{n} \cdot \vec{E} \times \vec{B}^\star\right] = \frac{c}{8\pi}k^4\left|\vec{p}\right|^2\sin^2\theta \tag{2.78}$$

where in the first step we have included a factor of 1/2 to give the time-average value, and in the second step, we assume that all components of $\vec{p}$ have the same phase.



Integrating over angles gives the total power

$$P = \frac{\omega^4 \left|\vec{p}\right|^2}{3c^3}. \tag{2.79}$$

(Note that we would get the same answer by applying Larmor's formula, since the dipole moment for a set of charges is $\vec{p} = \sum q_i \vec{r}_i$.)

2.6.2. Magnetic dipole and electric quadrupole

Now take the next term in the expansion. We only need to expand the exponent $k\left|\vec{r} - \vec{r'}\right|$ because this is $(kr)^{-1}$ times larger than the term in the expansion of $1/\left|\vec{r} - \vec{r'}\right|$. The expansion is $k\left|\vec{r} - \vec{r'}\right| = kr - k\vec{e}_r \cdot \vec{r'} + ...$, which gives

$$\vec{A}\left(\vec{r}\right) = -\frac{ik}{rc}e^{ikr}\int \vec{J}(\vec{r'})\vec{n}\cdot\vec{r'}d^3\vec{r'}. \tag{2.80}$$

We again evaluate this by integrating by parts. First, evaluate the surface term $\vec{\nabla}\cdot(r_i r_j \vec{J}) = r_i r_j \vec{\nabla}\cdot\ \vec{J} + (\vec{J}\cdot\vec{\nabla}r_i)r_j + (\vec{J}\cdot\vec{\nabla}r_j)r_i = r_i r_j \vec{\nabla}\cdot\ \vec{J} + J_i r_j + J_j r_i$ or

$$\frac{1}{2}\left(J_i r_j - J_j r_i\right) = -J_j r_i - \frac{1}{2} r_i r_j \vec{\nabla}\cdot\ \vec{J}. \tag{2.81}$$

Now dot this with $\vec{n}$,

$$\frac{1}{2}\left(\vec{r}(\vec{n}\cdot\vec{J}) - \vec{J}(\vec{r}\cdot\vec{n})\right) = \frac{1}{2}\vec{n}\times\left(\vec{r}\times\vec{J}\right) = -\vec{J}(\vec{r}\cdot\vec{n}) - \frac{1}{2}\vec{r}(\vec{r}\cdot\vec{n})\vec{\nabla}\cdot\ \vec{J}. \tag{2.82}$$

The first term on the RHS is the one we want, since it appears in the integral equation (2.80). Therefore,

$$\int \vec{J}\left(\vec{r'}\right)\vec{n}\cdot\vec{r'}d^3\vec{r'} = -i\frac{\omega}{2}\int \vec{r'}\left(\vec{r'}\cdot\vec{n}\right)\rho(\vec{r'})d^3\vec{r'} + \frac{1}{2}\int \vec{n}\times\left(\vec{J}(\vec{r'})\times\vec{r'}\right)d^3\vec{r'} \tag{2.83}$$

The two terms on the RHS represent electric quadrupole radiation and magnetic dipole radiation respectively.

Let's take the magnetic dipole term first. Its contribution to $\vec{A}$ is

$$\vec{A}(\vec{r}) = -\frac{ik}{rc}e^{ikr}\frac{1}{2}\vec{n} \times \int \vec{J}(\vec{r'}) \times \vec{r'}d^3\vec{r'} = \frac{ik}{r}e^{ikr}\vec{n} \times \vec{m} \tag{2.84}$$

which gives the radiation fields

$$\vec{E} = -k^2(\vec{n} \times \vec{m})\frac{e^{ikr}}{r} \qquad \vec{B} = k^2(\vec{n} \times \vec{m}) \times \vec{n}\frac{e^{ikr}}{r} \tag{2.85}$$

and radiated power

$$\frac{dP}{d\Omega} = \frac{\omega^4}{8\pi c^3}|\vec{m}|^2 \sin^2\theta \qquad P = \frac{\omega^4 |\vec{m}|^2}{3c^3}, \tag{2.86}$$

the (time-averaged) power radiated by an oscillating magnetic dipole. Since the magnetic dipole moment is of order v/c compared to the electric dipole moment, we see that the power emitted in this term is $\sim (v/c)^2$ times the electric dipole emission.

The electric quadrupole contribution to $\vec{A}$ is

$$\vec{A} = -\frac{ik}{rc}e^{ikr}\left(-\frac{i\omega}{2}\right)\int \vec{r'}(\vec{r'} \cdot \vec{n})\rho(\vec{r'})d^3\vec{r'}. \tag{2.87}$$

The integral in this expression is the first term of $(\mathbf{Q}_2)_{ij}n_j \equiv \vec{Q}(\vec{n})$. The remaining term $\propto \delta_{ij}$ vanishes when we take the cross product of $\vec{A}$ with $\vec{k} = \vec{n}$ to find the magnetic field. Therefore

$$\vec{B} = -\frac{ik^3}{6}\frac{e^{ikr}}{r}\vec{n} \times \vec{Q}(\vec{n}). \tag{2.88}$$

The radiated power is

$$\frac{dP}{d\Omega} = \frac{c}{288\pi}k^6\left|(\vec{n} \times \vec{Q}(\vec{n})) \times \vec{n}\right|^2 \qquad P = \frac{ck^6}{360}\sum_{ij}|(\mathbf{Q_2})_{ij}|^2 \propto \omega^6. \tag{2.89}$$

The power emitted is a factor of $\sim (kd)^2$ smaller than the electric dipole emission.

2.7. Applications of multipole emission

2.7.1. Spinning dust emission

For example, see Draine and Lazarian (1998). In the interstellar medium, dust grains become charged due to photoionization and collisions. In general, the charge distribution has a different center than the mass distribution, which implies a net dipole moment. Small molecules also have intrinsic dipole moments. If the rotation axis and dipole axis are misaligned by angle θ, the radiated power is $P = (2/3)(\omega^4 p^2 \sin^2\theta/c^3)$ (this is a factor of 2 larger than eq. [2.79] since for rotation you can think of two components of $\vec{p}$ perpendicular to the rotation axis that vary).

To estimate the rotation frequency and therefore frequency of the emitted radiation, we note that for thermal equilibrium we expect $(1/2)I\omega^2 = (3/2)k_B T$, and the simplest estimate is to assume $I = (2/5)Ma^2$ where $M = 4\pi a^3\rho/3$. Draine and Lazarian (1998) assume $\rho = 2$ g cm^{-3}. The result is

$$\nu = 5.6\times 10^9\ \mathrm{Hz}\ \left(\frac{a}{10^{-7}\ \mathrm{cm}}\right)^{-5/2}\left(\frac{T}{100\ \mathrm{K}}\right)^{1/2}. \tag{2.90}$$

Therefore, we expect radiation in the GHz range (wavelengths of ≈ 10 cm). For estimates of the dipole moment, see Draine and Lazarian (1998). There are two contributions: intrinsic dipole moments and grain charging. A typical value is a Debye.

This emission mechanism is used to explain the 15–90 GHz anomalous emission which was correlated with 100 μm from dust. An important question is whether the emission is polarized, which could arise if the grains align with the local B field for example (Lazarian and Draine 2000).

2.7.2. Radio pulsar spin down

The standard way to estimate the magnetic field of radio pulsars is to assume that the star spins down due to magnetic dipole radiation, that is

$$\frac{d}{dt}\left(\frac{1}{2}I\omega^2\right) = \frac{2}{3}\frac{\omega^4\mu^2}{c^3}\sin^2\theta. \tag{2.91}$$

The magnetic moment of the star is $\mu = BR^3$ where B is the surface magnetic field strength at the equator. For a neutron star, $I \approx MR^2/5$ (Lattimer and Schutz 2005; this is 1/2 the value for a constant density sphere). The magnetic field can then be written in terms of the spin period of the star, $P = 2\pi/\omega$, and the spin period derivative $\dot{P}$,

$$B = \left(\frac{3}{10}\frac{\dot{P}P}{4\pi^2}\frac{Mc^3}{R^4\sin^2\theta}\right)^{1/2} = 2.4\times 10^{19}\ \mathrm{G}\ (P\dot{P})^{1/2}\left(\frac{M}{1.4\ M_\odot}\right)^{1/2}\left(\frac{R}{10\ \mathrm{km}}\right)^{-2}\frac{1}{\sin\theta}. \tag{2.92}$$

Since $\dot{\omega}\propto\omega^3$, the *braking index* $n \equiv \ddot{\omega}\omega/\dot{\omega}^2$ is predicted to have the value 3. The measured values are all less than 3, although only a few have been measured so far. If the initial period is much smaller than the current period, the age of the pulsar is

$$\tau = \frac{P}{2\dot{P}} = 9\times 10^6\ \mathrm{yrs}\ \left(\frac{P}{1\ \mathrm{s}}\right)^2 \left(\frac{B}{10^{12}\ \mathrm{G}}\right)^{-2}. \tag{2.93}$$

In fact, the spin down of the pulsar is not as a vacuum dipole because the magnetosphere is filled with plasma! Even an aligned rotator ($\sin\theta = 0$) spins down, by driving a wind through the light cylinder. This is a complex theoretical problem that has begun to be solved only recently. Numerical simulations by Spitkovsky (2006) find

$$B = 2.6\times 10^{19}\ \mathrm{G}\ (P\dot{P})^{1/2}(1+\sin^2\theta)^{-1/2} \tag{2.94}$$

amazingly close to the vacuum dipole spin down value.

Summary and Further Reading

Here are the main ideas and results that we covered in this part of the course:

- *Radiation from an accelerated charge.* Retarded time and potentials. Velocity and radiation fields for a point charge. The velocity field always points to the *current* position of the charge.

- Poynting flux $\vec{S} = c\vec{E}\times\vec{B}/4\pi$. Larmor's formula

$$\frac{dP}{d\Omega} = \frac{q^2\dot{u}^2}{4\pi c^3}\sin^2\Theta \qquad P = \frac{2q^2\dot{u}^2}{3c^3}$$

- *Bremsstrahlung.* Flat spectrum with cutoff at $\omega = v/b$. Below the cutoff,

$$\frac{dW}{dt dV d\omega} = \frac{16e^6}{3c^3m^2v}n_e n_i Z^2\left[\frac{\pi g_{ff}}{\sqrt{3}}\right].$$

  The Gaunt factor $g_{ff}\approx \ln(b_{\rm max}/b_{\rm min})$. The physics setting $b_{\rm min}$ and $b_{\rm max}$.

- Thermal bremsstrahlung:

$$\epsilon_\nu^{ff} = 6.8\times 10^{-38}\ Z^2 n_e n_i T^{-1/2}e^{-h\nu/k_BT}\bar{g}_{ff}\ \mathrm{erg\ s^{-1}\ cm^{-3}\ Hz^{-1}}$$

$$\epsilon^{ff} = 1.4\times 10^{-27}\ T^{1/2}n_e n_i Z^2\bar{g}_B\ \mathrm{erg\ s^{-1}\ cm^{-3}}.$$

  Application to cluster gas.

- Free-free absorption:

$$\kappa_\nu^{ff} = 3.7 \times 10^8 \; \frac{Z^2 n_e n_i}{\rho T^{1/2} \nu^3} \left(1 - e^{-h\nu/k_B T}\right) \bar{g}_{ff}.$$

  Rosseland mean free-free opacity:

$$\kappa_R^{ff} = 1.7 \times 10^{-25} \; \frac{Z^2 n_e n_i}{\rho T^{7/2}} \bar{g}_R \text{ cm}^2 \text{ g}^{-1} \propto \rho T^{-3.5}.$$

  Self-absorption at low frequencies giving $\epsilon_\nu^{ff} \propto \nu^2$. Example: compact HII regions.

- *Multipole radiation.* Physics of the electric dipole approximation. Power radiated by oscillating electric and magnetic dipoles, and polarization of the radiation. $dP/d\Omega = (\ddot{p}^2/4\pi c^3)\sin^2\theta$, $P = 2\ddot{p}^2/3c^3$. Application to spinning charged dust grains and radio pulsars.

Reading

- Rybicki and Lightman, chapters 3 and 5. Longair p64 gives a nice pictorial argument for Larmor's formula. See also of course Jackson.
- Karzas & Latter 1961, ApJS 6, 167 calculate the free-free Gaunt factor.
- Draine & Lazarian 1998, and Lazarian & Draine 2000 calculate spinning dust emission. See Dickinson et al. 2006 ApJL for recent observations.

3. Compton Scattering

These are notes for part three of PHYS 642 Radiative Processes in Astrophysics. We cover Compton scattering and its applications. An excellent reference is the review article by Blumenthal and Gould (1970 Rev Mod Phys).

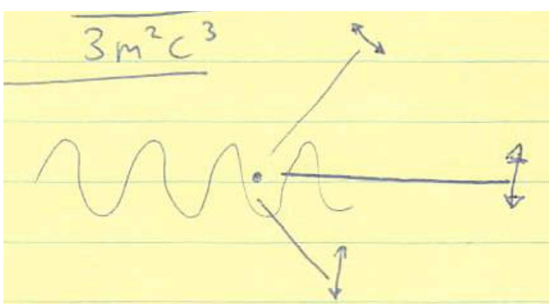
3.1. Thomson scattering

We mentioned earlier in the course that the cross-section for scattering of a photon by an electron is the *Thomson cross-section* $\sigma_T = 8\pi r_0^2/3$ where $r_0 = e^2/m_e c^2$ is the classical electron radius.

Rybick and Lightman give a simple derivation of the cross-section in section 3.4, that we go through here. We consider the response of a free electron to an incident electromagnetic wave. The force on the electron is $\vec{F} = e\vec{E}\sin\omega t = m\ddot{\vec{r}}$, and therefore the time-averaged acceleration is given by $\ddot{r}^2 = (eE/m)^2/2$. Substituting this into Larmor's formula gives the power radiated

$$\frac{dP}{d\Omega} = \frac{e^4}{8\pi m^2 c^3} E^2 \sin^2\theta \qquad P = \frac{e^4 E^2}{3m^2c^3}, \tag{3.1}$$

where the angle θ is measured relative to the electric field (and therefore acceleration) direction. The radiation is polarized in the plane of the incident wave:

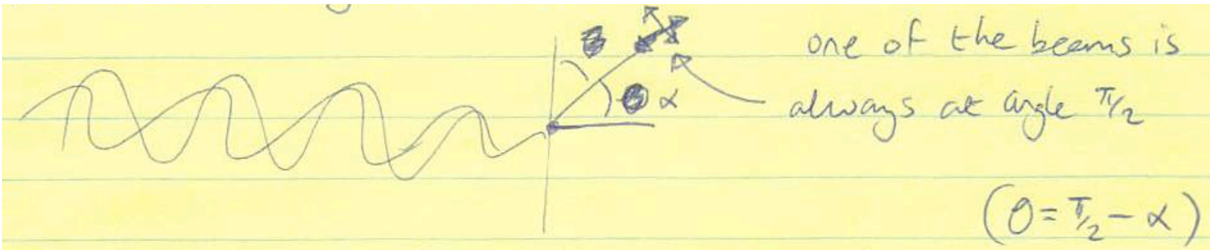


We can obtain the cross-section by dividing the radiated power by the incident flux. The incident flux is given by the time-average Poynting vector for the wave, $\langle S\rangle = cE^2/8\pi$, and therefore the differential cross-section and total cross-sections are

$$\frac{d\sigma}{d\Omega} = \frac{1}{\langle S\rangle}\frac{dP}{d\Omega} = r_0^2\sin^2\theta \qquad \sigma = \frac{8\pi}{3}r_0^2. \tag{3.2}$$

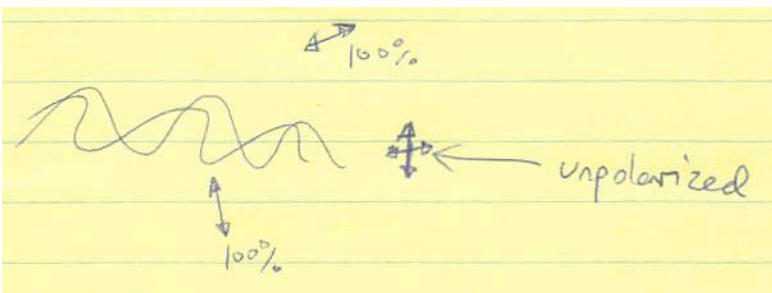
The numerical value is $\sigma_T = 6.63\times 10^{-25}\ \mathrm{cm}^2$.

An unpolarized beam can be thought of as a superposition of two uncorrelated orthogonal waves.



If we consider scattering at an angle α as shown in the diagram, one of the beams is always at angle $\pi/2$ compared to the incoming radiation, and so the differential cross-section is

$$\frac{d\sigma}{d\Omega} = \frac{1}{2}\left(\frac{d\sigma}{d\Omega}(\theta) + \frac{d\sigma}{d\Omega}\left(\frac{\pi}{2}\right)\right) = \frac{r_0^2}{2}\left(1 + \cos^2\alpha\right) \tag{3.3}$$



Note that the scattering is symmetric with respect to the forward and backward directions.

The time-averaged energy density in the incident wave is $E^2/8\pi$, so that the total power in equation (3.1) can be written as

$$P = \sigma_T c U_{\rm rad} \tag{3.4}$$

where $U_{\rm rad}$ is the radiation energy density. This is a general result, since the total Thomson cross-section does not depend on the direction or the polarization of the incoming radiation.

The Thomson cross-section is appropriate for photon energies $\epsilon \ll m_e c^2$ (that is $\epsilon \ll$ 511 keV). As the photon energy approaches and exceeds $m_e c^2$, there are two effects to worry about: a suppression of the cross-section and a change in photon energy on scattering due to electron recoil.

The first of these requires a quantum-mechanical calculation of the cross-section (look in any introductory book on quantum field theory), which gives the *Klein-Nishina cross-section*

$$\frac{d\sigma}{d\Omega} = \frac{r_0^2}{2}\frac{\epsilon_f^2}{\epsilon_i^2}\left(\frac{\epsilon_i}{\epsilon_f} + \frac{\epsilon_f}{\epsilon_i} - \sin^2\alpha\right). \tag{3.5}$$

It reduces to equation (3.3) when the scattering is elastic (final photon energy ϵ_f is equal to the initial photon energy ϵ_i).

The final photon energy ϵ_f is given by a consideration of the kinematics of the scattering, which we look at in the next section. The resulting total cross-section is given by Rybicki and Lightman equation (7.5). The limits are

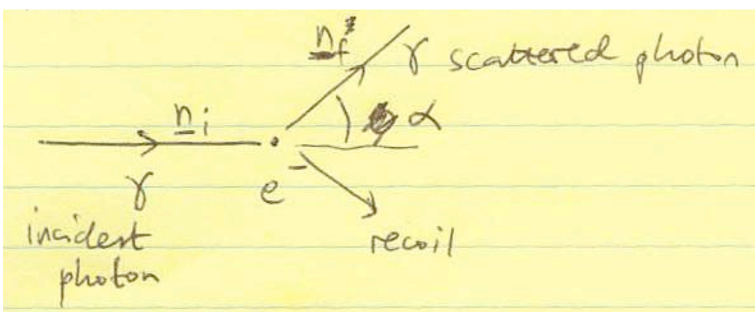
$$\sigma \approx \sigma_T(1 - 2x + ...) \qquad x \ll 1 \tag{3.6}$$

$$= \frac{3}{8}\sigma_T \frac{1}{x}\left(\ln 2x + \frac{1}{2}\right) \sim \sigma_T \left(\frac{m_e c^2}{\epsilon}\right) \qquad x \gg 1 \tag{3.7}$$

where $x = \epsilon_i/m_e c^2$. The cross-section is Thomson for low photon energies and suppressed at high photon energies.

3.2. Kinematics of Compton scattering

We first consider scattering of a photon from an electron at rest.



We write the initial and final 4-momenta of the photon as $\tilde{P}_i$ and $\tilde{P}_f$ and for the electron as $\tilde{Q}_i$ and $\tilde{Q}_f$. Then energy and momentum conservation is written as $\tilde{P}_i + \tilde{Q}_i = \tilde{P}_f + \tilde{Q}_f$. By expanding $\tilde{Q}_f^2 = (\tilde{P}_i + \tilde{Q}_i - \tilde{P}_f)^2$ and using $\tilde{Q}^2 = -m_e c^2$ and $\tilde{P}^2 = 0$, we find

$$\tilde{P}_i \cdot \tilde{P}_f = \tilde{Q}_i(\tilde{P}_i - \tilde{P}_f) \tag{3.8}$$

or

$$\frac{\epsilon_i \epsilon_f}{c^2}\left(-1 + \vec{n}_i \cdot \vec{n}_f\right) = m_e\left(-\epsilon_i + \epsilon_f\right). \tag{3.9}$$

In the last term we use the fact that $\tilde{Q}_i$ only has an energy component since the electron is initially at rest. Writing $\vec{n}_i \cdot \vec{n}_f = \cos\alpha$, we find

$$\epsilon_f = \frac{\epsilon_i}{1 + (\epsilon_i/m_e c^2)\left(1 - \cos\alpha\right)} \tag{3.10}$$

or in terms of photon wavelength we obtain the famous formula

$$\lambda_f - \lambda_i = \lambda_C(1 - \cos\alpha) \tag{3.11}$$

where $\lambda_C = h/m_e c$ is the Compton wavelength. Note that $\lambda_f > \lambda_i$ for all angles α, in other words the photon always loses energy in the collision.

3.3. Inverse Compton scattering

If the electron is moving with velocity v, energy can be transferred from the electron to the photon, which is known as *inverse Compton scattering.* In the electron rest frame, our previous result holds, but now written in terms of rest-frame variables which we indicate with a prime:

$$\epsilon'_f = \frac{\epsilon'_i}{1 + (\epsilon'_i/m_e c^2)\,(1 - \cos\alpha')}. \tag{3.12}$$

The angle $\alpha' = \theta'_f - \theta'_i$ is the scattering angle in the rest frame.

We just need to transform back into the lab frame. The angle θ_i is the initial angle between the electron and photon propagation directions, so that

$$\tilde{P}_i = \frac{\epsilon_i}{c}\,(1, \cos\theta_i, \sin\theta_i, 0)\,. \tag{3.13}$$

The Lorentz transform is

$$\tilde{P}'_i = \begin{pmatrix} \gamma & -\beta\gamma & 0 \\ -\beta\gamma & \gamma & 0 \\ 0 & 0 & 1 \end{pmatrix} \tilde{P}_i \tag{3.14}$$

giving

$$\tilde{P}'_i = \frac{\epsilon_i}{c}\,(\gamma(1 - \beta\cos\theta_i), -\gamma\beta + \gamma\cos\theta_i, \sin\theta_i, 0) \tag{3.15}$$

and

$$\epsilon'_i = \epsilon_i \gamma (1 - \beta\cos\theta_i). \tag{3.16}$$

The limits of this expression are (1) for $\theta_i \approx \pi$ (head on collision) $\epsilon'_i = \epsilon_i\gamma(1+\beta)$ or for large γ, $\epsilon'_i \approx 2\gamma\epsilon_i$, and (2) for $\theta_i \approx 0$ (photon approaches from behind) we get $\epsilon'_i = \epsilon_i\gamma(1-\beta) = \epsilon_i/(\gamma(1+\beta))$ or for large γ, $\epsilon'_i \approx \epsilon_i/2\gamma$. Similarly, the reverse transform gives

$$\epsilon_f = \epsilon'_f \gamma (1 + \beta\cos\theta'_f). \tag{3.17}$$

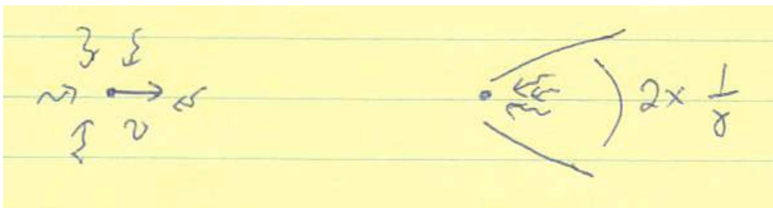
We see that the maximum energy we can expect is therefore $\epsilon_{f,\max} = 4\gamma^2\epsilon_i$.

The general rule is that the photon energies before scattering, in the electron rest frame, and after scattering are roughly in the ratios $1 : \gamma : \gamma^2$. A photon scattering from a relativistic electron can therefore undergo a tremendous increase in frequency, scattering radio photons into the optical to X-ray range for example, depending on the value of γ.

As well as being boosted in energy, the photon distribution is strongly beamed in the rest frame. Writing $\tilde{P}'_i = (\epsilon'_i/c)(1, \cos\theta'_i, \sin\theta'_i, 0)$, we see that $\cos\theta'_i\epsilon'_i/c = (\epsilon_i/c)(-\gamma\beta + \gamma\cos\theta_i)$ which gives the standard aberration formula (e.g. RL 4.86)

$$\cos\theta'_i = \frac{\cos\theta_i - \beta}{1 - \beta\cos\theta_i}. \tag{3.18}$$

(The same expression gives θ'_f in terms of θ_f.) Consider an isotropic distribution of photons in the lab frame. Half the photons have θ_i between π (head on collision) and $\pi/2$. In the rest frame, equation (3.18) gives $\cos\theta'_i = -\beta$ for $\theta_i = \pi/2$, or writing $\delta = \pi - \theta'_i$, we find that these same photons lie in a cone of half angle δ given by $\sin\delta = 1/\gamma$. Therefore for relativistic electrons most of the photons are close to head on in the rest frame.



If the rest frame photon energy satisfies $\epsilon'_i \ll m_e c^2$, equivalent to $\gamma\epsilon_i \ll m_e c^2$ in the lab frame, we can simplify the calculations by assuming elastic (Thomson) scattering in the rest frame. Then $\epsilon'_f = \epsilon'_i$, giving

$$\epsilon_f = \gamma^2 \epsilon_i (1 + \beta\cos\theta'_f)(1 - \beta\cos\theta_i) = \epsilon_i \frac{(1 - \beta\cos\theta_i)}{(1 - \beta\cos\theta_f)}. \tag{3.19}$$

For a head-on scattering with $\theta_i = \pi$ and $\theta_f = 0$, i.e. the photon turns around after scattering, we get $\epsilon_f/\epsilon_i = (1+\beta)/(1-\beta) = \gamma^2(1+\beta)^2 \approx 4\gamma^2$.

3.4. Power radiated in inverse Compton scattering

As a moving electron scatters photons, its energy decreases. Let's calculate the energy loss rate.

We assume that the scattering in the rest-frame of the electron is elastic ($\gamma h\nu \ll m_e c^2$). Then in the rest frame of the electron, the power radiated is given by equation (3.4),

$$\frac{dE'}{dt'} = \sigma_T c U'_{\rm rad} \tag{3.20}$$

but since dE/dt is a Lorentz invariant, this is also the power radiated in the lab frame.

Following Blumenthal and Gould (1970), RL use the fact that dn/ϵ is a Lorentz invariant[7] to write $U'_{\rm rad}$ in terms of the lab frame energy density $U_{\rm rad}$. The argument is

$$U'_{\rm rad} = \int \epsilon' dn' = \int \epsilon'^2 \frac{dn'}{\epsilon'} = \int \epsilon'^2 \frac{dn}{\epsilon} = \int \epsilon^2 \gamma^2 (1 - \beta\mu)^2 \frac{dn}{\epsilon}. \tag{3.21}$$

[7]This is related to the fact that the phase space density $dN/d^3\vec{p}d^3\vec{x}$ is invariant; see RL 7.2.

Averaging over angles gives (assume isotropic radiation field)

$$\frac{1}{2}\int_{-1}^{1} d\mu(1-\beta\mu)^2 = 1+\frac{\beta^2}{3} \tag{3.22}$$

and

$$U'_{\rm rad} = U_{\rm rad}\gamma^2\left(1+\frac{\beta^2}{3}\right). \tag{3.23}$$

There is another way to find $U'_{\rm rad}$, which is to transform the incident electric field into the rest frame. For an EM wave/photon travelling at angle θ relative to the electron, $E_x = -E\sin\theta$, $E_z = E\cos\theta$, and $B_y = -E$ (since $\vec{B} = \vec{n}\times\vec{E}$). Now transform these fields into the rest frame: $E'_x = E_x = -E\sin\theta$, $E'_z = \gamma(E_z+\beta B) = \gamma E(\cos\theta-\beta)$. The time-averaged energy density is

$$\frac{E'^2}{8\pi} = \frac{E^2}{8\pi}\left(1-\mu^2+\gamma^2(\mu-\beta)^2\right) = \frac{E^2}{8\pi}\gamma^2(1-\mu\beta)^2 \tag{3.24}$$

which is the same as previously.

Equation (3.20) gives the power in scattered photons which are added to the radiation field, but the scattering also removes energy from the radiation field at a rate $\sigma_T c U_{\rm rad}$ for an isotropic photon distribution[8]. The difference between these two rates must be supplied by the energy of the electron. Therefore the rate of energy loss of the electrons is given by

$$-\frac{dE_e}{dt} = \sigma_T c\left(U'_{\rm rad} - U_{\rm rad}\right) \tag{3.25}$$

or

$$-\frac{dE_e}{dt} = \sigma_T c U_{\rm rad}\left[\gamma^2\left(1+\frac{\beta^2}{3}\right)-1\right] = \frac{4}{3}\gamma^2\beta^2\sigma_T c U_{\rm rad}, \tag{3.26}$$

where we use $\gamma^2 - 1 = \gamma^2\beta^2$. This result is independent of the photon spectrum; applies for an isotropic distribution and Thomson scattering in the rest frame. The average fractional increase in photon energy is $(4/3)\gamma^2\beta^2$.

When the energy transfer in the electron rest frame becomes significant, Blumenthal and Gould (1970) showed that

$$-\frac{dE_e}{dt} = \frac{4}{3}\gamma^2\beta^2\sigma_T c U_{\rm rad}\left[1-\frac{63}{10}\frac{\gamma}{m_e c^2}\frac{\langle\epsilon^2\rangle}{\langle\epsilon\rangle}\right], \tag{3.27}$$

[8] You might wonder whether this formula would apply to an electron not at rest. Blumenthal and Gould 1970 show that the rate of scatterings for photons moving at angle θ with respect to the electron is $dn(\theta)\sigma_T(c - v\cos\theta)$. The cos term drops out when averaging over angles for an isotropic photon distribution.

where the averages $\langle\rangle$ are over the photon spectrum (so now we do care about the photon spectrum; the reason is that the amount of recoil depends on the photon energy).

For a power law distribution of electron energies, the total power is given by summing over the distribution of γ

$$P_{\rm tot} = \int \frac{dE}{dt}(\gamma)N(\gamma)d\gamma \tag{3.28}$$

and if $N(\gamma) = C\gamma^{-p}$ between $\gamma_{\rm min} \leq \gamma \leq \gamma_{\rm max}$ then

$$P_{\rm tot} = \frac{4}{3}\sigma_T c U_{\rm rad}\frac{C}{3-p}\left(\gamma_{\rm max}^{3-p} - \gamma_{\rm min}^{3-p}\right). \tag{3.29}$$

For a thermal distribution of non-relativistic electrons, the total power is

$$P_{\rm tot} = \left(\frac{4k_BT}{m_ec^2}\right)c\sigma_T n_e U_{\rm rad} \tag{3.30}$$

(which you can see since for a thermal gas $\langle\beta^2\rangle = 3k_BT/m_ec^2$ and for non-relativistic particles $\gamma \approx 1$).

3.5. The inverse Compton spectrum for single scattering of monochromatic photons

Previously, we derived the total power from inverse Compton across all frequencies. Now we turn to the spectrum of the scattered photons. For an isotropic monoenergetic photon distribution, Blumenthal and Gould (1970) calculate the spectrum in the ultrarelativistic limit $\gamma \gg 1$. In this limit, it is a good approximation to take the photons in the electron rest frame as having $\theta_i = \pi$ because of relativistic beaming. This means that there is only one angle to consider, the scattering angle in the rest frame α'. Each value of scattering angle maps onto a different final photon energy.

Using the distribution of scattering angles for Thomson scattering, and transforming back into the lab frame gives

$$\frac{dN}{dtd\epsilon_f} = \sigma_T cn(\epsilon_i)d\epsilon_i\frac{3f(x)}{4\gamma^2\epsilon_i} \tag{3.31}$$

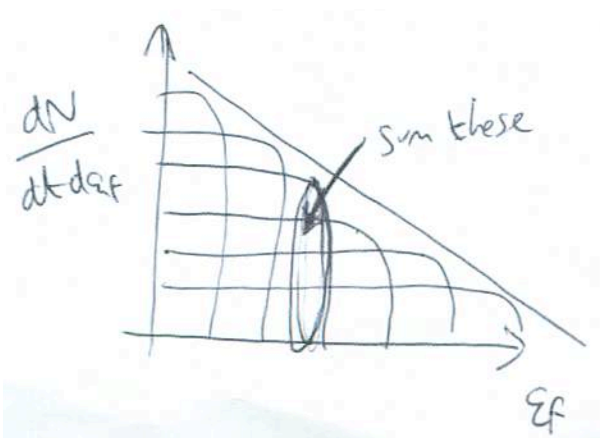
where $f(x) = 2x\ln x + x + 1 - 2x^2$ and $x = \epsilon_f/4\gamma^2\epsilon_i$. The function $f(x)$ is plotted in RL Figure 7.3b: it decreases smoothly from unity at $x = 0$ to zero at $x = 1$. This makes sense: we saw previously that the maximum photon energy is $\epsilon_f = 4\gamma^2\epsilon_i$. Note that $\int f(x)dx = 1/3$ and $\int xf(x)dx = 1/9$, so that we recover the correct expressions for the scattering rate dN/dt and power dE/dt on integrating over ϵ_f.

The spectrum of the scattered photon is therefore flat at low photon energies $\epsilon_f \ll 4\gamma^2\epsilon_i$, with a cutoff at $\epsilon_f = 4\gamma^2\epsilon_i$ ($x = 1$). (The spectrum for arbitrary γ, Klein-Nishina cross-section, and including the energy loss in the rest frame is given by Jones 1968).

If the electrons have a power law energy distribution, $N(\gamma) \propto \gamma^{-p} d\gamma$, the spectrum of the scattered photons is given at each final photon energy ϵ_f by summing the contributions from each γ

$$\frac{dN}{dt d\epsilon_f} \propto \int_{\gamma_{\rm min}} \gamma^{-p} d\gamma \gamma^{-2} \propto \gamma_{\rm min}^{-p-1} \propto \epsilon_f^{-(p+1)/2}, \tag{3.32}$$

where we use the fact that only electrons with $\gamma > \gamma_{\rm min} = (\epsilon_f/4\epsilon_i)^{1/2}$ have enough energy to contribute scattered photons with energy ϵ_f. The photon number spectrum is therefore a power law spectrum with index $(p+1)/2$. The energy spectrum is then $F_\nu \propto \nu^{-(p-1)/2}$ or an index $(p-1)/2$. We'll see later that the same result applies to synchrotron emission from a power law distribution of electrons.



3.6. Multiple scatterings

If multiple scatterings occur, the photon energy spectrum can be significantly affected. This is known as *Comptonization.* An important parameter is the *Compton y parameter*

$$y = \left(\text{average } \frac{\Delta\varepsilon}{\varepsilon} \text{ per scattering}\right) \times \left(\text{mean number of scatterings}\right)$$

which measures whether a photon will significantly change its energy when traversing a medium.

Let's look at each term separately. We saw previously that the average fractional change in photon energy on scattering is $(4/3)\gamma^2\beta^2$. For a thermal gas of non-relativistic electrons, this is $4k_BT/m_ec^2$ (since $m_e\langle v^2\rangle/2 = 3k_BT/2$). However, this ignores electron recoil (we assumed elastic scattering in the rest frame). We can estimate this from our earlier result

$$\epsilon'_f = \frac{\epsilon'_i}{1 + (\epsilon_i/m_ec^2)(1-\cos\alpha')} \approx \epsilon'_i\left(1 - \frac{\epsilon'_i}{m_ec^2}(1-\cos\alpha')\right). \tag{3.33}$$

Averaging over angles gives $\Delta\epsilon/\epsilon \approx -\epsilon/m_ec^2$. Therefore,

$$\frac{\Delta\epsilon}{\epsilon} = \left(\frac{4k_BT}{m_ec^2} - \frac{\epsilon}{m_ec^2}\right). \tag{3.34}$$

In the ultra-relativistic limit ($\gamma \gg 1$), $\Delta\epsilon/\epsilon \approx (4/3)\gamma^2$ (the recoil term is negligible in this limit).

The number of scatterings depends on the optical depth. We found in the section on radiative transfer that the number of scatterings is $\max(\tau,\tau^2)$, where $\tau \approx \rho\kappa_{es}R \approx n_e\sigma_T R$. Therefore for a thermal gas of non-relativistic electrons

$$y = \frac{4k_BT}{m_ec^2}\max(\tau,\tau^2). \tag{3.35}$$

How do multiple scatterings change the photon energy? If we start with a photon with initial energy $\epsilon_0 \ll k_BT$, then after N scatterings,

$$\epsilon = \epsilon_0\left(1 + \frac{4k_BT}{m_ec^2}\right)^N \approx \exp\left(\frac{4k_BT}{m_ec^2}N\right) \approx \exp\left(y\right). \tag{3.36}$$

When the photon energy reaches $\epsilon = 4k_BT$, equation (3.34) shows that the photon energy will no longer increase, that is we reach an equilibrium. By setting $\tau^2 = N$ and $\epsilon = 4k_BT$ in equation (3.36), we find the optical depth required to reach equilibrium is

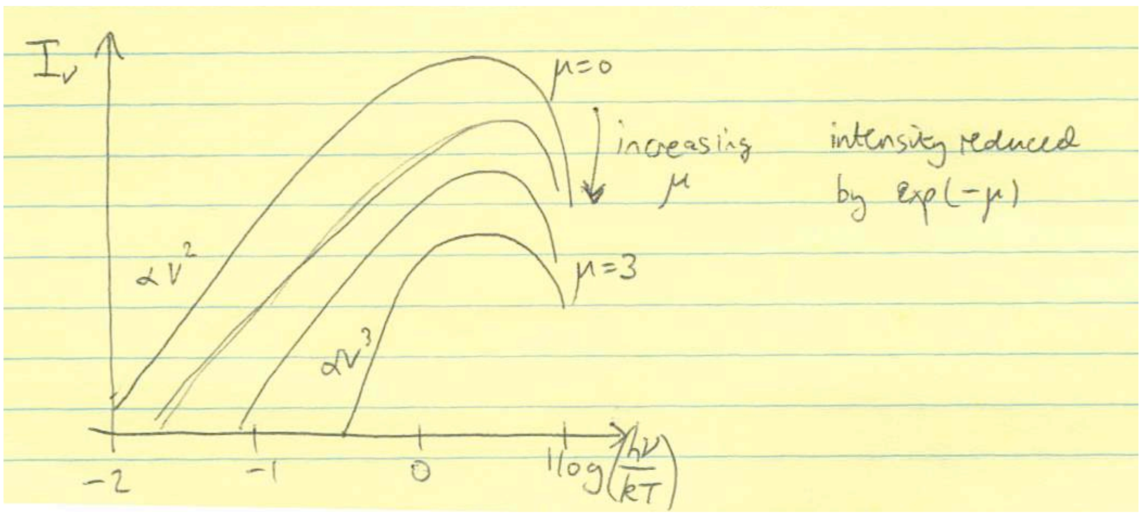
$$\tau = \left[\frac{m_ec^2}{4k_BT}\ln\left(\frac{4k_BT}{\epsilon_0}\right)\right]^{1/2}. \tag{3.37}$$

Because we are dealing with a fixed number of photons (scattering conserves photon number), we expect the equilibrium distribution of photons to be a Bose-Einstein distribution with a non-zero chemical potential μ, that is with energy density

$$U_\nu d\nu = \frac{8\pi h\nu^3}{c^2}\left[\exp\left(\frac{h\nu}{k_BT} + \mu\right) - 1\right]^{-1} d\nu. \tag{3.38}$$

The equivalent expression for a Planck distribution is $U_\nu = 4\pi B_\nu/c$, identical except $\mu = 0$ in that case. In a blackbody enclosure for example, photons can be created or destroyed as needed (e.g. by interaction at the walls) in order to reach thermal equilibrium, which gives $\mu = 0$.

The photon distribution is shown in the following plot. Except for small values of μ close to zero, the exponential term dominates, giving an overall suppression of the distribution by a factor $e^{-\mu}$, and a different scaling $\propto \nu^3$ at low frequency instead of $\propto \nu^2$.



For non-zero μ, the photon distribution is then $\propto \epsilon^2 e^{-\epsilon/k_B T}$, in which case it is straightforward to show that $\langle\epsilon\rangle = 3k_B T$ and $\langle\epsilon^2\rangle = 12(k_B T)^2$. The average change in photon energy on scattering is $\langle\Delta\epsilon\rangle = (4k_B T/m_e c^2)\langle\epsilon\rangle - \langle\epsilon^2\rangle/m_e c^2 = 0$, showing that this is indeed the equilibrium distribution.

Another case mentioned by RL (§7.5) that is interesting is multiple scattering by relativistic electrons with low optical depth, which is a way to produce a power law spectrum from a non-power law electron distribution! The energy amplification per scattering is $A = (4/3)\gamma^2 = 16(k_B T/m_e c^2)$ for a relativistic thermal gas. After k scatterings, $\epsilon_k = \epsilon_0 A^k$. For $\tau \ll 1$, the probability of having k scatterings is τ^k. Therefore the emergent spectrum is $I(\epsilon_k) = I(\epsilon_0)\tau^k = I(\epsilon_0)(\epsilon_k/\epsilon_0)^{-\alpha}$, where the power law index is $\alpha = -\ln\tau/\ln A$.

3.7. The Kompaneets equation

We discussed the expected equilibrium photon distribution for scattering in a thermal gas. Let's think about the approach to equilibrium. Write the number of photons with energy between ϵ and $\epsilon + d\epsilon$ as $N(\epsilon)$. Now ask, how does this distribution evolve with time as the scatterings occur? For each scattering, we can write down a probability distribution for a change in the photon energy $\epsilon \to \epsilon + \Delta$, i.e. $P(\epsilon, \Delta)d\Delta$ is the probability that a photon with energy ϵ changes its energy by an amount Δ. The normalization is $\int d\Delta P(\epsilon, \Delta) = 1$ (the photon must change its energy by some amount).

Then the evolution of the photon distribution is given by

$$N(\epsilon, t + \delta t) - N(\epsilon, t) = \int d\Delta \left[N(\epsilon - \Delta, t) P(\epsilon - \Delta, \Delta) - N(\epsilon, t) P(\epsilon, \Delta) \right] \tag{3.39}$$

where the timescale δt is a scattering timescale $\delta t = t_s = 1/n_e \sigma_T c$. For small Δ, we can expand

$$N(\epsilon - \Delta, t) \approx N(\epsilon, t) - \Delta \frac{\partial N}{\partial \epsilon} + \frac{\Delta^2}{2} \frac{\partial^2 N}{\partial \epsilon^2} + ... \tag{3.40}$$

$$P(\epsilon - \Delta, \Delta) \approx P(\epsilon, \Delta) - \Delta \frac{\partial P}{\partial \epsilon} + \frac{\Delta^2}{2} \frac{\partial^2 P}{\partial \epsilon^2} + ... \tag{3.41}$$

which gives

$$N(\epsilon, t + t_s) - N(\epsilon, t) = -\frac{\partial}{\partial \epsilon} \int d\Delta N P \Delta + \frac{1}{2} \frac{\partial^2}{\partial \epsilon^2} \int d\Delta N P \Delta^2. \tag{3.42}$$

Using this result and expanding $N(t + t_s) \approx N(t) + t_s \partial N / \partial t$, we obtain the Fokker-Planck equation

$$t_s \frac{\partial N}{\partial t} = -\frac{\partial}{\partial \epsilon} \left(\langle \Delta \rangle N \right) + \frac{1}{2} \frac{\partial^2}{\partial \epsilon^2} \left(N \langle \Delta^2 \rangle \right). \tag{3.43}$$

This is an advection-diffusion equation in energy space.

For a thermal distribution of electrons,

$$\langle \Delta \rangle = \epsilon \left(\frac{4 k_B T}{m_e c^2} - \frac{\epsilon}{m_e c^2} \right) \qquad \langle \Delta^2 \rangle = \epsilon^2 \left(\frac{2 k_B T}{m_e c^2} \right). \tag{3.44}$$

The second result is computed in RL §7.6 (compare RL eq. 7.54). Substituting these in and simplifying gives

$$t_s \frac{\partial N}{\partial t} = -\frac{\partial}{\partial \epsilon} \left[\left(2 \epsilon N - \epsilon^2 \frac{\partial N}{\partial \epsilon} \right) \left(\frac{k_B T}{m_e c^2} \right) - \frac{\epsilon^2 N}{m_e c^2} \right] \tag{3.45}$$

Now to agree with RL, we switch notation to $n(\omega)$ where $\omega^2 n(\omega) \propto N(\epsilon)$, and define $x = \hbar \omega / k_B T$. The result is

$$t_s \frac{\partial n}{\partial t} = \left(\frac{k_B T}{m_e c^2} \right) \frac{1}{x^2} \frac{\partial}{\partial x} \left[x^4 \left(\frac{\partial n}{\partial x} + n \right) \right]. \tag{3.46}$$

The equilibrium solution ($\partial n/\partial t = 0$) is $n \propto e^{-x}$ or $N \propto \epsilon^2 e^{-\epsilon/k_BT}$. This is the relativistic Maxwell-Boltzmann distribution we discussed earlier.

Why didn't we find the equilibrium distribution to be the Bose-Einstein distribution $n = (e^{x+\alpha} - 1)^{-1}$? We should have included extra $(1+n)$ factors to account for the fact that photons tend to mutual occupation of the same state (see RL eq. 7.48), i.e. the RHS of equation (3.39) should be

$$\int d\Delta \left[N(\epsilon - \Delta, t)P(\epsilon - \Delta, \Delta)(1 + N(\epsilon, t)) - N(\epsilon, t)P(\epsilon, \Delta)(1 + N(\epsilon - \Delta, t))\right]. \qquad (3.47)$$

As we discussed earlier the extra terms are only important for μ approaching zero, which indicates that multiple occupation of states is important. For non-zero μ the occupation number is small and the extra $1 + n$ terms are no longer important. The particles then have a Maxwell-Boltzmann distribution of energies.

With the extra terms, the analysis follows as before, but now the result is

$$t_s \frac{\partial n}{\partial t} = \left(\frac{k_B T}{m_e c^2}\right) \frac{1}{x^2} \frac{\partial}{\partial x} \left[x^4 \left(\frac{\partial n}{\partial x} + n + n^2\right)\right]. \qquad (3.48)$$

This is the *Kompaneets equation*, which has the equilibrium solution $n = (e^{x+\alpha} - 1)^{-1}$.

3.8. Example: Sunyaev-Zeldovich effect

The Sunyaev-Zeldovich or SZ effect is a small distortion of the spectrum of the cosmic microwave background (CMB) radiation due to inverse Compton scattering of CMB photons by hot electrons in Galaxy clusters (for a review see Carlstrom, Holder, and Reese 2002 ARAA).

We can calculate the effect by using the Kompaneets equation. Because $T_{\rm gas} \gg T_{\rm rad}$, the $\partial n/\partial x$ term dominates on the RHS. To see this, note that $x = h\nu/k_BT \ll 1$, and then you can show that $\partial n/\partial x \sim n/x$ for a Planck spectrum $n = (e^x - 1)^{-1}$. The Kompaneets equation we need to solve is therefore

$$\frac{1}{c}\frac{\partial n}{\partial t} = \left(n_e \sigma_T \frac{k_B T}{m_e c^2}\right) \frac{1}{x^2} \frac{\partial}{\partial x} \left(x^4 \frac{\partial n}{\partial x}\right). \qquad (3.49)$$

Following the path of a photon, $dl = c\,dt$, and so in terms of the Compton y parameter,

$$\frac{\partial n}{\partial y} = \frac{1}{x^2} \frac{\partial}{\partial x} \left(x^4 \frac{\partial n}{\partial x}\right). \qquad (3.50)$$

For cluster gas $y = \int n_e \sigma_T (k_B T/m_e c^2) dl \ll 1$, so we can calculate δn by inserting a Planck spectrum on the RHS and writing $\delta n \approx y \partial n/\partial y$, giving

$$\frac{\delta n}{n} = (e^x - 1)\frac{y}{x^2}\frac{\partial}{\partial x}\left(x^4 \frac{\partial}{\partial x}\left(\frac{1}{e^x - 1}\right)\right) = y\left(\frac{xe^x}{e^x - 1}\right)\left(x \coth\left(\frac{x}{2}\right) - 4\right), \tag{3.51}$$

which is Zeldovich and Sunyaev's original result. In the Rayleigh-Jeans part $x \ll 1$, $\delta n/n \approx -2y$, or

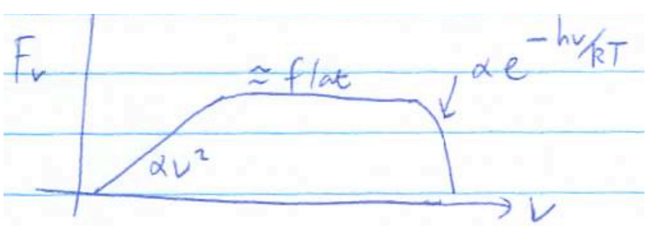
$$\frac{\Delta T}{T} = -2\frac{\sigma_T k_B}{m_e c^2}\int n_e T\, dl. \tag{3.52}$$

For a path length of 1 Mpc and a density $n_e = 10^{-2}$ cm^{-3}, I get a Thomson depth $\sim 10^{-2}$. The $k_B T/m_e c^2$ factor is 1/50 for $k_B T = 10$ keV. Therefore $\Delta T/T \sim 3 \times 10^{-4}$ or $\Delta T \sim 10^{-3}$ for $T = 3$K. The expected temperature decrement is therefore $\Delta T \sim$ mK. The integral in equation (3.52) is proportional to the integrated gas pressure of the cluster. The redshift independence of this signal makes it an important way to look for high redshift clusters. The Carlstrom et al. ARAA article has references to calculations of relativistic corrections to our formula, and also describes the "kinetic SZ" effect that is a temperature shift due to the peculiar velocity of the cluster (the spectrum remains Planckian but with $\Delta T/T \approx -\tau v/c$).

3.9. The spectrum of thermal gas

We next discuss the spectrum of thermal gas in which both free-free and scattering processes operate. See RL 7.7, Felten and Rees (1972), and Illarionov and Sunyaev (1972).

Recall that a finite region of gas emitting thermal Bremsstrahlung has a spectrum



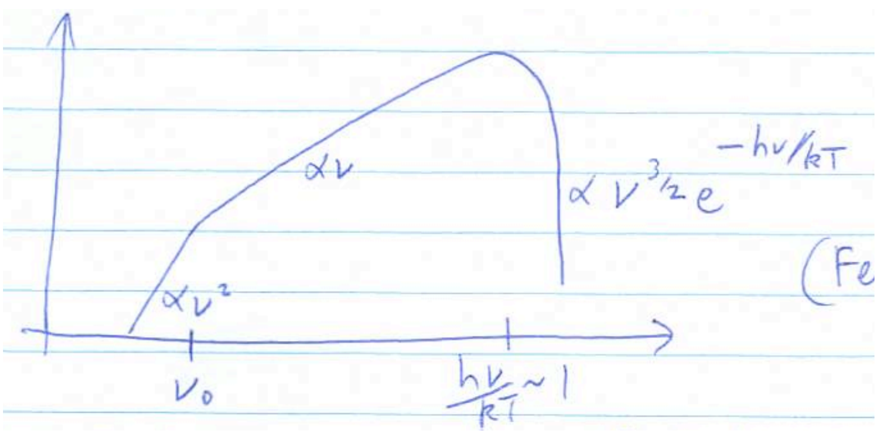
At low frequencies, the region becomes optically thick ($\kappa_{ff} \propto 1/\nu^2$) and the spectrum is Rayleigh-Jeans. At higher frequencies, the spectrum corresponds to optically thin bremsstrahlung, i.e. roughly flat with an exponential cutoff determined by the temperature of the gas.

What happens to this spectrum when we include electron scattering? We already mentioned the idea of thermalization depth at the end of Part I of these notes. Let's quickly review what we discussed there. With scattering and absorption included, the mean free path is $l_\nu = 1/(\alpha_\nu + \sigma_\nu)$, but only a fraction $\epsilon_\nu = \alpha_\nu/(\alpha_\nu + \sigma_\nu)$ of encounters result in absorption of a photon. The mean free path to absorption is $l^\star_\nu = \sqrt{N_\nu} l_\nu = l_\nu/\sqrt{\epsilon_\nu} = 1/\sqrt{\alpha_\nu(\alpha_\nu + \sigma_\nu)}$. If $\tau_\star \ll 1$, where $\tau_\star = L/l_\star$ is the effective optical thickness, we expect the luminosity to be $L_\nu \sim 4\pi B_\nu \alpha_\nu V$ (all photons escape). If $\tau_\star \gg 1$ (optically thick to absorption) then the photons come from a volume $l_\star A$ where A is the emitting area, and the resulting flux is $F_\nu \approx \pi B_\nu \sqrt{\epsilon_\nu}$, suppressed by a factor $\sqrt{\epsilon_\nu}$ compared to the flux from a blackbody.

How does this apply in this case? We start by assuming the electron scattering is elastic (no change in photon energy). At low frequencies, we expect $\kappa_{ff} \gg \kappa_{es}$ since $\kappa_{ff}/\kappa_{es} \propto 1/\nu^2$), implying a Rayleigh-Jeans spectrum at low frequencies. However, for frequencies for which $\kappa_{es} > \kappa_{ff}$, we expect the emissivity to be reduced by a factor

$$\sqrt{\epsilon_\nu} = \left(\frac{\kappa_{ff}}{\kappa_{es} + \kappa_{ff}}\right)^{1/2} \approx \left(\frac{\kappa_{ff}}{\kappa_{es}}\right)^{1/2} \propto \frac{1}{\nu} \tag{3.53}$$

where the last step is for the limit $\kappa_{es} \gg \kappa_{ff}$. Therefore, as long as the material is optically thick ($\tau_\star \gg 1$), we expect to see



(see Felten and Rees Fig 3a). This is known as a *modified blackbody* spectrum. Defining ν_0 to be the frequency where $\kappa_{ff}(\nu_0) = \kappa_{es}$, and writing $I_\nu = 2B_\nu/(1+\epsilon_\nu^{-1/2})$, we get $I_\nu \approx B_\nu$ for $\nu \ll \nu_0$ and $I_\nu = I_\nu^{MBB} = 2B_\nu(\kappa_{ff}/\kappa_{es})^{1/2}$ for $\nu \gg \nu_0$. The frequency dependence is

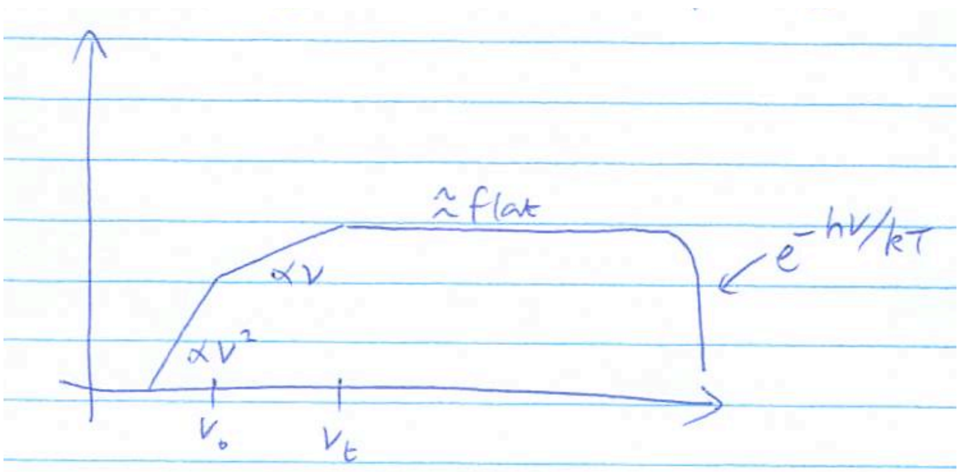
$$I_\nu^{MBB} \propto \left(\frac{x^3}{e^x - 1}\right)\left(\frac{1 - e^{-x}}{x^3}\right)^{1/2} \propto \frac{e^{-x/2}}{(e^x - 1)^{1/2}} x^{3/2} \tag{3.54}$$

with $x = h\nu/k_BT$. The limits are $I_\nu^{MBB} \propto x$ for $x \ll 1$ and $I_\nu^{MBB} \propto x^{3/2}\exp(-x)$ for $x \gg 1$.

If the medium has a finite optical thickness then at high enough photon frequency it will be optically thin to absorption. We define another frequency ν_t at which this transition occurs, i.e. where $l_\star = L$, or $\tau_a \tau_{es} = 1$ (actually $\tau_a(\tau_a + \tau_{es}) = 1$ but we are assuming that we are already in the regime where $\kappa_{es} > \kappa_a$ at $\nu = \nu_t$). Since $\kappa_{ff}(\nu_0) = \kappa_{es}$ but $\kappa_{ff}(\nu_t) = 1/\kappa_{es}$, then $(\nu_t/\nu_0)^2 = \tau_{es}^2$, or

$$\nu_t \approx \nu_0 \tau_{es}. \tag{3.55}$$

Above this frequency, the source is optically thin, and the overall spectrum is



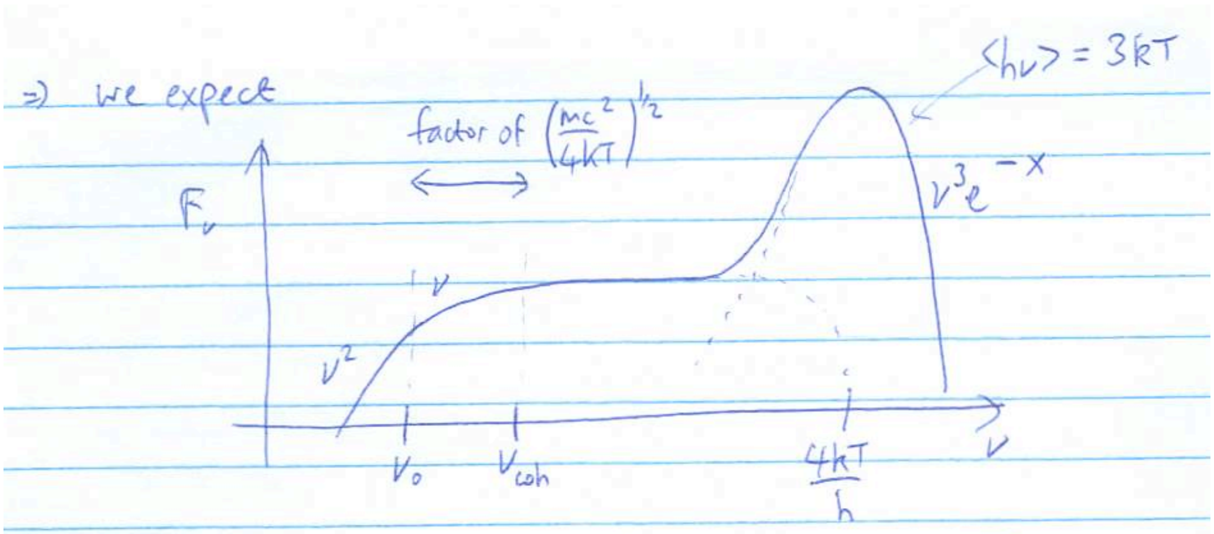
(Felten and Rees Fig 3c).

Now let's consider the possibility of non-elastic scattering, so the photons change frequency as they scatter. This will be important when $y \gg 1$ or $N(4k_BT/m_ec^2) \gg 1$ for N scatterings. Between absorptions, the number of scatterings is $\epsilon_\nu^{-1} = (\kappa_{ff} + \kappa_{es})/\kappa_{ff} \approx \kappa_{es}/\kappa_{ff}$, which increases $\propto \nu^2$ (assuming $x < 1$ so that $\kappa_{ff} \propto \nu^{-2}$). Therefore at high enough frequency, non-elastic scattering will become important. Since $\kappa_{ff}(\nu_0) = \kappa_{es}$ then $(\nu_{\rm coh}/\nu_0)^2(4k_BT/m_ec^2) = 1$ defines

$$\nu_{\rm coh} = \nu_0 \left(\frac{m_ec^2}{4k_BT}\right)^{1/2} \tag{3.56}$$

above which photon energy changes are significant. Another way to say this is that for $\nu > \nu_{\rm coh}$, the thermalization depth $l_\star$ becomes larger than the lengthscale that makes $y > 1$. (Note that if the y parameter for the whole medium $\tau_{es}^2(4k_BT/m_ec^2) < 1$ then incoherent scattering is never important at any frequency).

For $\nu > \nu_{\rm coh}$, the spectrum will saturate, and will take the Wien form $I_\nu = I_\nu^W = (2h\nu^3/c^2)e^{-\alpha}e^{-h\nu/k_BT}$ for $\alpha \gg 1$. The spectrum looks like:



The photons on the left part of the spectrum come from $\tau_{es} \lesssim (m_e c^2/4k_B T)^{1/2}$, and on the right part of the spectrum from deeper regions with $\tau_{es} \gtrsim (m_e c^2/4k_B T)^{1/2}$. The total flux in the Wien spectrum is

$$F^W = \pi \int I_\nu^W \, d\nu = \frac{12\pi e^{-\alpha}(k_B T)^4}{c^2 h^3}, \tag{3.57}$$

which is $(90e^{-\alpha}/\pi^4)\sigma T^4$. Roughly we can think of this as shifting all the bremsstrahlung emitted photons to energies $\approx k_B T$ or $F^W \approx l_\star k_B T \int (\epsilon_\nu^{ff}/h\nu) d\nu$ which allows the overall normalization to be calculated.

The spectrum saturates as a Wien spectrum for $y \gg 1$ and $x_{\rm coh} = h\nu_{\rm coh}/k_B T < 1$. For $x_{\rm coh} \gg 1$, inverse Compton effects are not important since all photons elastically scatter. In the intermediate range $x_{\rm coh} \sim 1$, the Comptonization does not saturate to a Wien spectrum when $y \gg 1$. RL have an argument for this case, as follows. We write down a steady state Kompaneets equation

$$0 = \left(\frac{k_B T}{m_e c^2}\right) \frac{1}{x^2} \frac{\partial}{\partial x} \left[x^4 (n' + n)\right] + Q(x) - \frac{n}{\max(\tau_{es}^2, \tau_{es})}. \tag{3.58}$$

In the first term, we neglect n^2 compared to n for $\alpha \gg 1$. The second term represents a source of photons with energy x, and the third term allows for escape of photons. For $x \gg 1$, $n' + n = 0$ gives $n \propto e^{-x}$. For $x \ll 1$, $n' \gg n$, and by balancing the first and third terms,

$$\frac{4n}{y} = \frac{1}{x^2} \frac{\partial}{\partial x} \left(x^4 n'\right). \tag{3.59}$$

A power law solution $n \propto x^m$ works with

$$m = -\frac{3}{2} \pm \sqrt{\frac{9}{4} + \frac{4}{y}}. \tag{3.60}$$

For $y \gg 1$ $m = 0, -3$ which gives either $I_\nu \propto x^3$ or constant. For $y \ll 1$, $I_\nu \propto \nu^{3+m}$. See Pozdniakov, Sobol, and Sunyaev (1983) for Monte Carlo calculations of spectra in this regime. I've included a Figure from their paper below. It shows the change in the spectrum with increasing y. I've also included a cartoon from that paper that shows astrophysical environments where Comptonization is likely to be important.

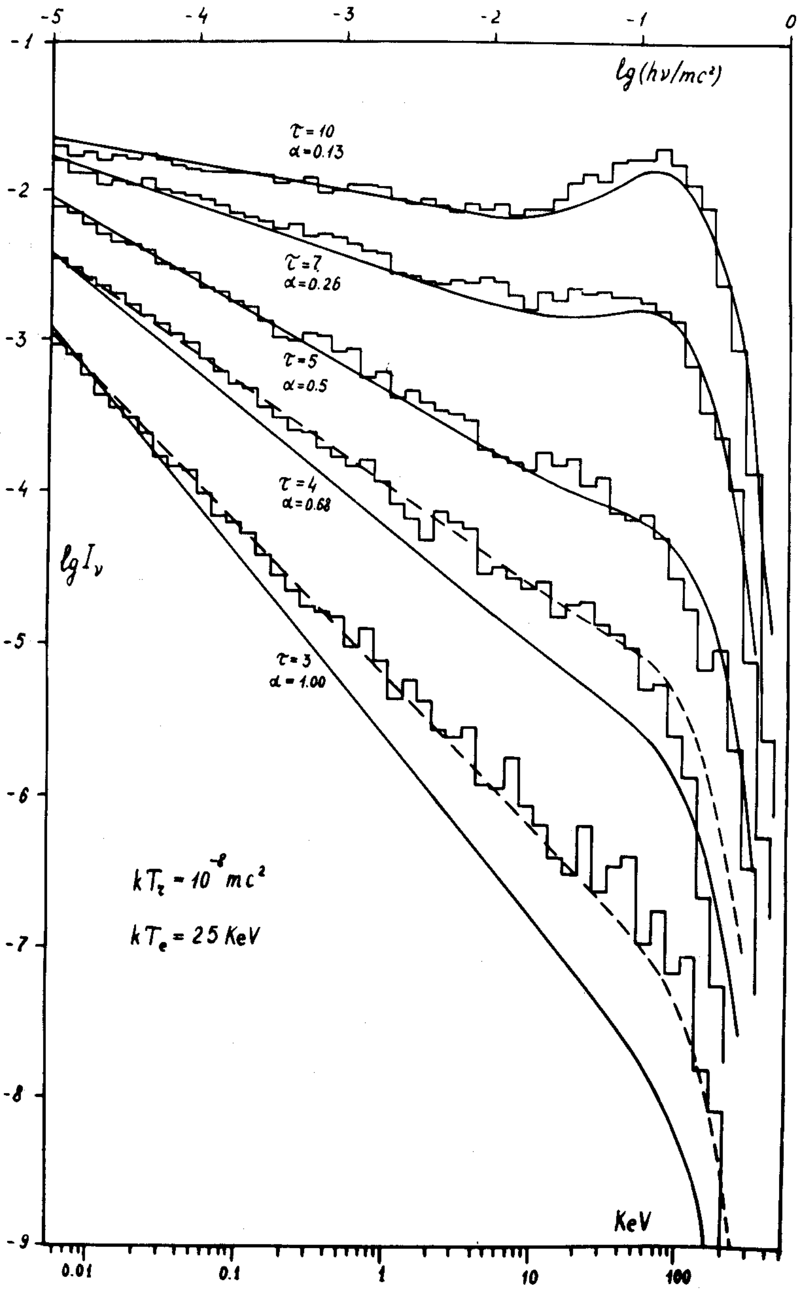


Figure 21 Comptonization of low-frequency photons in a spherical plasma cloud having kT_e = 25 keV. *Solid curves,* the analytic expression (4.5) with spectral index α given by Eq. (4.8); *dashed curves,* the analytic solution with α taken to agree with the low-frequency portion of the corresponding Monte Carlo spectrum. Central photon source.

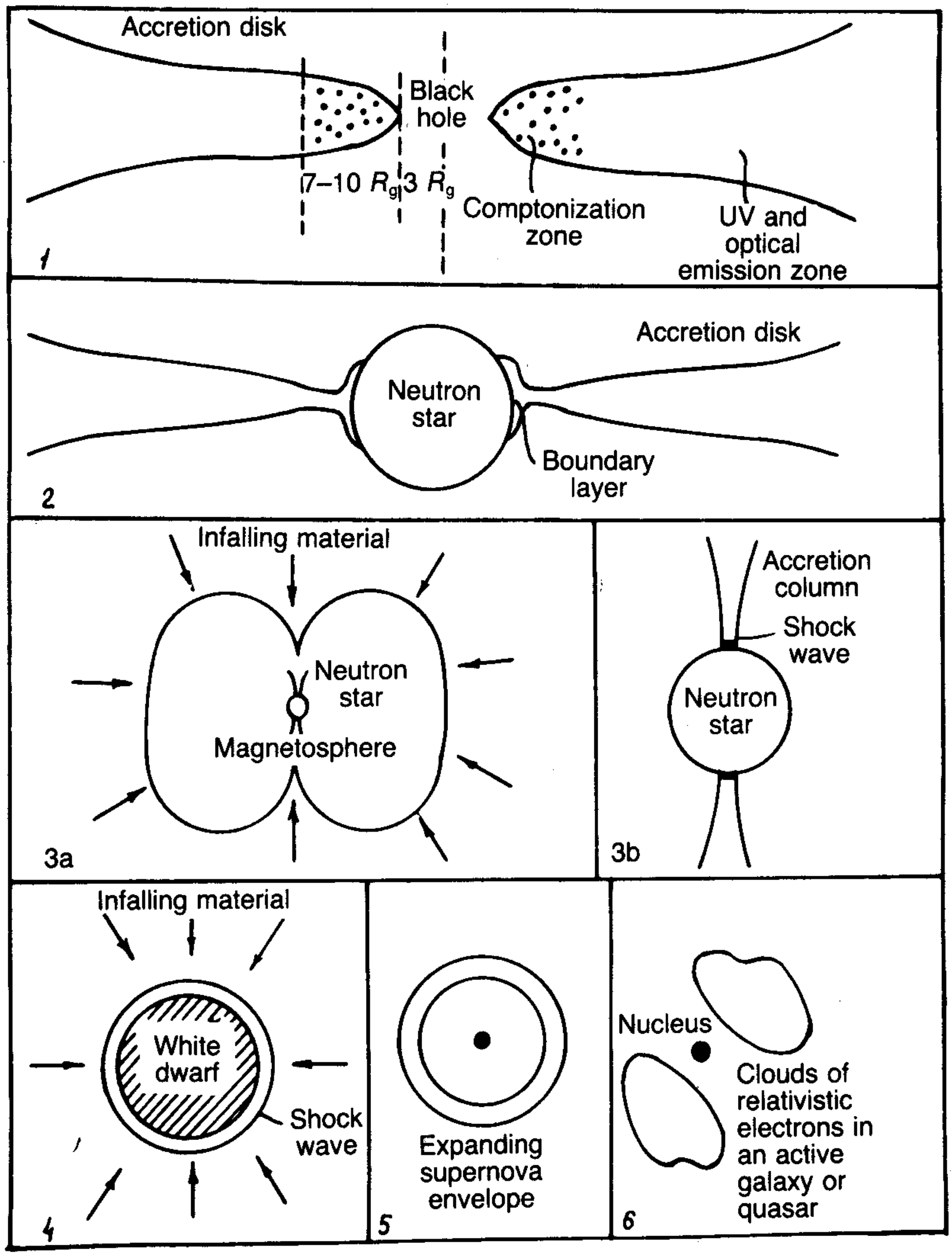


Figure 5 The principal astrophysical objects in which the Comptonization mechanism should operate efficiently.

Summary and Further Reading

Here are the main ideas and results that we covered in this part of the course:

- Thomson scattering cross-section. $\sigma_T = 8\pi r_0^2/3$, where $r_0 = e^2/m_e c^2$ is the classical electron radius. Differential cross-section for an unpolarized beam $d\sigma_T/d\Omega = (1/2)r_0^2(1+\cos^2\alpha)$, where α is the scattering angle. Radiated power $P = \sigma_T c U_{\rm rad}$. Klein-Nishina cross-section $\sigma \sim \sigma_T(m_e c^2/\epsilon_i)$ for initial photon energy $\epsilon_i \gg m_e c^2$.

- Thomson scattering opacity $\kappa_{es} = \sigma_T(1+X)/2m_p = 0.2(1+X)$ cm^2 g^{-1}.

- Kinematics of Compton scattering

$$\epsilon_f = \epsilon_i \left[1 + \frac{\epsilon_i}{m_e c^2}(1-\cos\alpha)\right]^{-1}.$$

  or $\lambda_f - \lambda_i = \lambda_c(1-\cos\alpha)$ with Compton wavelength $\lambda_c = h/m_e c$.

- Inverse Compton scattering

$$\epsilon_i' = \epsilon_i\gamma(1-\beta\cos\theta_i), \quad \epsilon_f = \epsilon_f'\gamma(1+\beta\cos\theta_f').$$

  Maximum photon energy $\epsilon_{f,\max} = 4\gamma^2\epsilon_i$. For $\epsilon_i' \ll m_e c^2$ or $\gamma\epsilon_i \ll m_e c^2$ can take $\sigma = \sigma_T$. Power radiated per electron

$$P = \frac{4}{3}\gamma^2\beta^2\sigma_T c U_\gamma,$$

  or for thermal non-relativistic electrons $P = (4k_BT/m_e c^2)c\sigma_T U_\gamma$.

- Inverse Compton spectrum. For a single electron number spectrum is

$$\frac{dN}{dt d\epsilon_f} = \sigma_T c n(\epsilon_i)\frac{d\epsilon_i}{\epsilon_i}\frac{3f(x)}{4\gamma^2}$$

  where $f(x) = 2x\ln x + x + 1 - 2x^2$, $x = \epsilon_f/4\gamma^2\epsilon_i$. Flat number spectrum with cutoff at $x \sim 1$. Mean photon energy $(4/3)\gamma^2\epsilon_i$. For a power law distribution of electrons $N(\gamma) \propto \gamma^{-p}$, $(dN/dV dt d\epsilon_f) \propto \epsilon_f^{-(p+1)/2}$.

- Multiple scatterings. Compton y parameter, $y = (\Delta\epsilon/\epsilon)\max(\tau,\tau^2)$. Mean energy change on scattering: $\Delta\epsilon/\epsilon = (4k_BT - \epsilon)$ (non-relativistic) or $(4/3)\gamma^2$ (relativistic). For $\epsilon \ll 4k_BT$, photon energy grows by $\exp(y)$. Equilibrium distribution is Bose-Einstein with finite μ. Kompaneets equation

$$t_s\frac{\partial n}{\partial t} = \left(\frac{k_BT}{m_e c^2}\right)\frac{1}{x^2}\frac{\partial}{\partial x}\left[x^4\left(\frac{\partial n}{\partial x} + n + n^2\right)\right].$$

- Applications. Sunyaev-Zeldovich effect: $\Delta T/T = -2\int n_e \sigma_T (k_B T/m_e c^2) dl$. Spectrum of thermal gas including free-free and scattering processes. Modified blackbody spectrum.

Reading

- RL chapter 7. Longair.
- Kinematic of Compton: see Blumenthal & Gould 1970, Rev Mod Phys, 42, 237
- Inverse Compton spectrum: Jones (1968) Phys Rev 167, 1159
- Sunyaev (1980) Sv A Lett, 6, 213 one of the original papers on SZ effect. See also the ARAA article by Carlstrom, Holder, & Reese (2002).
- Felten & Rees (1972), Illarionov & Sunyaev (1972) spectrum of thermal gas including free-free and electron scattering. Unsaturated Compton spectra are calculated by Pozdniakov, Sobol, & Sunyaev (1983).

4. Synchrotron Radiation

These are notes for part four of PHYS 642 Radiative Processes in Astrophysics. Synchrotron radiation is radiation from particles accelerated by magnetic fields. For non-relativistic electrons, the radiation is at the gyration frequency $\omega = eB/m_e c$ and is known as *cyclotron radiation*. However, for relativistic particles, the emission extends to higher frequencies, and we then describe the radiation as *synchrotron radiation*. We also include a discussion of Fermi acceleration of high energy particles, and the evolution of the energy spectrum.

4.1. Power radiated by a relativistic particle

First, we want to extend Larmor's formula to relativistic electrons. To do this, recall the "four-acceleration" $\tilde{a} = d\tilde{u}/d\tau$ where $\tilde{u} = (c, \vec{u})$ is the four-velocity, and $d\tau$ is the interval of proper time, $d\tau = ds/c$, $ds^2 = c^2dt^2 - |d\vec{x}|^2 = c^2dt^2/\gamma^2$.

Let's evaluate $\tilde{a}$ in the rest frame of the particle, where $d\tau = dt'$. First note that $d\gamma'/dt' = 0$ since

$$\frac{d\gamma'}{dt'} = \frac{d}{dt'}\left(1 - \frac{\vec{u}'\cdot\vec{u}'}{c^2}\right)^{-1/2} = \frac{\gamma^3}{2c^2}\frac{d}{dt'}\left(\vec{u}'\cdot\vec{u}'\right) = \frac{\gamma^3}{c^2}\vec{u}'\cdot\frac{d\vec{u}'}{dt'} = 0 \tag{4.1}$$

since $\vec{u}' = 0$ in the rest frame. Therefore $\tilde{a}' = d\tilde{u}'/dt' = (0, \vec{a}')$, and so Larmor's formula gives the power radiated in the rest frame

$$P = \frac{2q^2}{3c^3}\left|\vec{a}'\right|^2 = \frac{2q^2}{3c^3}\tilde{a}'\cdot\tilde{a}'. \tag{4.2}$$

The norm of a four vector is Lorentz invariant (the same in all reference frames), meaning that we can write the general form of Larmor's formula as

$$P = \frac{2q^2}{3c^3}\tilde{a}\cdot\tilde{a} \tag{4.3}$$

where $\tilde{a}$ is the four-acceleration in the frame of interest.

In the lab frame,

$$\tilde{a}\cdot\tilde{a} = -\left(c\gamma\frac{d\gamma}{dt}\right)^2 + \left(\gamma\frac{d}{dt}\left(\gamma\vec{u}\right)\right)^2. \tag{4.4}$$

Using $d\gamma/dt = \gamma^3\vec{u}\cdot\vec{a}/c^2$, this is

$$\tilde{a}\cdot\tilde{a} = \gamma^4\left|\vec{a}\right|^2 + \frac{\gamma^6\left(\vec{u}\cdot\vec{a}\right)^2}{c^2}. \tag{4.5}$$

Dividing the acceleration vector into components parallel and perpendicular to the velocity $\vec{a}_{\parallel} = \vec{u}(\vec{a} \cdot \vec{u})/u^2$ and $\vec{a}_{\perp} = \vec{a} - \vec{a}_{\parallel} = \vec{u} \times (\vec{a} \times \vec{u})/u^2$, Larmor's formula in the rest frame is

$$P = \frac{2q^2}{3c^3}\gamma^4 \left(a_{\perp}^2 + \gamma^2 a_{\parallel}^2\right). \tag{4.6}$$

4.2. Total synchrotron power

Now consider a particle with charge q moving in a magnetic field $\vec{B}$. The equations of motion are

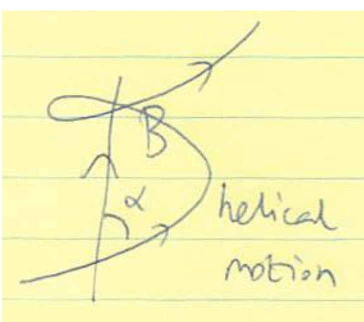
$$\frac{d}{dt}(\gamma m \vec{u}) = q\frac{\vec{u} \times \vec{B}}{c} \tag{4.7}$$

and

$$\frac{d}{dt}\left(\gamma m c^2\right) = q\vec{u} \cdot \vec{E}. \tag{4.8}$$

If the electric field is $\vec{E} = 0$, usually the case in astrophysical applications, then the energy of the particle is constant (γ is a constant) (recall that the magnetic field does no work on the particle because the force is always perpendicular to the velocity). The solution to equation (4.7) is *helical motion*: a constant velocity parallel to the magnetic field, $u_{\parallel} = \vec{u} \cdot \vec{B}/B$, and uniform circular motion in a plane perpendicular to $\vec{B}$, with gyration frequency

$$\omega_B = \frac{qB}{\gamma m c}. \tag{4.9}$$



The angle α is known as the *pitch angle*. The velocity perpendicular to the magnetic field is $u_{\perp} = u \sin\alpha$, so that $\alpha = \pi/2$ for pure circular motion ($u_{\parallel} = 0$).

The acceleration is $a_{\perp} = u_{\perp}\omega_B$, so that the total power is

$$P = \frac{2e^2}{3c^3}\gamma^4 \omega_B^2 u_{\perp}^2 = \frac{2}{3c} r_0^2 u_{\perp}^2 \gamma^2 B^2 \tag{4.10}$$

where $r_0 = (e^2/mc^2)$ is the classical electron radius. For a uniform distribution of pitch angles, the total power is

$$P = \frac{2}{3} r_0^2 c \gamma^2 \beta^2 B^2 \int \sin^2 \alpha \, \frac{d\Omega}{4\pi}. \tag{4.11}$$

The integral is 2/3, and the Thomson cross-section is $8\pi r_0^2/3$, giving the famous result

$$P = \frac{4}{3} \sigma_T c \beta^2 \gamma^2 U_B \tag{4.12}$$

where $U_B = B^2/8\pi$ is the magnetic energy density. Note the similarity to the inverse Compton power; the only difference is that the equation for inverse Compton power has U_γ rather than U_B. One way to think about synchrotron radiation that explains this similarity is as inverse Compton scattering of virtual photons in the magnetic field (see Blumenthal et al.).

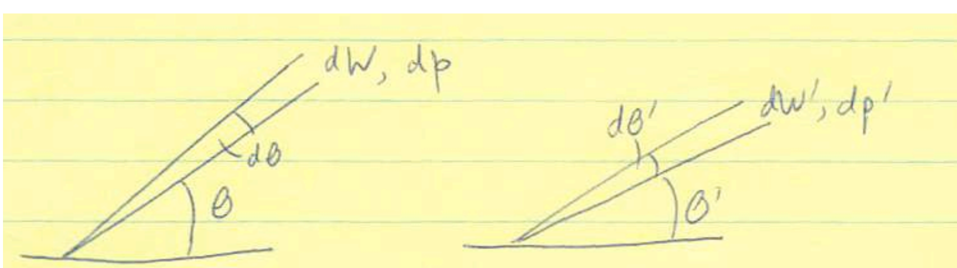
4.3. Angular distribution of received radiation

Before we can calculate the synchrotron spectrum, we have to think a little about what the angular distribution of radiation from a relativistic particle looks like. As you might expect, we'll find that the radiation is strongly focussed or beamed in the forwards direction.

In the electron rest frame, the power radiated is

$$\frac{dP'}{d\Omega'} = \frac{e^2 a'^2}{4\pi c^3} \sin^2 \Theta' \tag{4.13}$$

where Θ' is the angle between the acceleration vector and the emitted radiation, $\cos\Theta' = \vec{a}' \cdot \vec{n}$. How does this transform back into the lab frame?



Photons emitted into angle θ' in the rest frame travel at angle θ in the lab frame, where the angles are related by the aberration formula we used for Compton scattering,

$$\mu = \frac{\mu' + \beta}{1 + \beta\mu'} \rightarrow d\mu = \frac{d\mu'}{\gamma^2(1 + \beta\mu')^2}. \tag{4.14}$$

The energies are related by $dW = \gamma(dW' + cdp'_x) = \gamma(1 + \beta\mu')dW'$. Therefore,

$$\frac{dW}{d\Omega} = \frac{dW'}{d\Omega'}\gamma^3 \left(1 + \beta\mu'\right)^3 \tag{4.15}$$

and

$$\frac{dP}{d\Omega} = \frac{dP'}{d\Omega'}\left(\frac{dt'}{dt}\right)\gamma^3 \left(1 + \beta\mu'\right)^3 . \tag{4.16}$$

We need to relate the time interval in the lab frame dt to the time interval in the rest frame dt'. This is actually a subtle point. One way to do this would be to use time dilation: $\gamma dt' = dt$ (the radiation is emitted over a longer time interval as viewed in the lab frame). The resulting $dP/d\Omega$ is then known as the *emitted power*. However, because the particle is moving, a stationary observer in the lab frame actually measures a time interval $dt_A = \gamma(1-\beta\mu)dt'$. This is the same argument as led us to the κ factor in Compton scattering (recall the argument about measurements of a moving train). This choice gives the *received power* and is the one we'll use here.

Substituting dt'/dt_A into equation (4.16) and using $\vec{a}' \cdot \vec{a}' = \tilde{a}' \cdot \tilde{a}' = \tilde{a} \cdot \tilde{a}$ gives

$$\frac{dP}{d\Omega} = \frac{e^2}{4\pi c^3}\frac{a_\perp^2 + \gamma^2 a_\parallel^2}{(1 - \beta\mu)^4}\sin^2\Theta'. \tag{4.17}$$

What is Θ' in the lab frame? It helps to consider limiting cases. For acceleration parallel to the velocity, $\sin^2\Theta' = \sin^2\theta' = \sin^2\theta/\gamma(1 - \beta\mu)^2$, and so

$$\frac{dP_\parallel}{d\Omega} = \frac{e^2}{4\pi c^3}\frac{a_\parallel^2 \sin^2\theta}{(1 - \beta\mu)^6}. \tag{4.18}$$

For acceleration perpendicular to the velocity, $\cos\Theta' = \sin\theta'\cos\phi'$, and

$$\frac{dP_\perp}{d\Omega} = \frac{e^2 a_\perp^2}{4\pi c^3}\frac{1}{(1 - \beta\mu)^4}\left[1 - \frac{\sin^2\theta\cos^2\phi}{\gamma^2(1 - \beta\mu)^2}\right]. \tag{4.19}$$

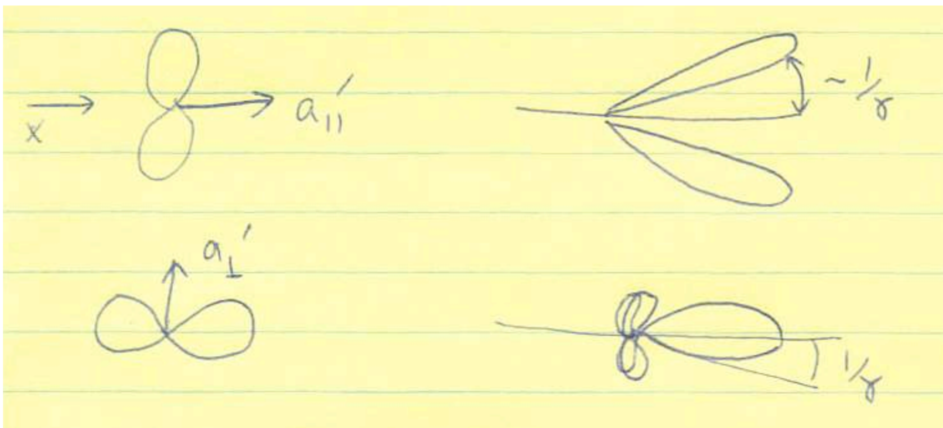
For large γ, these expressions can be rewritten using $\mu \approx 1 - \theta^2/2$, $\beta \approx 1 - 1/2\gamma^2$, $\gamma(1 - \beta\mu) \approx (1 + (\gamma\theta)^2)/2\gamma$, as

$$\frac{dP_\parallel}{d\Omega} = \frac{16 e^2 a_\parallel^2}{\pi c^3}\gamma^{10}\frac{(\gamma\theta)^2}{(1 + (\gamma\theta)^2)^6} \tag{4.20}$$

$$\frac{dP_\perp}{d\Omega} = \frac{4 e^2 a_\perp^2}{\pi c^3}\gamma^8\frac{1 - 2(\gamma\theta)^2\cos 2\phi + (\gamma\theta)^4}{(1 + (\gamma\theta)^2)^6}. \tag{4.21}$$

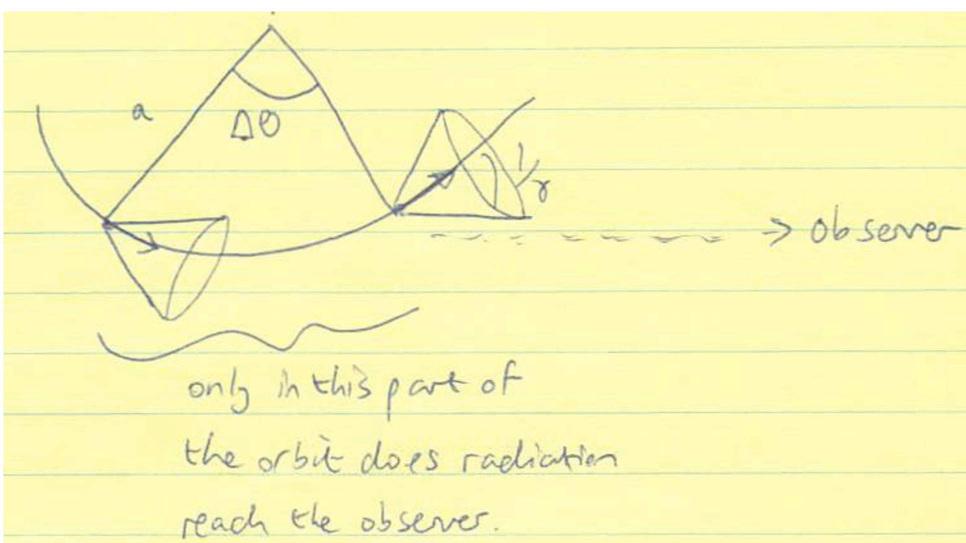
The important point is that these expressions depend on θ only through the combination $\gamma\theta$ and drop rapidly to zero for $\gamma\theta > 1$. The radiation is beamed into a cone of half angle $\sim 1/\gamma$.

For parallel and perpendicular acceleration the emission pattern looks like:



4.4. Simple treatment of synchrotron spectrum

We now use some simple arguments to get the basic form of the synchrotron spectrum from a relativistic electron with energy γ. The basic point is that the beaming gives rise to a broad frequency spectrum because we see only a short pulse of radiation each orbit when the beam is pointing at us.



To calculate the duration of the pulse, we first need to find the radius of curvature a of the orbit. Then the distance travelled by the particle while the line of sight lies in the cone of emission is $\Delta s = a\Delta\theta = 2a/\gamma$. To find a, we go back to the equation of motion. The velocity change is $|\Delta\vec{u}| = u\Delta\theta$ during a time $\Delta t = \Delta s/u$. Therefore,

$$\frac{\gamma m u^2 \Delta\theta}{\Delta s} = \frac{euB}{c}\sin\alpha. \tag{4.22}$$

But $\Delta\theta/\Delta s = 1/a$ giving

$$a = \frac{u}{\omega_B \sin\alpha} \tag{4.23}$$

and therefore

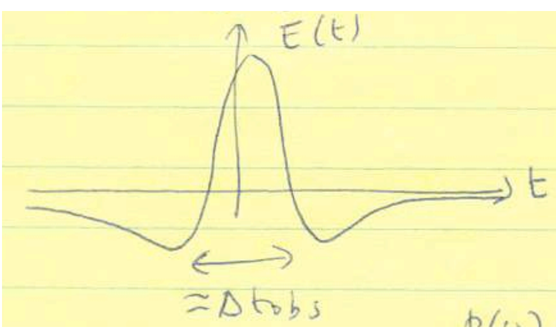
$$\Delta t = \frac{2}{\gamma\omega_B \sin\alpha}. \tag{4.24}$$

The observed time for a stationary observer is $\Delta t_{\rm obs} = (1-\beta)\Delta t = \Delta t/2\gamma^2$, giving

$$\Delta t_{\rm obs} = \frac{1}{\gamma^3\omega_B \sin\alpha}. \tag{4.25}$$

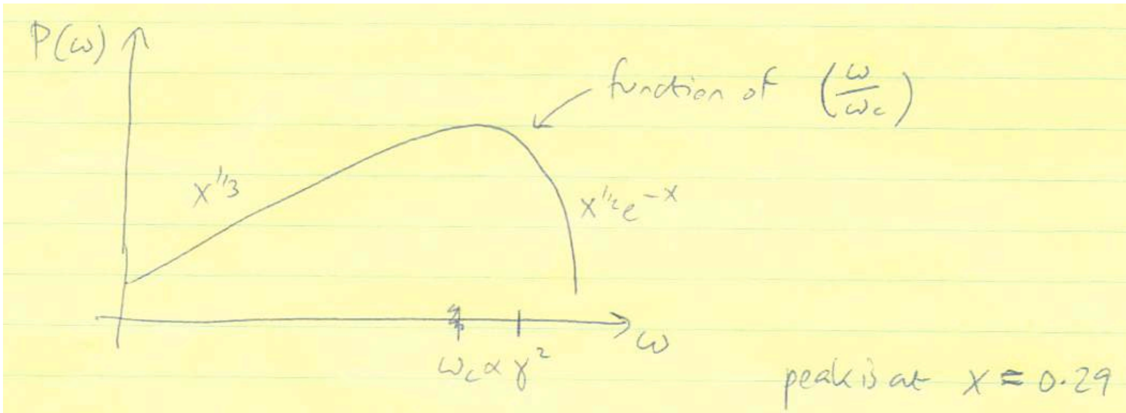
Note the observed duration of the pulse is shorter than the orbital time by a factor of γ^3: one power of γ from the beaming angle, and another two powers of γ from the train effect (duration of observed pulse).

The fact that the pulse has a short duration much smaller than the gyration time means that we can expect a broad frequency spectrum. The pulse looks like



and the spectrum is a function of $x = \omega/\omega_c$ where

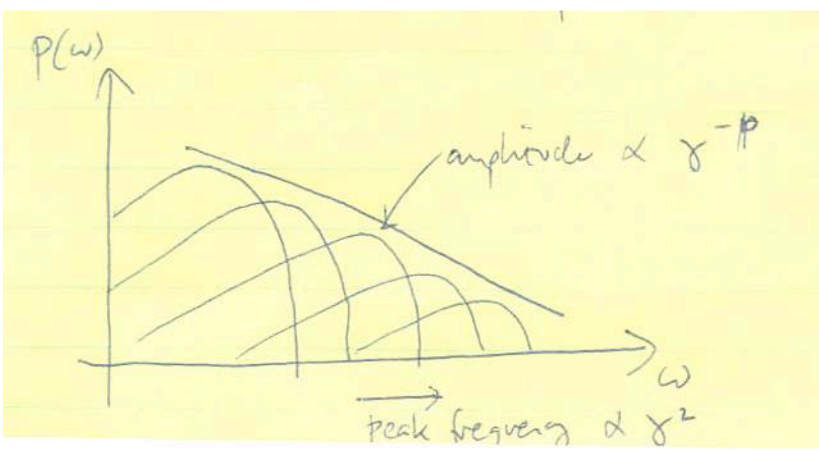
$$\omega_c = \frac{3}{2}\gamma^3\omega_B \sin\alpha \propto \gamma^2. \tag{4.26}$$



The low and high frequency behavior is shown in the figure.

Often, the electrons emitting the synchrotron radiation have a power-law distribution of energies, $N(E)dE \propto E^{-p}dE$ or $N(\gamma)d\gamma \propto \gamma^{-p}d\gamma$ (we'll discuss the origin of such *non-thermal* distributions later). In this case, the spectrum is given by an integral over the electron energy distribution

$$P(\omega) \propto \int d\gamma \gamma^{-p} F\left(\frac{\omega}{\omega_c}\right). \tag{4.27}$$



Integrals like this come up a lot, and the way to deal with them is to change variables from γ to $x = \omega/\omega_c \propto \gamma^{-2}$. The integral then becomes a dimensionless integral over x, i.e. just a number. If we do that here, we find

$$P(\omega) \propto \omega^{-(p-1)/2}. \tag{4.28}$$

The slope of the spectrum is the same as for inverse Compton scattering from a power-law distribution of electrons.

4.5. Detailed treatment of synchrotron spectrum

If you've looked in the books at the derivation of the synchrotron spectrum, you may be a little nervous. Shu warns "The formal manipulations required for synchrotron theory can get formidable" and Longair states "I am not aware of any particularly simple way of deriving [the synchrotron spectrum]..". In fact it's not that bad: in this section, we outline the detailed derivation of the synchrotron spectrum, focussing on the methodology and skipping a lot of the algebra. Here I follow the derivation given in Rybicki and Lightman 6.4, which in turn follows Jackson 14.6. Longair (volume 2) goes through it filling in more of the steps. The classic paper is Westfold (1959) ApJ.

We start with the expressions for the radiation fields we derived earlier,

$$\vec{E}_{\rm rad} = \frac{q}{c}\left[\frac{\vec{n}\times(\vec{n}-\vec{\beta})\times\dot{\vec{\beta}}}{\kappa^3 R}\right] \qquad \vec{B}_{\rm rad} = \vec{n}\times\vec{E}_{\rm rad} \tag{4.29}$$

where the notation [...] means evaluate at the retarded time $t = t' + R/c$ and $\kappa = 1 - \vec{n}\cdot\vec{u}/c$. Then

$$\frac{dW}{d\Omega} = \left[R^2\vec{n}\cdot\vec{S}\right] = \frac{c}{4\pi}\left[R\vec{E}\right]^2, \tag{4.30}$$

and the Fourier component at frequency ω is

$$\frac{dW}{d\omega d\Omega} = \frac{q^2}{4\pi^2 c}\left|\int\left[(\vec{n}\times(\vec{n}-\vec{\beta})\times\dot{\vec{\beta}})\kappa^{-3}\right]e^{i\omega t}dt\right|^2. \tag{4.31}$$

Previously, we worked in the limit $\beta \ll 1$, but now we need to keep the relativistic factors.

A standard trick is to integrate by parts. First, change variables in the integral to $t' = t - R(t')/c$. Since the source is very far from the observer, we can write $R \approx R_0 - \vec{n}\cdot\vec{r}(t')$. Then $dt = dt'(1 - \vec{n}\cdot\vec{\beta}) = \kappa dt'$, and the integral is

$$\frac{dW}{d\omega d\Omega} = \frac{q^2}{4\pi^2 c}\left|\int\frac{(\vec{n}\times(\vec{n}-\vec{\beta})\times\dot{\vec{\beta}})}{\kappa^2}e^{i\omega(t'-\vec{r}\cdot\vec{n}/c)}dt'\right|^2. \tag{4.32}$$

Now we can integrate by parts using

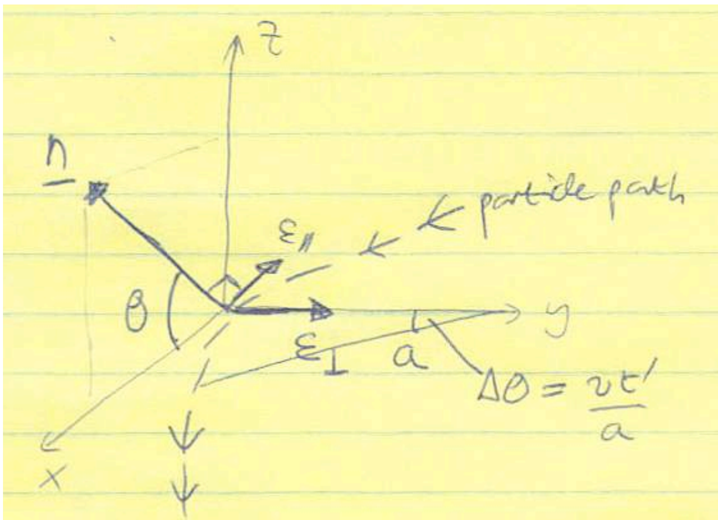
$$\frac{d}{dt'}\left(\frac{\vec{n}\times(\vec{n}\times\vec{\beta})}{1-\vec{\beta}\cdot\vec{n}}\right) = \frac{\vec{n}\times((\vec{n}-\vec{\beta})\times\dot{\vec{\beta}})}{(1-\vec{\beta}\cdot\vec{n})^2} \tag{4.33}$$

to get

$$\frac{dW}{d\omega d\Omega} = \frac{q^2}{4\pi^2 c}\left|\int dt'\ e^{i\omega(t'-\vec{r}\cdot\vec{n}/c)}\ \vec{n}\times(\vec{n}\times\vec{\beta})\right|^2 \tag{4.34}$$

which depends only on $\beta(t)$ and not $\dot{\beta}(t)$. This equation is the starting point for RL section 6.4.

Now we need to understand the particle path and can plug $\vec{\beta}(t)$ into equation (4.34) to evaluate the spectrum. The diagram is



The orbital plane of the particle is instantaneously (at $t' = 0$) in the x-y plane in this diagram, the acceleration is in the y direction at this moment, and the vector $\vec{n}$ points to the observer. We have defined a set of orthogonal axes $\vec{\epsilon}_{\parallel} = \vec{n} \times \vec{\epsilon}_{\perp}$. We will discuss the polarization in terms of these directions. The projected direction of the B field is along $\vec{\epsilon}_{\parallel}$.

The vector product $\vec{n} \times (\vec{\beta} \times \vec{n})$ is the component of $\vec{\beta}$ perpendicular to $\vec{n}$, which for $\beta = 1$ is

$$\vec{n} \times (\vec{\beta} \times \vec{n}) = \vec{\epsilon}_{\perp} \sin\left(\frac{ut'}{a}\right) - \vec{\epsilon}_{\parallel} \cos\left(\frac{ut'}{a}\right) \sin\theta \tag{4.35}$$

This formula is non-intuitive: for example if we take $\theta = 0$ (observer in the plane of the orbit) it vanishes (and therefore there is no contribution to the integral) when the particle is pointing directly at the observer ($t' = 0$).

The next step is to expand the angular factors since the cone of emission is very small for a relativistic particle. For example,

$$t' - \frac{\vec{n} \cdot \vec{r}(t')}{c} = t' - \frac{a}{c} \cos\theta \sin\left(\frac{ut'}{a}\right) \approx \frac{1}{2\gamma^2} \left[\left(1 + (\gamma\theta)^2\right) t' + \frac{c^2\gamma^2 t'^3}{3a^2} \right]. \tag{4.36}$$

Except for $\theta < 1/\gamma$ and $ct' < a/\gamma$, this factor in the exponent oscillates rapidly and its contribution to the integral vanishes. Therefore, we can integrate over the full particle path and only the contribution from small angles is included. Equation (4.35) can be expanded similarly, and the result is

$$\frac{dW}{d\omega d\Omega} = \frac{dW_\perp}{d\omega d\Omega} + \frac{dW_\parallel}{d\omega d\Omega} \tag{4.37}$$

where

$$\frac{dW_\perp}{d\omega d\Omega} = \frac{q^2\omega^2}{4\pi^2 c}\left|\int \frac{ct'}{a} \exp\left[\frac{i\omega}{2\gamma^2}\left(\theta_\gamma^2 t' + \frac{c^2\gamma^2 t'^3}{3a^2}\right)\right] dt'\right|^2 \tag{4.38}$$

$$\frac{dW_\parallel}{d\omega d\Omega} = \frac{q^2\omega^2\theta^2}{4\pi^2 c}\left|\int \exp\left[\frac{i\omega}{2\gamma^2}\left(\theta_\gamma^2 t' + \frac{c^2\gamma^2 t'^3}{3a^2}\right)\right] dt'\right|^2 \tag{4.39}$$

and we define the notation $\theta_\gamma^2 = 1 + (\theta\gamma)^2$.

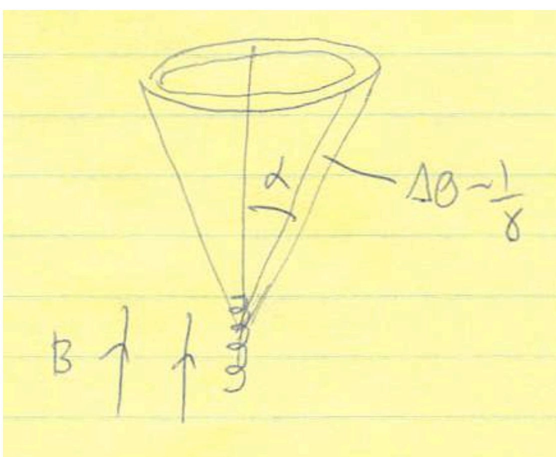
These results can be put into the form of standard functions if the integration limits go to $\pm\infty$, which is okay because the integrand dies away quickly for $t' > a/\gamma c$. The result is

$$\frac{dW_\perp}{d\omega d\Omega} = \frac{q^2\omega^2}{3\pi^2 c}\left(\frac{a\theta_\gamma^2}{c\gamma^2}\right)^2 K_{2/3}^2(\eta) \tag{4.40}$$

$$\frac{dW_\parallel}{d\omega d\Omega} = \frac{q^2\omega^2}{3\pi^2 c}\theta^2\left(\frac{a\theta_\gamma^2}{c\gamma}\right)^2 K_{5/3}^2(\eta) \tag{4.41}$$

where $\eta = \omega a\theta_\gamma^3/3c\gamma^3$.

Now integrate over the solid angle. The emission is in a cone



The solid angle integral is therefore $\int d\Omega \to \int_{-\infty}^{\infty} 2\pi \sin\alpha d\theta$, where once again we take the integration limits to be $\pm\infty$ on θ. This gives the total energy per orbit; the power is the energy per orbit multiplied by the orbital frequency $\omega_B/2\pi$. This gives the power in each polarization

$$P_\perp(\omega) = \frac{\sqrt{3}q^3 B\sin\alpha}{4\pi mc^2}\left[F(x) + G(x)\right] \tag{4.42}$$

$$P_{\parallel}(\omega) = \frac{\sqrt{3}q^3 B \sin\alpha}{4\pi mc^2}\left[F(x) - G(x)\right] \tag{4.43}$$

where $x = \omega/\omega_c$, $F(x) = x\int_x^\infty d\xi K_{5/3}(\xi)$ and $G(x) = xK_{2/3}(x)$. The total power in both polarizations is

$$P(\omega) = \frac{\sqrt{3}q^3 B \sin\alpha}{2\pi mc^2} F(x). \tag{4.44}$$

The following integrals of $F(x)$ and $G(x)$ are useful

$$\int_0^\infty x^\mu F(x)dx = \frac{2^{\mu+1}}{\mu+2}\Gamma\left(\frac{\mu}{2}+\frac{7}{3}\right)\Gamma\left(\frac{\mu}{2}+\frac{2}{3}\right) \tag{4.45}$$

$$\int_0^\infty x^\mu G(x)dx = 2^\mu\Gamma\left(\frac{\mu}{2}+\frac{4}{3}\right)\Gamma\left(\frac{\mu}{2}+\frac{2}{3}\right). \tag{4.46}$$

For a power-law distribution of electron energies $N(\gamma) = N(\gamma_1)(\gamma/\gamma_1)^{-p}$, the frequency spectrum is given by

$$\frac{dP}{dVd\omega} = \int_{\gamma_1}^{\gamma_2} N(\gamma)d\gamma\frac{dP}{d\omega}. \tag{4.47}$$

For $\gamma_2 \gg \gamma_1$ and $p > 1$, the number density of electrons is $n_e = N(\gamma_1)\gamma_1/(p-1)$, giving

$$\frac{dP}{dVd\omega} = \frac{\sqrt{3}q^3 B \sin\alpha}{2\pi mc^2}\frac{(p-1)n_e}{\gamma_1}\int\left(\frac{\gamma}{\gamma_1}\right)^{-p} F(x)d\gamma. \tag{4.48}$$

Now as we discussed earlier, changing integration variable from γ to

$$x = \frac{\omega}{\omega_c} = \gamma^{-2}\left(\frac{2\omega mc}{3eB\sin\alpha}\right) \tag{4.49}$$

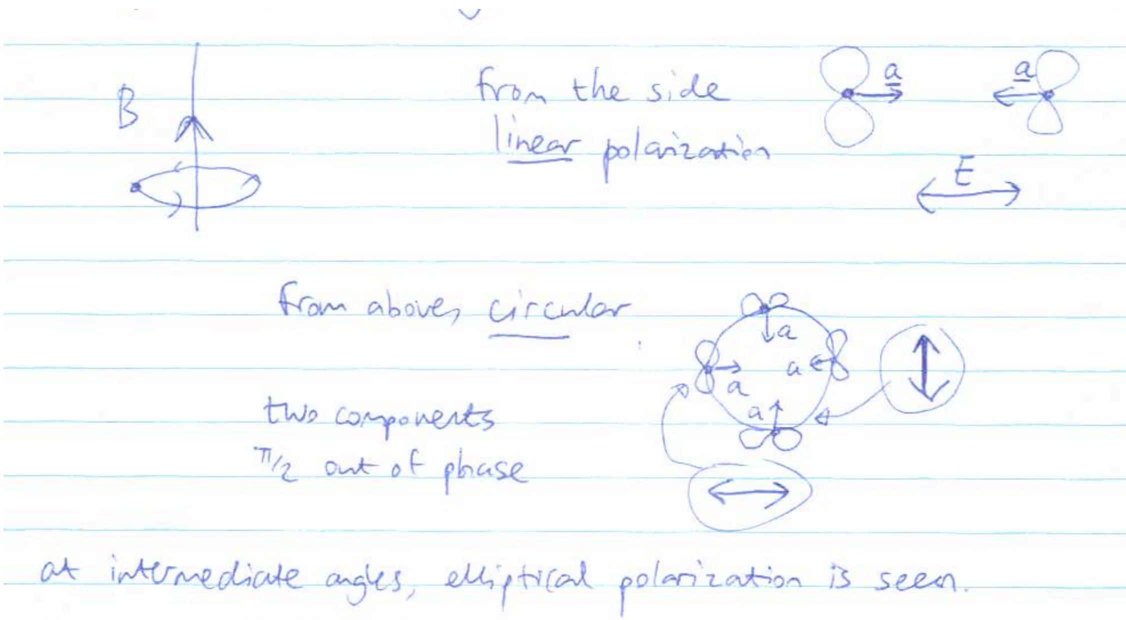
gives

$$\frac{dP}{dVd\omega} \propto B^{(p+1)/2}\omega^{-(p-1)/2}. \tag{4.50}$$

4.6. Important features of synchrotron radiation and some applications

Let's briefly highlight here some important aspects of synchrotron radiation. These are worked out in more detail in HW4.

- **Polarization**. One of the distinguishing features of synchrotron radiation is that it is strongly-polarized. For cyclotron emission, this is straightforward to see from the following diagram



For synchrotron, since we only see the radiation when the particle is moving within $1/\gamma$ of towards us, we see linear polarization with the dominant component being the polarization perpendicular to $\vec{B}$. For a single electron, 7 times more power is radiation in the perpendicular than in the parallel polarization. For a power law electron energy distribution, the degree of polarization is $\Pi = (P_\perp - P_\parallel / P_\perp + P_\parallel) = (p+1)/(p+7/3)$.

An application of this is to map out the magnetic fields of spiral galaxies.

- **Energetics**. Radio galaxies show strong synchrotron emission in radio ($\nu \lesssim 300$ GHz or $\lambda \gtrsim 1$ mm). Jets from the central AGN terminate in huge radio lobes which have luminosities $L \sim 10^{45}$ erg s^{-1}. The total energy implied for a lobe of ≈ 50 kpc across is $\sim 10^{60}$ erg in particles and fields.

  An important result is the *minimum energy argument* of Burbidge (1959). For a given synchrotron luminosity, you can show that the minimum energy requires corresponds to the case where the energies of particles and fields are roughly equal, known as *equipartition* of energy between particles and fields.

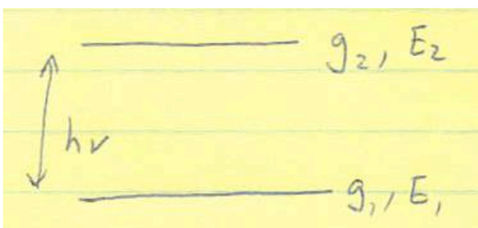
- **Cooling**. The cooling time for the radio lobes in radio galaxies is very short, $\sim 10^7$ years, implying that the energy must be continuously replenished. An important competition occurs between synchrotron and inverse Compton, the ratio of the power emitted from each process is just the ratio of magnetic and photon energy densities $P_{\rm synch}/P_{\rm IC} = U_B/U_\gamma$. *Synchrotron self-Compton* is the process in which inverse Compton scattering of synchrotron photons occurs by the same electrons that radiated the synchrotron photons. An *inverse Compton catastrophe* is believed to occur when U_γ

exceeds U_B, which corresponds to a brightness temperature of $\sim 10^{12}$ K (e.g. Readhead 1994; Kelleman and Pading-Toth 1969).

- **The transition from cyclotron to synchrotron**. We have assumed relativistic electrons here $\gamma \gg 1$ so that the radiation is strongly beamed. As we mentioned earlier for non-relativistic electrons the emission is at the cyclotron frequency. As v/c increases, the harmonics of the cyclotron frequency are excited, eventually merging to produce the synchrotron spectrum we have calculated (see Rybick and Lightman 6.6).

4.7. Synchrotron self-absorption

First, a reminder of *Einstein coefficients*. Consider a 2-level system



The three Einstein coefficients that describe transitions between the two levels are: A_{21} the rate (probability per unit time) of spontaneous emission of a photon, $B_{12}\bar{J}$ is the rate of absorption of a photon, $B_{21}\bar{J}$ is the rate of stimulated emission, where

$$\bar{J} = \int_0^\infty J_\nu \phi(\nu) d\nu \tag{4.51}$$

is the integral of the mean intensity over the line profile $\phi(\nu)$ ($\int d\nu\phi(\nu) = 1$). Now, in thermal equilibrium, the rates of downwards transitions (2 to 1) must balance the upwards transitions (1 to 2) or

$$n_1 B_{12}\bar{J} = n_2 A_{21} + n_2 B_{21}\bar{J}. \tag{4.52}$$

But in thermal equilibrium, $\bar{J} = B_\nu$ (the line profile is narrow compared to the scale on which B_ν varies) and $n_2/n_1 = (g_2/g_1)e^{-h\nu/k_BT}$. These three conditions imply relations between the Einstein coefficients,

$$A_{21} = B_{21}\frac{2h\nu^3}{c^2} \qquad g_1 B_{12} = g_2 B_{21}. \tag{4.53}$$

Although thermal equilibrium is assumed in deriving these relations, it is not required for them to hold.

The absorption coefficient α_ν and emission coefficient j_ν can be written in terms of the Einstein coefficients. First, the emission coefficient which describes the rate at which photons are added to the radiation field independent of the incident radiation we relate to A_{21}

$$j_\nu = \frac{h\nu}{4\pi} n_2 A_{21} \phi(\nu). \tag{4.54}$$

Since the stimulated emission is proportional to $\bar{J}$, we include it in the absorption coefficient as a negative absorption term. The net absorption is

$$n_1 B_{12} \bar{J} \left(1 - \frac{n_2}{n_1}\frac{B_{21}}{B_{12}}\right) = n_1 B_{12} \bar{J} \left(1 - e^{-h\nu/k_B T}\right), \tag{4.55}$$

where in the last step we use the level populations in thermal equilibrium. Note that this does not require full thermal equilibrium to be valid, but only LTE; we assume that the level densities are thermally-populated but the matter does not have to be in equilibrium with the photons. Therefore it is often a good assumption. The term in brackets is referred to as the *correction for stimulated emission*. We have seen it previously in the expression for free-free absorption. The absorption coefficient is then

$$\alpha_\nu = \frac{h\nu}{4\pi} n_1 B_{12} \phi(\nu)(1 - e^{-h\nu/k_B T}). \tag{4.56}$$

Now we can apply these ideas to synchrotron absorption. The idea is to use the Einstein relations to obtain the synchrotron absorption coefficient from the emission coefficient that we have already calculated. The main difference is that rather than a two-level system, in this case we must consider transitions between states in the continuum. The emission coefficient can be written

$$j_\nu = \frac{1}{4\pi} \int d^3\vec{p} f(p) P_\nu(p) \tag{4.57}$$

where $f(p)$ is the density of states as a function of electron momentum p, and $P_\nu(p)$ is the power emitted at frequency ν by an electron with momentum p (something we have calculated in previous sections). Then applying the Einstein relations, we get the absorption coefficient

$$\alpha_\nu = \frac{c^2}{8\pi h\nu^3} \int d^3\vec{p} \left[f(\vec{p}^\star) - f(\vec{p})\right] P_\nu(p) \tag{4.58}$$

where $p^\star$ is the electron momentum corresponding to an energy $\epsilon + h\nu$.

As a check, we can put in a thermal (Maxwell-Boltzmann) distribution for the electron distribution, $f(p) = Ce^{-\epsilon/k_B T}$. Then $f(p^\star) = Ce^{-\epsilon/k_B T} e^{h\nu/k_B T}$, which gives

$$\alpha_\nu = \frac{c^2}{2h\nu^3}(e^{h\nu/k_B T} - 1)\frac{1}{4\pi} \int d^3\vec{p} f(p) P_\nu(p) = \frac{j_\nu}{B_\nu}. \tag{4.59}$$

And so we see that we recover Kirchoff's law $j_\nu = \alpha_\nu B_\nu$. You may recall that we used Kirchoff's law directly to get the free-free absorption coefficient from the free-free emissivity.

For synchrotron, we have to be more careful because the electron population is not necessarily in thermal equilibrium.

Now let's evaluate α_ν for a power law distribution of electron energies, $N(E) \propto E^{-p}$. We will assume that $h\nu \ll \gamma m_e c^2$, $\gamma \gg 1$ and that $f(p)$ is isotropic. First we change variables in the integral from momentum to energy $E = pc$, $N(E)dE = f(p)4\pi p^2 dp$,

$$\alpha_\nu = \frac{c^2}{8\pi h\nu^3} \int E^2 dE P_\nu(E) \left[\frac{N(E-h\nu)}{(E-h\nu)^2} - \frac{N(E)}{E^2} \right]. \tag{4.60}$$

For $h\nu \ll E$, we Taylor expand the first term in the square brackets, to get

$$\alpha_\nu = -\frac{c^2}{8\pi\nu^2} \int E^2 dE P_\nu(E) \frac{\partial}{\partial E} \left(\frac{N}{E^2} \right). \tag{4.61}$$

Evaluating this for $N(E) = E^{-p}$ gives

$$\alpha_\nu = \frac{(p+2)c^2}{8\pi\nu^2} \int dE P_\nu(E) \frac{N}{E}. \tag{4.62}$$

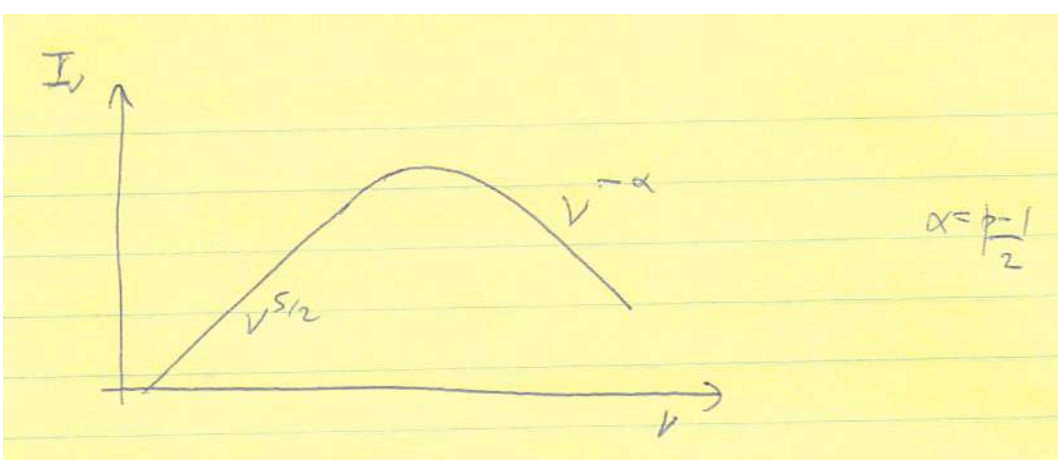
But $j_\nu = \int dE P_\nu(E) N(E)$, so the source function is

$$S_\nu = \frac{j_\nu}{\alpha_\nu} = \frac{2\nu^2}{c^2} \frac{1}{p+2} \frac{\int dE N(E) P_\nu(E)}{\int dE N(E) P_\nu(E)/E}. \tag{4.63}$$

Now put in the expression for $P_\nu(E) \propto F(x)$ and change integration variables from $E = \gamma m_e c^2$ to $x = \omega/\omega_c$ as we did earlier. This makes the integrals dimensionless, and shows us the scaling with ν,

$$S_\nu = \frac{2\nu^2}{c^2} \frac{mc^2}{p+2} \left(\frac{4\pi\nu mc}{3eB\sin\alpha} \right)^{1/2} \frac{\int dx x^{(p-3)/2} F(x)}{\int dx x^{(p-3)/2} F(x) x^{1/2}} \propto \nu^{5/2}. \tag{4.64}$$

The overall synchrotron spectrum therefore looks like



turning over from the optically thin scaling $\propto \nu^{-(p-1)/2}$ at high frequency to $\propto \nu^{5/2}$ at low frequencies where the source becomes optically thick to synchrotron self-absorption.

By comparing this with $S_\nu = (2\nu^2/c^2)k_BT_b$, we see that the brightness temperature is

$$\frac{k_BT_b}{m_ec^2} = \left(\frac{\omega}{\frac{3}{2}\frac{eB}{m_ec}\sin\alpha}\right)^{1/2} \left\{\frac{1}{p+2}\frac{\int dx x^{(p-3)/2}F(x)}{\int dx x^{(p-3)/2}F(x)x^{1/2}}\right\}. \tag{4.65}$$

If we look at frequency ω, that emission is primarily coming from electrons with energy γ_ω such that $\omega = \omega_c = \gamma_\omega^2(3eB/2m_ec)\sin\alpha$, so we can write

$$\frac{k_BT_b}{m_ec^2} = \gamma_\omega \left\{\frac{1}{p+2}\frac{\int dx x^{(p-3)/2}F(x)}{\int dx x^{(p-3)/2}F(x)x^{1/2}}\right\}. \tag{4.66}$$

This then gives us a way to think of the self-absorbed spectrum. At each frequency, we calculate the γ of the electrons that radiate the peak of their spectrum at that frequency, i.e. we calculate γ_ω. The brightness temperature at that frequency is then given by $k_BT \approx \gamma_\omega m_ec^2 \propto \nu^{1/2}$ and therefore $S_\nu \propto \nu^{5/2}$. This is the back of the envelope understanding of the spectrum that is often given in the books.

Summary and Further Reading

Here are the main ideas and results that we covered in this part of the course:

- *Cyclotron radiation* from non-relativistic electrons. Cyclotron frequency $\omega_c = eB/m_ec =$, or $f_c = 2.8$ MHz (B/G).

- *Synchrotron.* Relativistic gyration frequency $\omega_B = eB/\gamma m_ec$. Power radiated per electron $P = (4/3)\gamma^2\beta^2\sigma_TcU_B$. Distinction between "emitted" and "received" power. Angular distribution of radiation from a relativistic charge – beaming into angle $1/\gamma$. Characteristic frequency of radiation $\omega_c = (3/2)\gamma^3\omega_B\sin\alpha$ (roughly 1/pulse duration). Broad spectrum $\propto x^{1/3}$ ($x = \omega/\omega_c \ll 1$) or $x^{1/2}e^{-x}$ ($x \gg 1$), peaks at $x \approx 0.3$. Exact form

$$P_\perp(\omega) = \frac{\sqrt{3}e^3B\sin\alpha}{4\pi m_ec^2}\left[F(x)+G(x)\right] \qquad P_\perp(\omega) = \frac{\sqrt{3}e^3B\sin\alpha}{4\pi m_ec^2}\left[F(x)-G(x)\right]$$

- For a *power law distribution of electrons*, $dP/dVd\omega \propto B^{(p+1)/2}\omega^{-(p-1)/2}$. Typically, $p \approx 2.5$ giving $\alpha = (p-1)/2 \approx 0.7$.

- *Einstein coefficients.* $A_{21} = B_{21}(2h\nu^3/c^2)$, $g_1 B_{12} = g_2 B_{21}$. Correction to absorption coefficient for stimulated emission. $j_\nu = (h\nu/4\pi)n_2 A_{21}\phi(\nu)$, $\alpha_\nu = (h\nu/4\pi)n_1 B_{12}\phi(\nu)(1 - e^{-h\nu/k_BT})$.

- *Synchrotron self-absorption.* $S_\nu \propto \nu^{5/2}$, $k_B T_b \approx \gamma m_e c^2$ where γ is such that $\omega_c = \omega$.

- *Important features of synchrotron radiation.* Polarization as a characteristic feature of synchrotron radiation. Minimum energy argument. Cooling timescales for synchrotron and inverse Compton. The inverse Compton catastrophe. Application to radio galaxies, magnetic fields of spiral galaxies.

Reading

- Synchrotron derivation: RL section 6.4, Jackson 14.6, Longair volume 2. Westfold (1959) ApJ.

- Minimum energy argument, Burbidge 1959. Inverse Compton catastrophe, Readhead 1994, Kelleman & Pauliny-Toth 1969.

5. Fermi Acceleration and Plasma Effects

These are notes for part five of PHYS 642 Radiative Processes in Astrophysics. We discuss two distinct topics: the origin of power-law distributions of particle energy, in particular Fermi acceleration in strong shocks as a generic origin of power-law distributions, and propagation of electromagnetic waves through plasma, including the effects of dispersion and Faraday rotation.

5.1. General way to make a power law spectrum of particle energies

A characteristic spectral index for synchrotron is $\alpha \approx 0.7$ ($F_\nu \propto \nu^{-0.7}$) which implies $p \approx 2.5$ ($N(\gamma) \propto \gamma^{-2.5}$) (recall that $\alpha = (p-1)/2$). This raises the question of why the electron distribution is a power law and in particular why p has this value.

We already have a clue as to how get a power law from our discussion of Compton scattering by optically thin relativistic electrons (see also Rybicki and Lightman 7.5). There, the photon energy increased by the same fractional amount $\Delta\epsilon/\epsilon$ on each scattering, but each subsequent scattering is less likely by a factor of the optical depth τ. The resulting spectrum is a power law.

Here the idea is similar: imagine a scattering process for particles in which the energy of the particle changes by $\Delta\epsilon/\epsilon = B$ on average, but there is also a probability per collision that the particles escape $P_{\rm esc}$. If the mean time between collisions is t_c, then the particle energy obeys $d\epsilon/dt = \epsilon B/t_c$ or $\epsilon = \epsilon_0 e^{Bt/t_c}$. On the other hand, the number of particles drops according to $dN/dt = -NP_{\rm esc}/t_c$ so that $N = N_0 e^{-tP_{\rm esc}/t_c}$. Dividing these distributions gives the energy distribution

$$\frac{dN}{d\epsilon} = \frac{dN}{dt}\frac{dt}{d\epsilon} \propto e^{-P_{\rm esc}t/t_c} e^{-Bt/t_c} \propto e^{-(Bt/t_c)(1+P_{\rm esc}/B)} \propto \left(\frac{\epsilon}{\epsilon_0}\right)^{-(1+P_{\rm esc}/B)}. \tag{5.1}$$

A power law distribution of energies is the result, with index $p = 1 + P_{\rm esc}/B$.

5.2. Second order Fermi acceleration

Originally, Fermi suggested a mechanism in which particles would be accelerated on colliding from moving clouds of gas. For example, consider the collision of a particle with a cloud moving with velocity V to the right. The particle travels to the left with velocity v and makes an angle θ with respect to the normal to the surface.

The way to analyze this collision is to move into the frame of the cloud. The cloud is much more massive than the particle and so in that frame we can think of the particle reversing it's momentum but not changing its energy. The incoming energy and momentum in the rest frame of the cloud are $E' = \gamma_V(E + pV\cos\theta)$ and $p'_x = \gamma_V(p\cos\theta + VE/c^2)$, where γ_V refers to γ calculated using the velocity V of the cloud. In the collision, the energy is unchanged, but the x-component of momentum reverses. Now transforming back, the energy of the particle post-collision in the original frame is $E'' = \gamma_V(E' + Vp'_x)$ or

$$E'' = E\gamma_V^2\left[1 + 2\frac{pV}{E}\cos\theta + \frac{V^2}{c^2}\right] \tag{5.2}$$

or to second order in V/c,

$$\frac{\Delta E}{E} = \frac{2vV}{c^2}\cos\theta + 2\left(\frac{V}{c}\right)^2. \tag{5.3}$$

To average over angles, we note that the rate of encounters of the particle with a cloud depends on whether the collision is a trailing collision or head-on collision. Specifically, the rate of encounters is $\propto 1 + (V/c)\cos\theta)$. Therefore

$$\langle\frac{\Delta E}{E}\rangle = \frac{\int_{-1}^{1} d\mu(1 + (V/c)\mu)(2(V/c)^2 + (2Vv/c^2)\mu)}{\int_{-1}^{1} d\mu(1 + (V/c)\mu)} \tag{5.4}$$

which for $v = c$ is $\langle\Delta E/E\rangle = (8/3)(V/c)^2$, second order in V/c.

The fact that the acceleration is only second order makes it inefficient, but perhaps a more important objection to this mechanism is that the values of $P_{\rm esc}$ and B, and therefore the power law index p, are not expected to be generic. They would presumably depend on the environment that the particle happens to find itself in.

5.3. First order Fermi acceleration by strong shocks

This is a mechanism in which particles scatter back and forth across a shock. Consider a strong shock[9] in gas with $\gamma = 5/3$. You may recall that in the frame of the shock, the incoming supersonic matter is slowed by the shock to a speed $u_s/4$, and compressed by a factor of 4 ($\rho_2 = 4\rho_1$). Here is a picture of the shock in the lab frame and in the frame moving with the shock:

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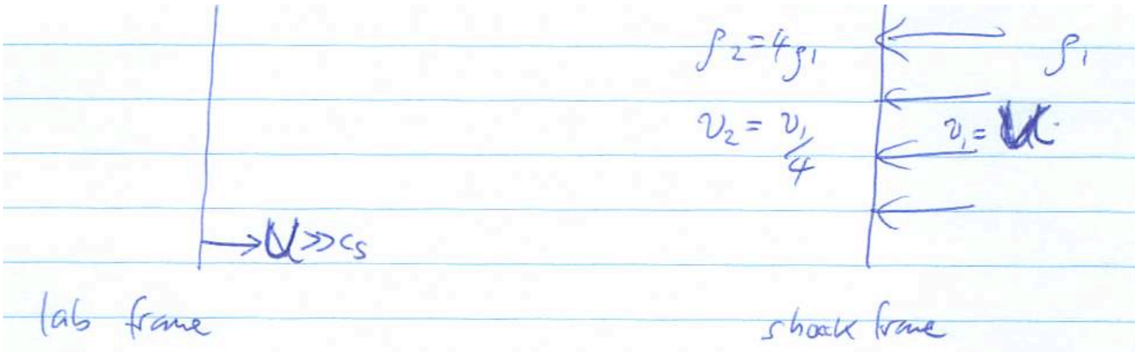
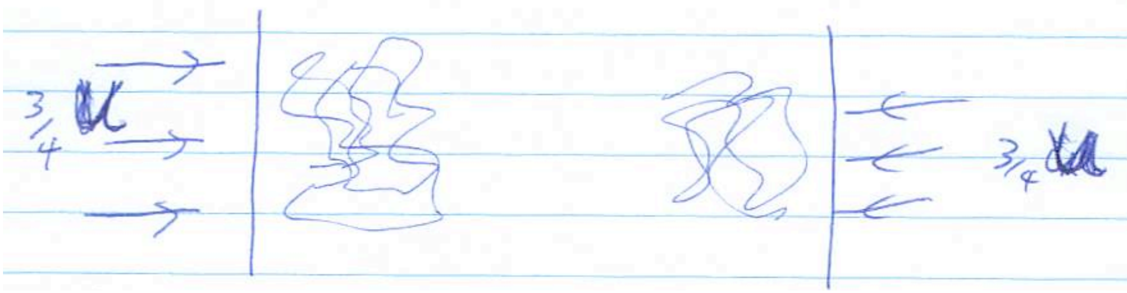
[9]These results come from applying conservation of mass, momentum and energy across the shock. Mass conservation is $\rho_1 u_1 = \rho_2 u_2$. For a strong shock, the force balance is $P_2 \approx \rho_1 u_1^2$.



The idea is that a relativistic particle which has an isotropic momentum distribution on one side of the shock (ie. is "at rest" on average relative to the fluid) moves across the shock. What does it see? If we draw a picture of the shock in a frame moving with the upstream or with the downstream fluid, we see



The particle that crosses the shock encounters fluid moving towards it with speed $3u_s/4$ *no matter which direction it crosses the shock*. This is the key to this mechanism: the particle always undergoes a head-on collision, and so the resulting acceleration is first order. After crossing the shock, the particle scatters and becomes isotropic in the new fluid, and in doing so gains energy.

To calculate the energy gain, we can use our results for second order Fermi acceleration, but allowing for the fact that p'_x vanishes after the scattering rather than being reversed. The first order term is therefore $\Delta E/E = (Vv/c^2)\cos\theta$ where here V corresponds to $3u_s/4$, the velocity of the incoming fluid. We assume the particle is relativistic, and set $v = c$. Averaging over angles $\int_0^1 d\mu 2\mu(V\mu/c) = 2V/3c$ (the factor of 2 in the integral comes from normalizing the probability over incoming angles). For a round trip crossing from the downstream region into the upstream region, and back again, the energy gain is $B = \delta E/E = 4V/3c = u_s/c$.

The escape probability $P_{\rm esc}$ comes from the probability that a particle will be swept downstream from the shock rather than crossing the shock again. The rate at which particles cross the shock is $(1/2)\int_0^1 d\mu Nc\mu = Nc/4$ in either direction. The rate at which particles are advected away is $Nu_s/4$. Therefore $P_{\rm esc} \approx u_s/c$.

Putting these values of B and $P_{\rm esc}$ into our result for the particle spectrum, we predict

$$N(E)dE \propto E^{-2}dE \tag{5.5}$$

very close to observed values (which are $p \approx 2$–3). The importance of this mechanism is that it provides a generic way to make a power law distribution with about the right index, that should apply across many astrophysical environments (shocks are ubiquitous). Corrections to the index p come from changing the shock compressibility (for example a radiative shock can have $\rho_2 \gg \rho_1$) or from including the effect of the pressure of the accelerated particles on the shock itself.

5.4. Evolution of the particle spectrum

The evolution of the particle spectrum is described by a *diffusion loss* equation

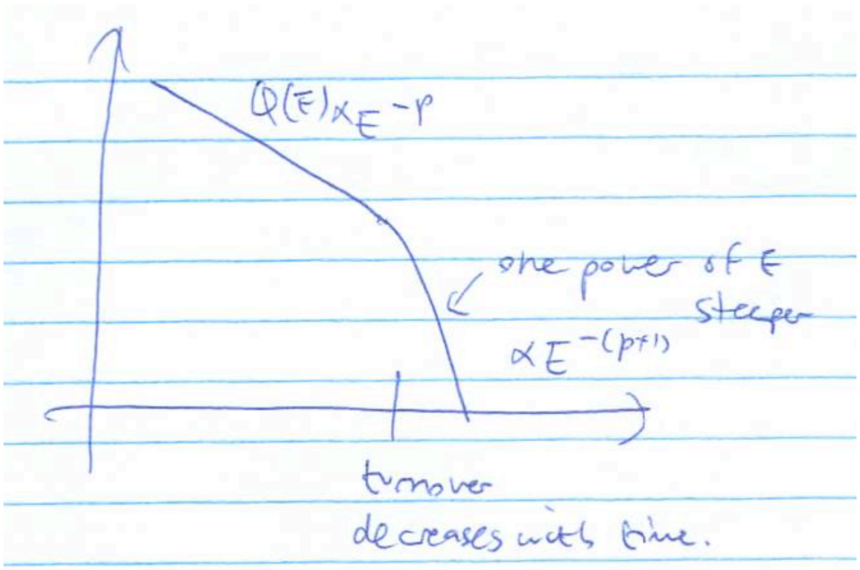
$$\frac{dN(E)}{dt} = \frac{d}{dE}\left[b(E)N(E)\right] + Q(E,t) + D\nabla^2 N(E) \tag{5.6}$$

where the particles are injected at a rate $Q(E)$, cool according to $dE/dt = -b(E)$, and the diffusion term represents spatial diffusion, for example diffusion of particles away from a source. A derivation can be found in Longair (I am using his notation and following his approach here), it is similar to our derivation of the Kompaneet's equation earlier.

Now let's solve this equation and look for a steady-state solution ($dN/dt = 0$) for a spatially uniform distribution of sources (so that we can ignore the spatial diffusion term). We take the injection term to be $Q(E) = \kappa E^{-p}$. Then $d(bN)/dE = -Q(E)$ has a solution (assuming $N \to 0$ for large energies)

$$N(E) = \frac{\kappa E^{-(p-1)}}{(p-1)b(E)}. \tag{5.7}$$

If synchrotron losses set the cooling rate of particles, $b(E) \propto E^2$, and we expect $N(E) \propto E^{-(p+1)}$. Therefore for particles that have had sufficient time to cool, the energy spectrum of the particles will be steeper than the initial spectrum by one power of E. For lower energy particles that have not yet had time to cool, the spectrum will correspond to the injection spectrum. Therefore we expect a break:



The energy of the break tells you about the lifetime of a given source.

5.5. Propagation of electromagnetic waves in a plasma: dispersion

We now turn to a different topic and consider the propagation of EM waves through plasma. In other words, we begin to ask what effect does intervening gas have on the radiation from an astrophysical source as it propagates towards us? First consider an unmagnetized plasma. In that case, there are two effects: only waves with frequencies larger than the plasma frequency can propagate, and the group velocity depends on the frequency, leading to dispersion.

To see what electromagnetic waves look like in a plasma, we must solve Maxwell's equations, in particular

$$\frac{\partial \vec{E}}{\partial t} = c\vec{\nabla}\times\,\vec{B} - 4\pi\vec{J} \qquad \frac{\partial \vec{B}}{\partial t} = -c\vec{\nabla}\times\,\vec{E}. \tag{5.8}$$

For a plasma, we must include the term $4\pi\vec{J}$ in Ampere's law since the electric field in the wave drives a current. We will see that this leads to a difference in the speed of the wave compared to in vacuum. We will look for a solution $\vec{E} = \vec{e_x}E_x e^{-i\omega t}e^{i\vec{k}\cdot\vec{r}}$.

To determine the current density $\vec{J}$, we write the equation of motion for the electrons $-i\omega m_e v_x = -eE_x$ which gives $v_x = -ieE_x/m_e\omega$. We ignore the motion of the ions, which is valid since $Am_p \gg m_e$ and the corresponding velocity is much lower. The current is

$\vec{J} = -en_e\vec{v_e}$, or

$$J_x = i\frac{n_e e^2}{m_e \omega} E_x. \tag{5.9}$$

Note that the current and electric field are $\pi/2$ out of phase, and therefore $\langle \vec{J} \cdot \vec{E} \rangle = 0$: there is no net work done on the plasma by the wave. Therefore the wave propagates without dissipation.

The next step is to write $\vec{\nabla} \times \vec{B} \propto \vec{E}$ using the fact that $\vec{J} \propto \vec{E}$,

$$i\vec{k} \times \vec{B} = \frac{4\pi \vec{J}}{c} - \frac{i\omega \vec{E}}{c} = -\frac{i\omega}{c}\epsilon \vec{E} \tag{5.10}$$

where we have introduced the dielectric constant of the plasma ϵ,

$$\epsilon = 1 - \frac{4\pi n_e e^2}{m_e}\frac{1}{\omega^2} = 1 - \frac{\omega_p^2}{\omega^2} \tag{5.11}$$

which defines the *plasma frequency* $\omega_p^2 = 4\pi n_e e^2/m_e$.

Now we solve for the disperson relation for the wave. Maxwell's equations are

$$i\vec{k} \times \vec{B} = -\frac{i\omega}{c}\epsilon \vec{E} \qquad i\vec{k} \times \vec{E} = \frac{i\omega}{c}\vec{B} \tag{5.12}$$

which gives

$$\vec{k} \times (\vec{k} \times \vec{E}) = -\left(\frac{\omega}{c}\right)^2 \epsilon \vec{E} \tag{5.13}$$

which is the dispersion relation for a transverse wave ($\vec{k}.\vec{E} = 0$) with $\omega^2 = c^2k^2/\epsilon$, or

$$k = \frac{1}{c}\sqrt{\omega^2 - \omega_p^2}. \tag{5.14}$$

We see that for $\omega > \omega_p$, k is real and the wave propagates, but for $\omega < \omega_p$, k is imaginary and the wave evanesces. Physically, what's going on is that at high frequency, the electrons don't have time to accelerate, and therefore the current is small and has little effect on the wave. At low frequencies, a significant current can build up during a wave period that acts to "short out" the wave (recall that there is no static electric field inside a conductor, ie. the electric field vanishes in the limit $\omega \to 0$). Putting in numbers, we get a useful formula

$$f_p = \frac{\omega_p}{2\pi} = 9.0 \text{ kHz } \left(\frac{n_e}{\text{cm}^{-3}}\right)^{1/2}. \tag{5.15}$$

A terrestrial application is to the Earth's ionsphere in which $n_e \sim 10^4 - 10^5 \text{ cm}^{-3}$ and radio waves with frequencies below a few MHz are reflected back towards the Earth's surface.

The dispersion relation gives the wave phase speed as $\omega/k = c/\sqrt{\epsilon} = c/n > c$ where the refractive index $n = \sqrt{\epsilon}$. The group velocity is $\partial\omega/\partial k = cn < c$, and depends on frequency. Therefore electromagnetic waves in a plasma are dispersive, and waves of different frequencies from a source will arrive at different times. This effect is quantified by the *dispersion measure* DM, which is defined as follows. First we write down the travel time of a wave from a source at distance d,

$$t = \int_0^d \frac{dl}{cn} \approx \int_0^d \frac{dl}{c}\left(1 + \frac{1}{2}\left(\frac{\omega_p}{\omega}\right)^2\right) \tag{5.16}$$

where we take $\omega \gg \omega_p$. The delay due to the plasma is

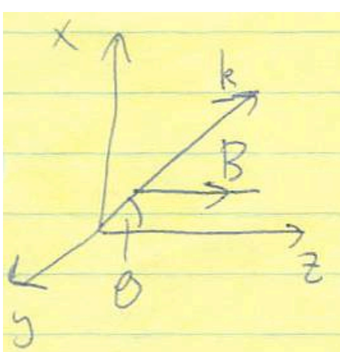
$$\Delta t = \frac{2\pi e^2}{m_e c}\frac{1}{\omega^2}\int_0^d dl\; n_e = \frac{2\pi e^2}{m_e c}\frac{1}{\omega^2}\mathrm{DM} \approx 3\ \mathrm{ms}\ \left(\frac{\mathrm{DM}}{\mathrm{pc\ cm^{-3}}}\right)\left(\frac{1\ \mathrm{GHz}}{\nu}\right)^2 \tag{5.17}$$

where DM is the dispersion measure, the integrated electron density along the line of sight. The units of dispersion measure are usually taken to be $\mathrm{cm^{-3}}$ pc.

This effect is seen in the arrival times of pulses from radio pulsars, and is often used as a measure of distance to the pulsar. Given a model to the electron density in the Galaxy, the distance can be inferred from the measured DM.

5.6. Faraday rotation

Now we include a magnetic field in the plasma. We choose our axes so that the magnetic field lies in the z-direction, and consider a wave propagating in the x-z plane



The equations of motion for an electron are

$$-i\omega m_e v_x = -eE_x - \frac{ev_yB}{c} \qquad -i\omega m_e v_y = -eE_y + \frac{ev_xB}{c} \qquad -i\omega m_e v_z = -eE_z \tag{5.18}$$

where $\vec{E}$ is the electric field in the wave, and $\vec{B}$ is the magnetic field in the plasma. We assume that the magnetic field in the wave $\vec{B}'$ is much smaller than the field in the plasma.

We solve for the current as before, and find

$$\begin{aligned} J_x &= \frac{\omega_p^2}{4\pi} \frac{1}{\omega_c^2 - \omega^2} \left(-i\omega E_x - \omega_c E_y \right) \\ J_y &= \frac{\omega_p^2}{4\pi} \frac{1}{\omega_c^2 - \omega^2} \left(-i\omega E_y + \omega_c E_x \right) \\ J_z &= \frac{i}{\omega} \frac{\omega_p^2}{4\pi} E_z \end{aligned} \qquad (5.19)$$

where ω_p is the plasma frequency and $\omega_c = eB/m_e c$ is the cyclotron frequency.

The effect of the magnetic field is that $\vec{J}$ is no longer in the same direction as $\vec{E}$ because of deflections by the magnetic field. Instead of a scalar dielectric constant, we instead must introduce a *dielectric tensor* such that

$$i\vec{k} \times \vec{B}' = \frac{4\pi \vec{J}}{c} - \frac{i\omega \vec{E}}{c} = -\frac{i\omega}{c} \epsilon \cdot \vec{E} \qquad (5.20)$$

where

$$\epsilon = \begin{pmatrix} S & -iD & 0 \\ +iD & S & 0 \\ 0 & 0 & P \end{pmatrix} \qquad (5.21)$$

with

$$S = 1 - \frac{\omega_p^2}{\omega^2 - \omega_c^2} \qquad D = -\frac{\omega_p^2 \omega_c}{\omega(\omega^2 - \omega_c^2)} \qquad P = 1 - \frac{\omega_p^2}{\omega^2}. \qquad (5.22)$$

(Note that for no magnetic field ($\omega_c = 0$), $S = P$ and $D = 0$, reducing to the result we had earlier.)

What does this mean for wave propagation? Consider an electromagnetic wave propagating along the magnetic field, $\vec{k} = k\vec{e_z}$, $E_z = 0$. The dispersion relation is given by $\vec{k} \times (\vec{k} \times \vec{E}) = -(\omega^2/c^2)\epsilon \cdot \vec{E}$, which in components is

$$-k^2 E_x = -\frac{\omega^2}{c^2}(SE_x - iDE_y) \qquad -k^2 E_y = -\frac{\omega^2}{c^2}(iDE_x - SE_y) \qquad (5.23)$$

or in terms of $n = ck/\omega$,

$$(S - n^2)E_x = iDE_y \qquad (S - n^2)E_y = -iDE_x \qquad (5.24)$$

giving

$$(n^2 - S)^2 = D^2 \Rightarrow n_\pm^2 = S \pm D. \qquad (5.25)$$

There are two different solutions for n. Writing them out in terms of the various frequencies,

$$n_+^2 = 1 - \frac{\omega_p^2}{\omega(\omega - \omega_c)} \qquad n_-^2 = 1 - \frac{\omega_p^2}{\omega(\omega + \omega_c)}. \qquad (5.26)$$

To see what these two modes look like, plug each value of n back into equations (5.24). The mode with $n = n_+$ has $E_x = -iE_y$ and the mode with $n = n_-$ has $E_x = +iE_y$.

We see therefore that the modes are *left and right circularly polarized waves travelling with different speeds*. Physically, this is because the electrons "want to" spiral around the field line in a particular direction. The wave that rotates in the right direction propagates more easily.

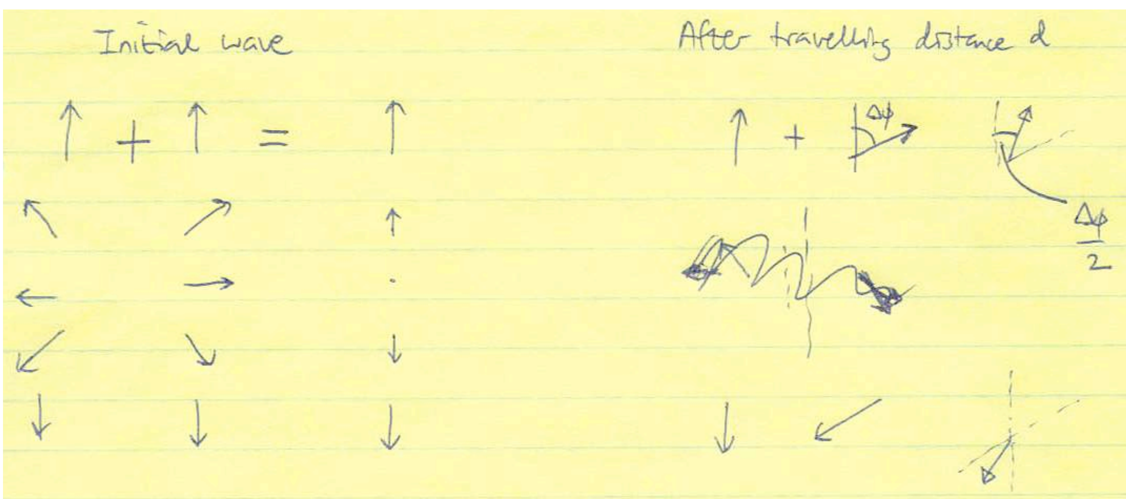
For $\omega \gg \omega_c, \omega_p$, we can expand

$$n_\pm \approx 1 - \frac{1}{2}\frac{\omega_p^2}{\omega^2}\left(1 \pm \frac{\omega_c}{\omega}\right). \tag{5.27}$$

As the two polarizations propagate they build up a phase difference

$$\Delta\phi = \int dl\, \Delta k = \int dl\, \frac{\omega}{c}\Delta n = \int_0^d \frac{dl}{c}\frac{\omega_c \omega_p^2}{\omega^2}. \tag{5.28}$$

The effect on a linearly polarized wave is that the angle of polarization rotates by an angle $\Delta\phi/2$. It is easiest to see this geometrically, as in the following diagram:



We define the *rotation measure* RM as a measure of this rotation. The plane of polarization rotates by

$$\Delta\theta = \frac{1}{2}\int_0^d \frac{dl}{c}\frac{\omega_c\omega_p^2}{\omega^2} = \frac{e^3}{2\pi m_e^2 c^2}\frac{1}{\nu^2}\int_0^d dl\, n_e B_\parallel \tag{5.29}$$

where we write $B_\parallel$ because although here we have consider only propagation along the direction of B, the result holds more generally in which case the component of the magnetic field along the line of sight $B_\parallel$ appears in the integral. The *rotation measure* RM is defined so that $\Delta\theta = \mathrm{RM}\,\lambda^2$. It gives a measure of the electron-density-weighted line of sight magnetic field.

Summary and Further Reading

Here are the main ideas and results that we covered in this part of the course:

- *Origin and evolution of electron spectrum.* How to generate a power law energy distribution. Fermi acceleration. Scattering across a strong shock front gives $p \approx 2$. A break in the electron spectrum indicates the energy where the cooling time equals the age of the population.

- *Electromagnetic waves in a plasma.* Plasma frequency $\omega_p^2 = 4\pi n_e e^2/m_e$, or $f_p = 9\ \mathrm{kHz}\ n_e^{1/2}$. Refractive index $n^2 = 1 - \omega_p^2/\omega^2$. Dispersion $\Delta t = (2\pi e^2/m_e c)(\mathrm{DM}/\omega^2)$, where $\mathrm{DM} = \int n_e\ dl$. Faraday rotation. Rotation measure $\Delta\theta = \mathrm{RM}\,\lambda^2$, $\mathrm{RM} \propto \int n_e B_{\parallel}\ dl$.

Reading

- Electron spectrum: good discussions are in Longair and Kulsrud 12.7.

- Electron model for Galaxy used to interpret DM measurements: Taylor & Cordes 1993

6. Atoms and Molecules

These are notes for part six of PHYS 642 Radiative Processes in Astrophysics. We now turn to bound electrons in atoms and molecules.

6.1. The Saha equation

We first want to understand the level of ionization of different species *in thermal equilibrium* for different temperature and density conditions. Consider the ionization and recombination of hydrogen[10]

$$H^+ + e^- \rightleftharpoons H^0$$

The ionization potential is $\chi_H = 13.6\ \mathrm{eV} = (m_p + m_e - m_H)c^2$ (for now, we ignore levels other than the ground state and assume that recombination occurs into the ground state only).

In equilibrium, the chemical potentials satisfy[11]

$$\mu_+ + \mu_- = \mu_0. \tag{6.1}$$

For an ideal gas of particles the chemical potential is

$$\mu = k_B T \ln\left(\frac{n}{g n_Q}\right) + mc^2, \tag{6.2}$$

where "n-quantum" is

$$n_Q = \left(\frac{2\pi m k_B T}{h^2}\right)^{3/2} \tag{6.3}$$

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[10] It's worth reminding ourselves of the astronomical notation here: the neutral hydrogen atom H^0 is referred to as HI, singly ionized hydrogen H^+ is known as HII, etc.

[11] In general for a chemical reaction

$$aA + bB \rightleftharpoons cC + dD$$

in which a particles of species A react with b particles of B to make c particles of C and d particles of D, $a\mu_A + b\mu_B = c\mu_C + d\mu_D$ in equilibrium. To see this, start with the Gibbs free energy $G = E - TS + PV$ which gives $dG = -SdT + VdP + \mu dN$. In equilibrium $dG = 0$ at constant pressure and temperature gives $\sum \mu_i dN_i = 0$ from which the relation between the chemical potentials follows.

and g is a degeneracy factor that counts the spin states (or other internal states of the particles).

Therefore,

$$(m_e + m_p)c^2 + k_B T \ln \left[\frac{n_e}{g_e n_{Q,e}} \frac{n_p}{g_p n_{Q,p}} \right] = m_H c^2 + k_B T \ln \left(\frac{n_H}{g_H n_{Q,H}} \right) \tag{6.4}$$

or rearranging we find

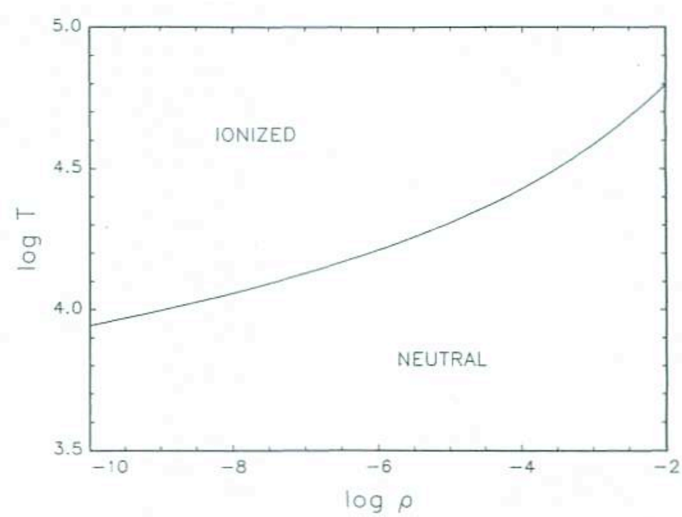
$$\frac{n_e n_p}{n_H} = \frac{g_p g_e}{g_H} \left(\frac{2\pi m_e k_B T}{h^2} \right)^{3/2} \exp\left(-\frac{\chi_H}{k_B T} \right) \tag{6.5}$$

which is the *Saha equation*. For hydrogen ionization, the spin degeneracy factors cancel, since $g_e = 2$, $g_p = 2$ and $g_H = 4$ (the hydrogen atom has 4 possible spin states; 3 triplet and 1 singlet state).

The half-ionization point is when the *ionization fraction* $y = n_+/n = 1/2$. Since the plasma is neutral $n_e = n_+$, so that

$$\frac{y^2}{1-y} = \frac{1}{n} \left(\frac{2\pi m_e k_B T}{h^2} \right)^{3/2} e^{-\chi_H/k_B T} = \frac{4 \times 10^{-9}}{\rho} T^{3/2} e^{-1.6\times 10^5/T}. \tag{6.6}$$

For high temperatures $k_B T \gg \chi_H$ $y \to 1$ and the hydrogen is fully ionized; for low temperatures $k_B T \ll \chi_H$ $y \to 0$ and the hydrogen is neutral. The plot below taken from Hansen and Kawaler shows the half-ionization point as a function of ρ and T.



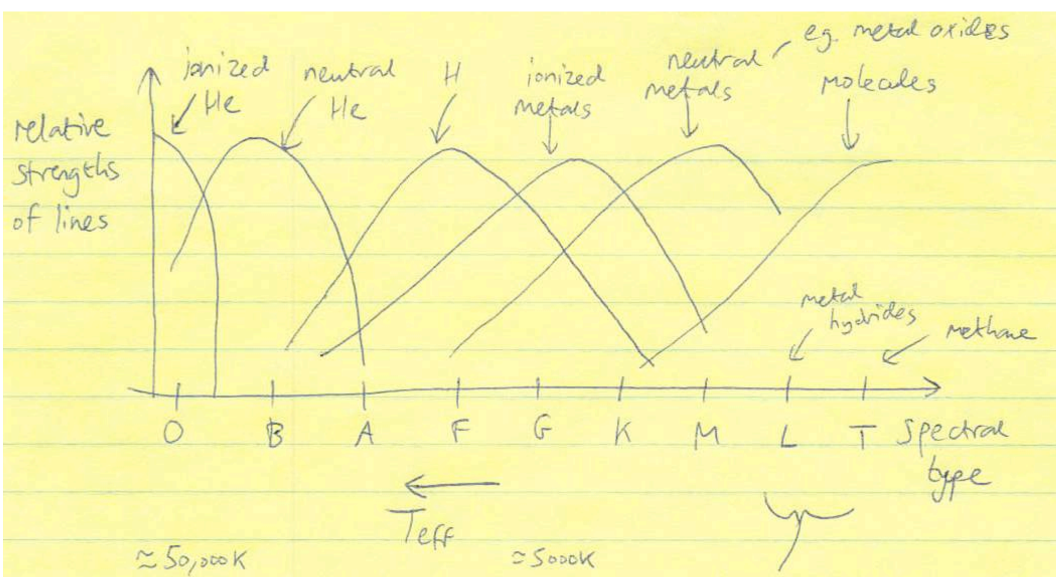
It is important to note that 13.6 eV corresponds to a temperature of $T = 1.6 \times 10^5$ K. Typically, the half-ionization point is a factor of ten below that value, $T \approx 10^4$ K. The reason for this is that there is a large entropy gain in ionizing the atom, since two particles are created rather than one. Even though there is an energy cost for ionizing the atom, the free energy of the system $E - TS$ is lowered because of the large entropy increase.

We considered only the ground state here, but in general we must include all the energy levels of the atom. The chemical potential is then of the same form as equation (6.2) but the factor g is replaced by the partition function[12] summed over all internal energy levels $Z = \sum_k g_k e^{-\beta E_k}$ where E_k is the energy and g_k the degeneracy of level k. For ionization of an atom between ionization states i and $i+1$, the Saha equation is then

$$\frac{n_{i+1} n_e}{n_i} = \frac{2 Z_{i+1}}{Z_i} \left(\frac{2\pi m_e k_B T}{h^2} \right)^{3/2} e^{-\chi_i / k_B T} \tag{6.7}$$

which is the form in which it is usually given. Here χ_i is the energy required to ionize species i and form species $i+1$ by removing an electron from the ground state of species i.

An immediate application of the Saha equation is to understand the sequence of spectral types. Stars are divided into spectral types based on the species whose absorption lines appear in their spectra.



[12] To see this, use the fact that the occupation number of a classical gas is $e^{-(\epsilon_i - \mu)/k_B T}$ (the classical limit of the Fermi-Dirac or Bose-Einstein distributions). Then

$$N = V\, e^{\mu/k_B T} \int \frac{4\pi p^2 dp}{h^3} e^{-p^2/2mk_BT} \sum_i g_i e^{-E_i/k_B T}$$

where the sum over i is over internal energy levels E_i with degeneracies g_i. This can be integrated and solved for μ.

6.2. Line profiles and curve of growth

Several different processes determine the shape of spectral lines. An excellent and detailed discussion is in Mihalas "Stellar Atmospheres" chapter 9. Here we briefly discuss three contributions.

The first is *natural linewidth* which arises due to the finite lifetime of a state. One way to think of it is in terms of the uncertainty principle $\Delta E \sim \hbar/\Delta t$. The line profile is a *Lorentzian* profile[13]

$$\phi(\nu) = \frac{\Gamma/4\pi^2}{(\nu-\nu_0)^2 + (\Gamma/4\pi)^2} \tag{6.8}$$

where ν_0 is the frequency of the line center (the normalization here is $\int d\nu\phi(\nu) = 1$).

Collisions between particles give rise to *collisional broadening*. This also has a Lorentzian profile, in which $\Gamma \rightarrow \Gamma + 2\nu_c$ where ν_c is the collision frequency, and is only important if the time between collisions is shorter than the natural lifetime of the state. A simple model is to assume that collisions lead to abrupt phase changes in the emitted radiation. The emitted radiation is made up of pieces with finite length T, which have a Fourier transformed electric field

$$E(\omega, T) = \int_0^T e^{i(\omega_0-\omega)t}dt = \frac{\exp\left[i(\omega_0-\omega)T\right] - 1}{i(\omega-\omega_0)}. \tag{6.9}$$

The idea is then to average over a probability distribution of collision times to get the power,

$$P(\omega) \propto \frac{1}{2\pi}\int_0^\infty E^\star E\,\mathrm{Prob}(T)dT \tag{6.10}$$

and with $\mathrm{Prob}(T)dT \propto e^{-T/\tau}dT$ this gives a Lorentzian profile with $\Gamma = 2/\tau$. More sophisticated theory is discussed in Mihalas' book. For example, the kind of interactions occurring (van der Waals, Stark effect etc.) must be included. There is also a line shift which is not predicted by the Lorentz model but is treated by more sophisticated models.

Doppler broadening occurs because a moving atom absorbs at a slightly different frequency than an atom at rest because of Doppler shifts. We should write

$$\phi(\nu) \propto \frac{1}{\left(\nu(1-\mu v/c) - \nu_0\right)^2 + (\Gamma/4\pi)^2} \tag{6.11}$$

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[13] One way to get this is to repeat the calculation we did for the Thomson cross-section but for an electron in a harmonic potential (see Rybicki and Lightman). You may also be familiar with the Lorentzian form of cross-sections in particle or nuclear physics in which a reaction proceeds via a short-lived excited state that decays into different channels.

where the photon frequency seen by the atom is $\nu(1-\mu v/c)$. Averaging this expression over a Maxwellian velocity distribution gives the *Voigt profile*. We first simplify by writing

$$\nu\left(1-\frac{v\mu}{c}\right)-\nu_0=\nu-\nu_0\left(1+\frac{v\mu}{c}\right)-(\nu-\nu_0)\frac{v\mu}{c} \tag{6.12}$$

and drop the last term which is second order in v/c. Effectively, this says that to first order in v/c we just need to Doppler shift the line center. Then average over the particle velocity and direction

$$\phi(\nu)=\left(\frac{\Gamma}{4\pi}\right)\frac{1}{\sqrt{2\pi}}\left(\frac{m}{k_BT}\right)^{3/2}\int_0^\infty\int_{-1}^1\frac{e^{-mv^2/2k_BT}v^2dvd\mu}{(\nu-\nu_0-\nu_0v\mu/c)^2+(\Gamma/4\pi)^2} \tag{6.13}$$

(which is a convolution of a Maxwellian and Gaussian). We write the result as

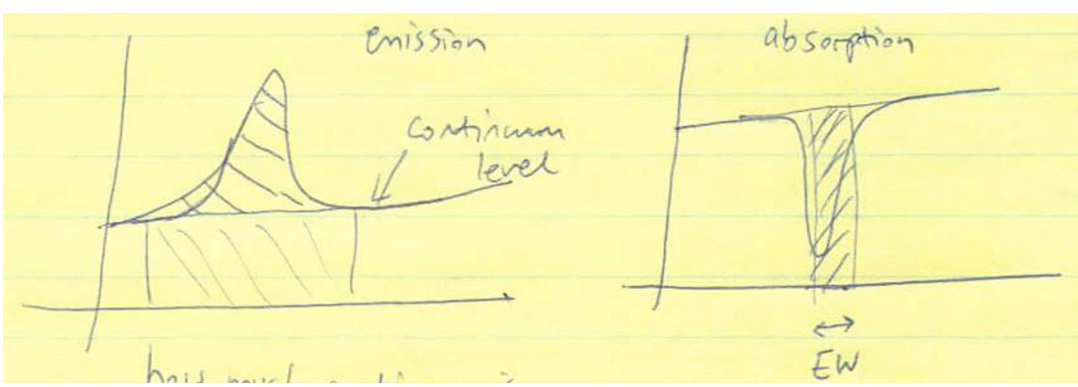
$$\phi(\nu)=\frac{1}{\sqrt{\pi}\Delta\nu_D}H\left(a,\frac{\nu-\nu_0}{\Delta\nu_D}\right) \tag{6.14}$$

where $\Delta\nu_D=\nu_0(2k_BT/mc^2)^{1/2}\sim\nu_0\langle v\rangle/c$ is the Doppler width, $a=\Gamma/4\pi\Delta\nu_D$ compares the two widths, and

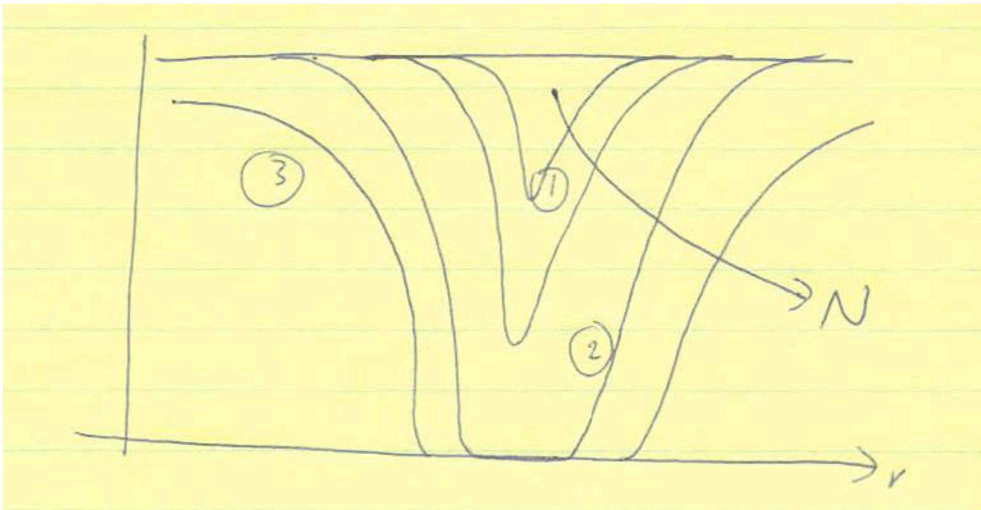
$$H(a,u)=\frac{a}{\pi}\int_{-\infty}^{\infty}\frac{e^{-y^2}dy}{a^2+(u-y)^2} \tag{6.15}$$

is the *Voigt function*. This gives the general shape of spectral lines.

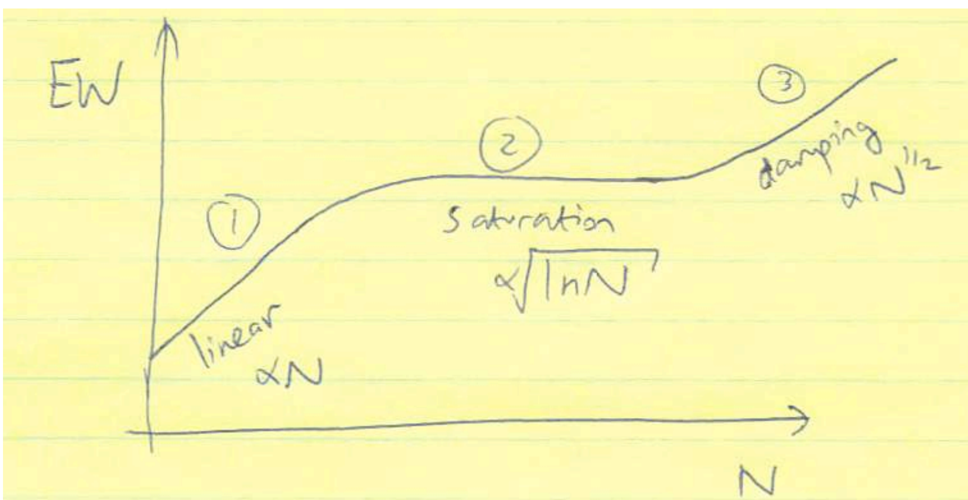
The line shape gives rise to a distinctive *curve of growth*, which is a plot of the equivalent width of a spectral line against the column of absorbers. The equivalent width is a measure of the total absorption or emission in a line, and is calculated as the width of the underlying continuum that would be needed to make up the emission or absorption in the line.



As the number of absorbers increases along the line of sight, measured by the number of absorbers per unit area N, the absorption increases, as in the following plot:



The curve of growth is a plot of EW against N which has a distinct shape determined by the Voigt profile.



The EW at first increases linearly with N, but then flattens off as the line becomes saturated (EW$\propto \sqrt{\ln N}$), later rising again as the Lorentzian wings of the Voigt profile become important (EW$\propto \sqrt{N}$).

6.3. A reminder of hydrogen-like atoms

Let's give a few reminders about hydrogen-like atoms that are useful for estimates. The binding energy is

$$\chi_H = \frac{1}{2}\alpha^2 Z^2 m_e c^2 = 13.6 \text{ eV } Z^2, \tag{6.16}$$

the Bohr radius is

$$a_0 = \frac{\hbar}{m_e c}\frac{1}{Z\alpha} = \frac{0.53\ \text{Å}}{Z} \tag{6.17}$$

where

$$\alpha = \frac{e^2}{\hbar c} = \frac{1}{137} \tag{6.18}$$

is the fine structure constant. The classical electron radius is $r_0 = \alpha^2 Z a_0$ The spacing between energy levels is

$$\Delta E_{nm} = \chi_H \left(\frac{1}{n^2} - \frac{1}{m^2}\right). \tag{6.19}$$

Also, now is a good time to remind ourselves of the absorption line series of hydrogen. The Lyman series corresponds to transitions to $n = 1$. The Ly limit is 912Å, Lyα ($n = 2$ to $n = 1$) is 1216Å, Lyβ ($n = 3$ to $n = 1$) is 1025Å. The Balmer series corresponds to transitions to $n = 2$. The Balmer limit is 364.6 nm, Hα ($n = 3$ to $n = 2$) is 656.3 nm, Hβ ($n = 4$ to $n = 2$) is 486.1 nm.

6.4. Calculation of radiative transitions

In this section, we summarize the main steps in calculating radiative transitions in atoms. We start with *Fermi's Golden Rule* from time-dependent perturbation theory in non-relativistic quantum mechanics. We consider two eigenstates of the Hamiltonian $|\mathrm{i}\rangle$ and $|\mathrm{f}\rangle$, e.g. $\hat{H}_0 |\mathrm{i}\rangle = E_i^0 |\mathrm{i}\rangle$, and perturb the Hamiltonian $\hat{H} = \hat{H}_0 + \hat{H}_1$. The transition rate from state i to f is

$$R_{if} = \frac{2\pi}{\hbar} \int \prod_k \frac{d^3 p_k}{h^3} \left|\langle \mathrm{f} \mid \hat{\mathrm{H}}_1 |\mathrm{i}\rangle\right|^2 \delta\left(E_i^0 - E_f^0 - \sum_k E_k\right) \delta\left(\vec{p}_i^0 - \vec{p}_f^0 - \sum_k \vec{p}_k\right) \tag{6.20}$$

where the product k is over all final state particles, and the delta-functions enforce energy and momentum conservation. This is often abbreviated as

$$R_{if} = \frac{2\pi}{\hbar} |M_{fi}|^2 \rho(E) \tag{6.21}$$

where M_{fi} is the matrix element connecting the initial and final states, and $\rho(E)$ represents the phase space of the outgoing particles.

To write down $\hat{H}_1$ for radiative transitions, we need the Hamiltonian of a (non-relativistic) charged particle in an electromagnetic field,

$$H = \frac{(\vec{p} - q\vec{A}/c)^2}{2m} + q\phi \tag{6.22}$$

where $\vec{A}$ and ϕ are the vector and scalar potentials. This Hamiltonian recovers the correct equations of motion for the charged particle. Expanding the first term, we write

$$\hat{H}_0 = \frac{\hat{p}^2}{2m} - e\phi \tag{6.23}$$

and

$$\hat{H}_1 = \frac{e\hat{p}\cdot\hat{A}}{m_e c} + \frac{e^2}{2m_e c^2}\hat{A}\cdot\hat{A} \tag{6.24}$$

(here we have chosen the Coulomb gauge $\vec{\nabla}\cdot\vec{A} = 0$ so that $[\hat{A}, \hat{p}] = 0$). These terms represent one and two photon processes. Their ratio is $(n_\gamma a_0)^{-1/2}$ where n_γ is the photon density, so that the first term dominates as long as the number of photons per atom is small.

The Hamiltonian for the electromagnetic field is

$$H_{EM} = \frac{1}{8\pi}\int d^3\vec{r}\left(E^2 + B^2\right) = \frac{1}{8\pi}\int d^3\vec{r}\left[\left(-\frac{1}{c}\frac{\partial A}{\partial t}\right)^2 + \left(\vec{\nabla}\times\vec{A}\right)^2\right]. \tag{6.25}$$

The photon field is quantized by writing

$$\hat{A} = \sum_{\vec{k},\lambda}\left(\frac{2\pi\hbar c^2}{\omega}\right)^{1/2}\left[\hat{a}_{\vec{k},\lambda}\vec{\epsilon}\left(\vec{k},\lambda\right)e^{i(\vec{k}\cdot\vec{r}-\omega t)} + \hat{a}^\dagger_{\vec{k},\lambda}\vec{\epsilon}\left(\vec{k},\lambda\right)e^{-i(\vec{k}\cdot\vec{r}-\omega t)}\right] \tag{6.26}$$

where $\hat{a}$ and $\hat{a}^\dagger$ are annilation and creation operators, satisfying $[\hat{a}_{\vec{k},\lambda}, \hat{a}^\dagger_{\vec{k},\lambda}] = \delta_{\vec{k},\vec{k}'}\delta_{\lambda,\lambda'}$. Here, $\vec{k}$ is the photon wavevector and λ is the spin. This should be familiar to you if you think back to the harmonic oscillator. Just as in that case, in terms of these operators, the Hamiltonian is

$$\hat{H} = \sum_{\vec{k},\lambda}\hbar\omega\left(\hat{a}_{\vec{k},\lambda}\hat{a}^\dagger_{\vec{k},\lambda} + \frac{1}{2}\right), \tag{6.27}$$

and we also have

$$\hat{a}_{\vec{k},\lambda}\left|n_{\vec{k},\lambda}\right\rangle = \sqrt{n_{\vec{k},\lambda}}\left|n_{\vec{k},\lambda} - 1\right\rangle \qquad \hat{a}^\dagger_{\vec{k},\lambda}\left|n_{\vec{k},\lambda}\right\rangle = \sqrt{n_{\vec{k},\lambda} + 1}\left|n_{\vec{k},\lambda} + 1\right\rangle. \tag{6.28}$$

The idea then is that the annhilation and creation operators inside the $\hat{A}$ factors in the perturbing Hamiltonian $\hat{H}_1$ couple the photon states in $|\mathrm{i}\rangle$ and $|\mathrm{f}\rangle$. For example, $\langle 1 \mid \langle\phi_n \mid \hat{H}_1\,|\phi_m\rangle\;|0\rangle$ represents spontaneous emission from state m to n with the emission of a photon.

Now we make a *dipole approximation.* Substituting our expression for $\hat{A}$ into $\hat{H}_1$, we see that we have to evaluate

$$\langle \mathrm{f} \mid \mathrm{e}^{-\mathrm{i}\vec{\mathrm{k}}\cdot\vec{\mathrm{r}}}\vec{\epsilon}\cdot\hat{\mathrm{p}}\,|\mathrm{i}\rangle \tag{6.29}$$

where we drop the prefactor, including the $\hat{a}$ and $\hat{a}^\dagger$ operators that couple the photon states. The approximation is to recognize that $kr \sim (\omega/c)a_0 \sim Z\alpha/2 \ll 1$ and so $e^{i\vec{k}\cdot\vec{r}} \sim 1$. Just as in the classical dipole approximation, we ignore phase changes across the source. Secondly, we write

$$\begin{aligned}
\vec{\epsilon}\cdot\langle \mathrm{f} \mid \hat{\mathrm{p}} \,|\mathrm{i}\rangle &= m\vec{\epsilon}\cdot\langle \mathrm{f} \mid \frac{\mathrm{d}\hat{\mathrm{r}}}{\mathrm{dt}} \,|\mathrm{i}\rangle \\
&= \frac{im}{\hbar}\vec{\epsilon}\cdot\langle \mathrm{f} \mid [\hat{\mathrm{H}},\hat{\mathrm{r}}] \,|\mathrm{i}\rangle \\
&= \frac{im}{\hbar}(E_f^0 - E_i^0)\vec{\epsilon}\cdot\langle \mathrm{f} \mid \hat{\mathrm{r}} \,|\mathrm{i}\rangle \\
&= im\omega\vec{\epsilon}\cdot\langle \mathrm{f} \mid \hat{\mathrm{r}} \,|\mathrm{i}\rangle \;.
\end{aligned} \tag{6.30}$$

Now we have everything we need to evaluate the rate. We discuss two aspects: first, the spatial integrals which give rise to *selection rules* for the transitions, and then estimate the rate of bound-bound transitions.

6.5. Selection rules from the spatial integral

If the initial and final states are hydrogen-like states, then we need to evaluate

$$\langle\psi_m \mid \vec{\epsilon}\cdot\vec{r}\,|\psi_k\rangle = \int_0^\infty dr\, r^2\, R^\star_{n_f l_f}(r)\, r\, R_{n_i l_i}(r) \int d\Omega\, Y^\star_{l_f m_f}(\theta,\phi)\vec{\epsilon}\cdot\vec{e}_r\, Y_{l_i m_i}(\theta,\phi). \tag{6.31}$$

For the angular integral, it is useful to write

$$\begin{aligned}
\vec{\epsilon}\cdot\vec{e}_r &= \epsilon_x \sin\theta\cos\phi + \epsilon_y \sin\theta\sin\phi + \epsilon_z\cos\phi \\
&= \sqrt{\frac{4\pi}{3}}\left(\epsilon_z Y_{1,0} + \frac{-\epsilon_x + i\epsilon_y}{\sqrt{2}}Y_{1,1} + \frac{\epsilon_x + i\epsilon_y}{\sqrt{2}}Y_{1,-1}\right),
\end{aligned} \tag{6.32}$$

which reduces the angular integral to an integral of three Y_{lm}'s,

$$\int d\Omega\, Y^\star_{l_f,m_f}(\theta,\phi) Y_{l,m} Y_{l_i,m_i}(\theta,\phi). \tag{6.33}$$

Selection rules arise because these integrals are non-vanishing only for particular combinations of initial and final quantum numbers. For example, the azimuthal angular integration is

$$\int d\phi\, e^{-im_f\phi} e^{im\phi} e^{im_i\phi} = 2\pi\delta(m - m_f + m_i) \tag{6.34}$$

which tells us that $m_f - m_i = m = 1, 0$, or -1. Choosing our z axis to lie along the photon propagation direction, $\epsilon_z = 0$, we see that $m = \pm 1$ only, or

$$\Delta m = m_f - m_i = \pm 1 \tag{6.35}$$

which is our first selection rule. For example a transition to the ground state has $m_f = 0$ and therefore must have $m = -m_i$. If $m_i = 1$ then $m = -1$ and the photon has a polarization $\epsilon_x + i\epsilon_y$ (third term in eq. [6.32]), carrying away the angular momentum of the initial state.

Another selection rule that is simple to see is that the parity must change so that the radial integral is non-vanishing $\int d^3\vec{r}\,\psi_f^\star \vec{r}\psi_i$. Since the parity is $(-1)^l$, we conclude that l must change, and the angular integral gives the constraint

$$\Delta l = \pm 1 \tag{6.36}$$

(e.g. you could check the simple case $l_f = 0$ and use the fact that $Y_{0,0} = 1/\sqrt{4\pi}$ =constant).

Selection rules for multi-electron atoms are more complex; see Rybicki and Lightman for details. It is important to note that the selection rules we derived here can be violated by higher order processes, e.g. magnetic dipole or electric quadrupole transitions.

6.6. Bound-bound transitions

We can now use our results so far to estimate a typical bound-bound transition rate:

$$R_{if} \sim \frac{2\pi}{\hbar}\left[\frac{e}{m_e c}\left(\frac{2\pi\hbar c^2}{\omega}\right)^{1/2}\alpha m_e c\right]^2 \frac{4\pi(\hbar\omega)^2}{c^3 h^3} \tag{6.37}$$

where we use equation (6.26) to estimate the size of $\hat{A}$, and $p \sim \alpha m_e c$ for hydrogen, and ignore the overlap integrals; we're just trying to estimate the magnitude here. The final factor is the phase space for the outgoing photon. Cancelling factors gives

$$R_{if} \sim \frac{2\alpha^2\omega e^2}{\hbar c} \sim 2\alpha^3\omega. \tag{6.38}$$

Now assume a transition to the hydrogen ground state to get the energy scale, i.e. $\hbar\omega \sim (1/2)\alpha^2 m_e c^2$. Therefore

$$R_{if} \sim \alpha^5 \frac{m_e c^2}{\hbar} \sim \alpha^4 \frac{c}{a_0} \sim 10^{10}\ \mathrm{s}^{-1}. \tag{6.39}$$

If we had kept the Z scalings, we would find $R_{if} \propto Z^4$.

A concrete example is the $2p \to 1s$ transition in hydrogen (Ly α). The energy is $\hbar\omega_{21} = (3/8)\alpha^2 m_e c^2$. The radial integral in that case is

$$\int_0^\infty R_{10}^\star(r) R_{21}(r) r^3 dr = \frac{24}{\sqrt{6}} \left(\frac{2}{3}\right)^5 a_0 \tag{6.40}$$

where we use $R_{10}(r) = 2e^{-r/a_0}/a_0^{3/2}$ and $R_{21} = re^{-r/2a_0}/(\sqrt{24}a_0^{5/2})$. Averaging over m states $(1/3)\sum_{m=-1}^{1} R_{2p\to 1s}(m)$ gives the final result

$$R_{2p\to 1s} = A_{21} = \left(\frac{2}{3}\right)^8 \frac{m_e c^2}{\hbar} \alpha^5 = 0.6 \times 10^9 \text{ s}^{-1} \tag{6.41}$$

(the Einstein A coefficient for the $2p \to 1s$ transition). This is just over an order of magnitude below our simple estimate.

The Einstein relations allow us to get the cross-section for the reverse process, bound-bound absorption. Using the results we wrote down relating j_ν to the Einstein coefficient A and α_ν to the Einstein coefficient B (see the section on synchrotron self-absorption), we have

$$\sigma_{12} = \frac{h\nu}{4\pi}\phi(\nu) B_{12} = \frac{g_2}{g_1}\phi(\nu)\frac{c^2}{8\pi\nu^2} A_{21} = \frac{g_2}{g_1}\phi(\nu)\frac{\lambda^2}{8\pi} A_{21} \tag{6.42}$$

where we use the Einstein relations to relate B_{12} to A_{21}. Substituting in our expression for A_{21} and ν_{21} gives

$$\sigma_{12} = f \frac{\pi e^2}{m_e c}\phi(\nu) \tag{6.43}$$

where the *oscillator strength* $f = (g_2/g_1)(2^{13}/3^{10})$.

We expect the same scalings for A and σ for other transitions, but the prefactor will be different, depending on the overlap of the wavefunctions. For example, for transitions to the ground state, $f_{1n} \propto 1/n^3$.

6.7. Bound-free transitions: photoelectric effect

Now consider absorption of a photon that excites an electron into the continuum, the *photoelectric effect.* Fermi's Golden Rule for this case is

$$R = \frac{2\pi}{\hbar} \int \frac{d^3\vec{p}_e}{h^3} |M_{fi}|^2 \delta\left(\hbar\omega - \chi - \frac{p_e^2}{2m}\right) \tag{6.44}$$

where we integrate over the phase space of the outgoing electron, and χ is the ionization energy. This can be simplified if we write $d^3\vec{p} = d\Omega p^2 dp = d\Omega m p_e dE_e$, and integrate over energy using the delta function, giving

$$R = \frac{2\pi}{\hbar}\int d\Omega \frac{mp_e}{h^3}\left|M_{fi}\right|^2 \tag{6.45}$$

where $p_e = 2m\sqrt{\hbar\omega - \chi}$. Writing $dR/d\Omega = cd\sigma/d\Omega$, we obtain an expression for the differential cross-section

$$\frac{d\sigma}{d\Omega} = \frac{2\pi}{\hbar c}\frac{mp_e}{h^3}\left|M_{fi}\right|^2. \tag{6.46}$$

The matrix element is the same as previously,

$$M_{fi} = \frac{e}{m_e c}\left(\frac{2\pi\hbar c^2}{\omega}\right)^{1/2}\int d^3\vec{r}\,\psi_f^\star\,\vec{\epsilon}\cdot\vec{p}\,e^{i\vec{k}\cdot\vec{r}}\,\psi_i. \tag{6.47}$$

As an example, let's take the initial state to be the hydrogen ground state,

$$\psi_i = \frac{1}{\sqrt{\pi}}\left(\frac{1}{a_0}\right)^{3/2} e^{-r/a_0} \tag{6.48}$$

and we represent the electron part of the final state as a plane wave (Born approximation)

$$\psi_f \propto e^{i\vec{p}_e\cdot\vec{r}/\hbar}, \tag{6.49}$$

giving

$$\langle \mathrm{f} \mid \vec{\epsilon}\cdot\hat{\mathrm{p}}\mathrm{e}^{\mathrm{i}\vec{\mathrm{k}}\cdot\vec{\mathrm{r}}}\left|\mathrm{i}\right\rangle = \vec{\epsilon}\cdot\vec{\mathrm{p}}_\mathrm{e}\langle \mathrm{f} \mid \mathrm{e}^{\mathrm{i}\vec{\mathrm{k}}\cdot\vec{\mathrm{r}}}\left|\mathrm{i}\right\rangle, \tag{6.50}$$

or

$$\left|M_{fi}\right|^2 = \left(\frac{e}{m_e c}\right)^2\frac{2\pi\hbar c^2}{\omega}\frac{1}{\pi a_0^3}\left(\vec{\epsilon}\cdot\vec{p}_e\right)^2\left|\int d^3\vec{r}\,e^{i(\vec{k}-\vec{p}_e/\hbar)\cdot\vec{r}}e^{-r/a_0}\right|^2. \tag{6.51}$$

The integral is

$$\int d^3\vec{r}\,e^{-i\vec{\Delta}\cdot\vec{r}}e^{-mur} = \frac{8\pi\mu}{(\mu^2+\Delta^2)^2}, \tag{6.52}$$

which gives (after simplifying)

$$\frac{d\sigma}{d\Omega} = 32a_0^2\left(\frac{p_e c}{\hbar\omega}\right)\left(\frac{\vec{\epsilon}\cdot\vec{p}_e}{m_e c}\right)^2\frac{1}{(1+a_0^2\Delta^2)^4}. \tag{6.53}$$

where

$$\hbar\vec{\Delta} = \hbar\vec{k} - \vec{p}_e \tag{6.54}$$

is the momentum transfer and recall that energy conservation implies $E_e = p_e^2/2m_e = \hbar\omega - \chi$.

A simplification can be made if we assume $\hbar\omega \gg \chi$ or $E_e \approx \hbar\omega$, then

$$\frac{d\sigma}{d\Omega} = 2\sqrt{2}\alpha^8 a_0^2\left(\frac{E_e}{m_e c^2}\right)^{-7/2}\frac{\sin^2\theta}{\left(1 - v_e\cos\theta/c\right)^4} \tag{6.55}$$

where the photon direction defines the z-axis ($\theta = 0$). Keeping the scaling with Z for hydrogen-like atoms, we find $d\sigma/d\Omega \propto Z^5$. Integrating over angles we find the total cross-section is

$$\sigma_{bf} = \frac{2^8\pi}{3}\alpha a_0^2 \left(\frac{\chi}{\hbar\omega}\right)^{7/2} \tag{6.56}$$

(which agrees with Rybicki and Lightman eq. [10.53]). The general integrated cross-section is written in terms of the semi-classical result derived by Kramers with a multiplying Gaunt factor (Karzas and Latter 1961)

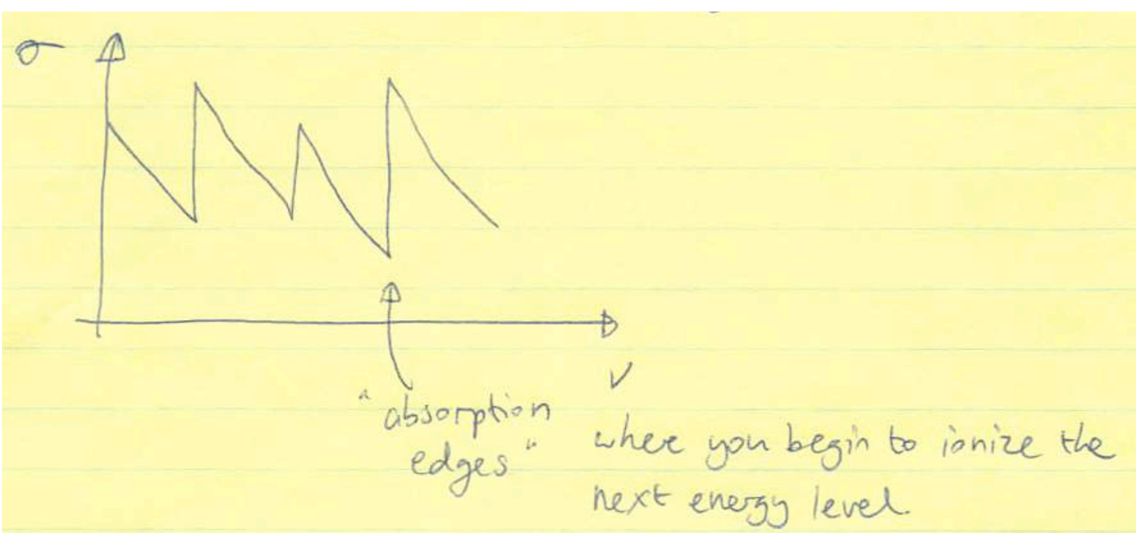
$$\sigma_{bf} = \frac{64\pi}{3\sqrt{3}} n g_{bf} \alpha \left(\frac{a_0}{Z}\right)^2 \left(\frac{\omega_n}{\omega}\right)^3 \tag{6.57}$$

where n is the energy level, $\hbar\omega_n = \chi_n = \alpha^2 m_e c^2 Z^2/2n^2$, and g_{bf} is the bound-free Gaunt factor. We might have guessed $\sigma \sim (\pi a_0^2)\alpha$ (geometrical cross-section multiplied by interaction strength), but there is also an additional correction term

$$\left(\frac{\omega}{\omega_n}\right)^{-3} \sim \left(\frac{v_e^2}{\alpha^2 Z^2 c^2}\right)^{-3} \tag{6.58}$$

which comes from the overlap between the two wavefunctions. The factor inside the brackets is roughly the velocity of the ejected electron divided by the orbital velocity.

As a function of frequency, the cross-section looks like



Note that the rough scaling between absorption edges is $\sigma_{bf} \propto 1/\nu^3$ which is the same as the free-free cross-section σ_{ff}. This means that the thermally-averaged opacity is also a Kramers' opacity $\kappa_{bf} \propto \rho T^{-7/2}$.

6.8. Bound-free transitions: recombination and the Milne relation

The inverse of the photoelectric effect is *recombination*. To calculate this we can use considerations of thermal equilibrium just as we did when deriving the Einstein relations.

We start by deriving the *Milne relation*,

$$\frac{\sigma_{bf}}{\sigma_{fb}} = \left(\frac{mcv}{h\nu}\right)^2 \frac{g_e g_+}{2g_n}. \tag{6.59}$$

To see this, first write down the number of recombinations per unit time and volume

$$n_+ n_e \sigma_{fb} v f(v) dv \tag{6.60}$$

and the rate of photoionizations per unit volume

$$n_n \sigma_{bf} c \left[\frac{4\pi I_\nu}{h\nu c}\right] \left(1 - e^{-h\nu/k_B T}\right) d\nu \tag{6.61}$$

where the factor in square brackets is the photon number density, and we include a correction for stimulated emission or "stimulated recombination" in this case. In thermal equilibrium, they must be equal

$$1 = \frac{n_+ n_e}{n_n} \frac{\sigma_{fb}}{\sigma_{bf}} \frac{cvh\nu}{4\pi} \frac{f(v)}{I_\nu} \frac{dv}{d\nu}. \tag{6.62}$$

Now we use results for thermal equilibrium, $I_\nu = B_\nu$ (Planck for photons),

$$f(v) = 4\pi \left(\frac{m}{2\pi k_B T}\right)^{3/2} v^2 e^{-mv^2/2k_B T} \tag{6.63}$$

(Maxwell-Boltzmann for the electrons), the Saha equation for $n_+ n_e / n_n$, and since $mv^2/2 = h\nu - \chi$, $mvdv = hd\nu$ and therefore $dv/d\nu = h/mv$. Simplifying, we arrive at the Milne relation, equation (6.59).

We can apply this relation to get σ_{fb} from σ_{bf}. The quantity usually given is the *recombination coefficient* (units are $\mathrm{cm^3\ s^{-1}}$)

$$\langle v\sigma_{fb}\rangle = \int v f(v) \sigma_{fb} dv. \tag{6.64}$$

Using the expression for σ_{bf} and the Milne relation with $g_n = 2n^2$, $g_e = 2$, $g_+ = 1$, gives

$$\langle v\sigma_{fb}\rangle = 3.262 \times 10^{-6}\, M(n, T) \tag{6.65}$$

where

$$M(n, T) = \frac{e^{\chi_n/k_B T}}{n^3 T^{3/2}} E_1\left(\frac{\chi_n}{k_B T}\right) \tag{6.66}$$

(see Cillié 1932) or summed over levels n,

$$\langle v\sigma_{fb}\rangle = 5.197\times 10^{-14}\ \lambda^{1/2}\left(0.4288+\frac{1}{2}\ln\lambda+0.469\lambda^{-1/3}\right)\ \mathrm{cm^3\ s^{-1}} \tag{6.67}$$

where $\lambda = 13.6\ \mathrm{eV}/k_BT = 1.579\times 10^5\ \mathrm{K}/T$ (Seaton 1959).

6.9. Summary of bound-bound and bound-free cross-sections

Let's summarize our results so far in a form that is useful for making simple estimates of quantities. First, *bound-bound* transitions. We have

$$A_{21}\approx Z^2\alpha^3\omega\approx Z^4\alpha^5\frac{m_ec^2}{\hbar}\approx 10^{10}Z^4\ \mathrm{s^{-1}}, \tag{6.68}$$

although we saw that the wavefunction overlap makes this an overestimate, for example $2p\rightarrow 1s$ in hydrogen is $6\times 10^8\ \mathrm{s^{-1}}$. The absorption cross-section is

$$\sigma = \frac{g_2}{g_1}\phi(\nu)A_{21}\frac{\lambda^2}{8\pi} \tag{6.69}$$

or at the line center,

$$\sigma\approx\frac{\lambda^2}{8\pi}\frac{A_{21}}{\Delta\nu} \tag{6.70}$$

where $\Delta\nu$ is the width of the line. For example, for Lyα ($n = 1$–2) $\lambda = 1216$Å. If we take $\Delta\nu\approx A_{21}$, then $\sigma\sim\lambda^2/8\pi = 6\times 10^{-12}\ \mathrm{cm^2}$.

For *bound-free* absorption, we have

$$\sigma_{bf}\sim\frac{\alpha}{n^5}\left(\frac{a_0}{Z}\right)^2\left(\frac{\omega_n}{\omega}\right)^3\approx\frac{7\times 10^{-18}\ \mathrm{cm^2}}{n^5}\left(\frac{\omega_n}{\omega}\right)^3. \tag{6.71}$$

The recombination coefficient is $\langle v\sigma_{fb}\rangle\sim 10^{-13}\ \mathrm{cm^3\ s^{-1}}$ and the cross-section is

$$\sigma_{fb}\approx 10^{-22}\ \mathrm{cm^2}\ \left(\frac{\omega_n}{\omega}\right)\left(\frac{\hbar\omega_n}{mv^2/2}\right)\frac{1}{n^3}. \tag{6.72}$$

6.10. Application: stellar opacities

We are now (almost) in a position to understand stellar opacities. Both free-free and bound-free absorption have $\kappa_\nu\propto 1/\nu^3$ which leads to a Kramers type opacity law, $\kappa\propto\rho T^{-7/2}$. Although not obvious why, bound-bound transitions also have a Kramers type opacity law. The one opacity we have not talked about is H^- opacity which is important in the atmospheres of stars with about a solar mass.

The handouts show the main contributions to opacity as a function of density and temperature.

6.11. Application: X-ray absorption by the ISM

X-rays are absorbed while travelling through the ISM by the photoelectric effect on metals. The papers to look at are Morrison and McCammon (1983) and Wilms, Allen, and McCray (2000). Using our earlier results, we expect

$$\sigma_{bf}\,(1\ \mathrm{keV}) \sim 10^{-17}\ \mathrm{cm}^2\ \left(\frac{10\ \mathrm{eV}}{1\ \mathrm{keV}}\right)^3 \frac{Z^4}{n^5} \sim 3\times 10^{-23}\ \mathrm{cm}^2\frac{Z^4}{n^5}. \qquad (6.73)$$

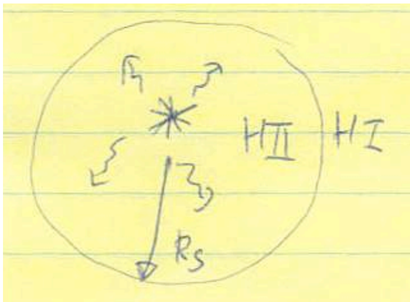
For carbon, $Z^4 \sim 1000$ but its abundance relative to hydrogen by number is $\sim 10^{-3}$, so we expect $\sim 3\times 10^{-23}$ cm^2 at 1 keV (per hydrogen) which agrees well with Figure 1 of Wilms et al. (2000). This doesn't work out so well for iron, which has $Z^4 \sim (26)^4 \sim 10^6$ and a number abundance $\sim 10^{-4}$, which gives $\sim 3\times 10^{-21}$ at 1 keV. Iron absorption is important at several keV and above, but the number at 1 keV is the one to compare against the Wilms et al. plot, again it compares reasonably well.

In X-ray astronomy, the absorption is measured in terms of the hydrogen column (with assumed metal abundances). We see that $\sigma \sim 10^{-22}$ cm^2 giving order unity optical depth for $N_H \sim 10^{22}$ cm^{-2}.

Dust also contributes to the absorption – see the discussion in Wilms et al. (2000) who conclude that it is not a large factor.

6.12. Application: Strömgen sphere; HII regions

Consider a hot star (O or B spectral type) in a region of constant density gas. The radiation from the star ionizes the surrounding gas to form an HII region.



To calculate the size of the HII region, first consider the different components of the ionizing radiation field ($\nu > \nu_1$ where $h\nu_1 = \chi_H$). The stellar radiation has a flux

$$F_{s\nu} = \frac{L_\nu}{4\pi r^2} e^{-\tau_\nu} \tag{6.74}$$

where τ_ν is the optical depth to photoionization,

$$d\tau_\nu = (1-x) n_H \sigma_{bf} dr \tag{6.75}$$

where x is the ionization fraction and n_H is the number density of hydrogen atoms (ionized or neutral). This flux satisfies

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2 F_{s\nu}\right) = -(1-x) n \sigma_{bf} c U_{s\nu}. \tag{6.76}$$

The diffuse ionizing radiation from captures onto the $n=1$ level satisfies

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2 F_{d\nu}\right) = -(1-x) n_H \sigma_{bf} c U_{d\nu} + 4\pi j_{d\nu}. \tag{6.77}$$

Now add these two components and integrate $\int_{\nu_1}^\infty d\nu/h\nu$

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2 \int_{\nu_1}^\infty \frac{d\nu}{h\nu}(F_{d\nu}+F_{s\nu})\right) = 4\pi \int_{\nu_1}^\infty \frac{j_{d\nu}}{h\nu} d\nu - (1-x) n_H c \int_{\nu_1}^\infty d\nu \sigma_{bf} \frac{(U_{d\nu}+U_{s\nu})}{h\nu}. \tag{6.78}$$

On the RHS the first term is $x n_e n_H \alpha_1$ where α_1 is the recombination rate onto the ground state. The second term is the photoionization rate. In steady state, the rate of ionizations must be balanced by the rate of recombinations,

$$(1-x) n_H c \int \frac{\sigma_{bf} U_\nu}{h\nu} d\nu = n_H x n_e \alpha \tag{6.79}$$

where now we write α, the recombination coefficient for recombination onto all levels, rather than α_1. In other words, the photoionizations that occur in a steady-state HII region are all balanced by a recombination, but not all recombinations are to the ground state. The flux of ionizing photons satisfies

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\int_{\nu_1}^{\infty}\frac{d\nu}{h\nu}(F_{d\nu}+F_{s\nu})\right) = -xn_Hn_e\alpha_2 \tag{6.80}$$

where α_2 is the recombination coefficient for recombination onto all levels $n \geq 2$. If all recombinations were to the ground state, the number of ionizing photons would be conserved as the photons were destroyed by photoionization and produced by recombination; recombinations to levels other than the ground state are the sink of ionizing photons.

The radius of the HII region is the radius at which all of the ionizing photons produced by the star have been absorbed and converted into diffuse photons. The integral of equation (6.80) is

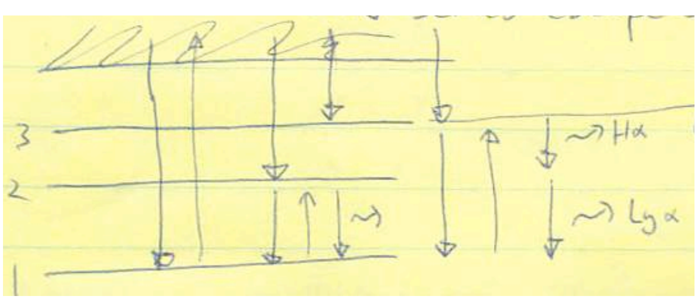
$$\frac{4\pi}{3}R_S^3 n_e n_H \alpha_2 = \dot{N}_\gamma \tag{6.81}$$

where $\dot{N}_\gamma$ is the rate at which the central star produces ionizing photons, and we've assumed $x = 1$ and α_2 is constant inside the HII region. This gives the radius of the *Strömgen sphere*

$$R_S = \left(\frac{3\dot{N}_\gamma}{4\pi\alpha_{bf}n^2}\right)^{1/3} = \frac{7\ \mathrm{pc}}{(n/100)^{2/3}}\left(\frac{10^{-13}}{\alpha_2}\right)^{1/3}\left(\frac{\dot{N}_\gamma}{5\times10^{49}\ \mathrm{s}^{-1}}\right)^{1/3} \tag{6.82}$$

where we've used a value of $\dot{N}_\gamma$ appropriate for an O5 main sequence star.

For the photons with $\nu < \nu_1$ produced by recombinations into the $n = 2$ and higher levels, there are two limiting cases that are usually discussed. The first, *case A* is when the HII region is optically thin to line photons. The second, *case B* is when the HII region is optically thick to Ly series photons (corresponding to a transition from $n = 1$ to a higher level). The other series escape since the populations of levels $n = 2, 3, ...$ are tiny compared to the population of the ground state.



6.13. Collisional excitation and deexcitation

If the electron density is high enough, collisions of atoms with electrons can cause transitions between atomic levels. We write the rate of collision-induced transitions from j to k $R_{jk} = n_e \gamma_{jk}$ where γ_{jk} has units cm^3 s^{-1}. Now include these processes in the argument for the Einstein relations. In steady-state,

$$n_j \left[\sum_k n_e \gamma_{jk} + \sum_k B_{jk} \bar{J} + \sum_{k<j} A_{jk} \right] = \sum_k n_k \left(n_e \gamma_{kj} + B_{kj} \bar{J} \right) + \sum_{k>j} n_k A_{kj}. \qquad (6.83)$$

In thermal equilibrium, the terms involving collisions must balance, and since $n_k/n_j = (g_k/g_j) e^{-E_{jk}/k_B T}$ we must have

$$\frac{\gamma_{jk}}{\gamma_{kj}} = \frac{g_k}{g_j} e^{-E_{jk}/k_B T}. \qquad (6.84)$$

The coefficient γ_{jk} is an average of σv over the velocity distribution of the electrons

$$\gamma_{jk} = 4\pi \left(\frac{m}{2\pi k_B T} \right)^{3/2} \int_0^\infty v^3 \sigma_{jk}(v) e^{-mv^2/2k_B T} dv. \qquad (6.85)$$

In thermal equilibrium we must have detailed balance

$$n_j f(u_j) du_j u_j \sigma_{jk} = n_k f(u_k) du_k u_k \sigma_{kj} \qquad (6.86)$$

where u_j is the velocity before the collision and $u_k < u_j$ the velocity after the collision, $(1/2) m u_k^2 = (1/2) m u_j^2 - E_{jk}$, which gives

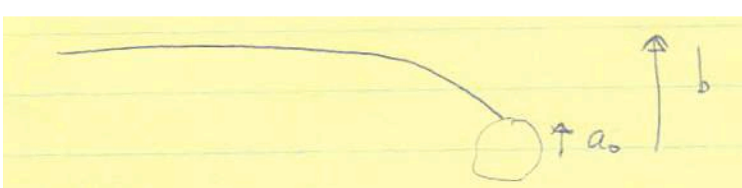
$$g_j u_j^2 \sigma_{jk} = g_k u_k^2 \sigma_{kj} \qquad (6.87)$$

analogous to the Milne relation.

The cross-section is written in terms of the *collision strength* $\Omega(j,k)$ as

$$\sigma_{jk} = \frac{\pi}{g_j} \left(\frac{\hbar}{m_e u_j} \right)^2 \Omega(j,k) \qquad (6.88)$$

where $\Omega(j,k)$ comes from a QM calculation of the cross-section and is typically of order unity. We can get the scalings from a classical argument which is similar to the idea of gravitational focusing. Consider an electron approaching an ion with charge Z and radius a_0. The impact parameter b is such that the electron just hits the ion.



Conservation of angular momentum and energy conservation gives

$$bv_i = av_f \qquad v_f^2 = v_i^2 + \frac{2Ze^2}{ma_0}. \tag{6.89}$$

The cross-section is

$$\sigma = \pi b^2 = \pi a_0^2 \left(1 + \frac{2Ze^2}{ma_0 v_i^2}\right). \tag{6.90}$$

The second term in the brackets dominates: using $a_0 = (\hbar/m_e c)(1/Z\alpha)$, we find it is $2Z^2\alpha^2 c^2/v_i^2 \sim 10Z^2(T/10^4\,\mathrm{K})^{-1}$, where we write $v_i \approx (3k_BT/m_e)^{1/2} = 6.7$ km s$^{-1}T^{1/2}$. The cross-section is then

$$\sigma \approx \pi a_0^2 \left(\frac{2Ze^2}{ma_0 v_i^2}\right) = 2\pi \left(\frac{\hbar}{m_e v_i}\right)^2. \tag{6.91}$$

Averaging over the electron velocity distribution using $\int_0^\infty u du e^{-l^2u^2} = 1/2l^2$ gives the deexcitation rate as

$$\gamma_{kj} = \frac{h^2\Omega(j,k)}{g_k(2\pi m_e)^{3/2}(k_BT)^{1/2}} = 8.63\times 10^{-6}\ \mathrm{cm^3\ s^{-1}}\frac{\Omega(j,k)}{g_kT^{1/2}}. \tag{6.92}$$

The excitation rate coefficient is given by[14] $g_j\gamma_{jk} = g_k\gamma_{kj}e^{-E_{jk}/k_BT}$.

6.14. Line diagnostics of temperature and density

Consider a two-level system. In equilibrium,

$$n_1 n_e \gamma_{12} + n_1 B_{12}\bar{J} = n_2 n_e \gamma_{21} + n_2 A_{21} + n_2 B_{21}\bar{J} \tag{6.93}$$

which gives

$$\frac{n_2}{n_1} = \frac{n_e\gamma_{12} + B_{12}\bar{J}}{n_e\gamma_{21} + B_{21}\bar{J} + A_{21}} = \frac{n_e\gamma_{12}/A_{21} + B_{12}\bar{J}/A_{21}}{1 + \bar{J}B_{21}/A_{21} + n_e\gamma_{21}/A_{21}}. \tag{6.94}$$

Now using the Einstein relations $B_{21}/A_{21} = c^2/2h\nu^3$, $B_{12}/A_{21} = (g_2/g_1)(c^2/2h\nu^3)$ and $\gamma_{12} = \gamma_{21}(g_2/g_1)e^{-E_{12}/k_BT}$, we get

$$\frac{n_2}{n_1} = \frac{(g_2/g_1)e^{-E_{12}/k_BT}\left(n_e\gamma_{12}/A_{21} + e^{E_{12}/k_BT}c^2\bar{J}/2h\nu^3\right)}{1 + c^2\bar{J}/2h\nu^3 + n_e\gamma_{21}/A_{21}}. \tag{6.95}$$

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[14] Using the relation in eq. [6.84] that we derived using arguments about thermal equilibrium. You can also directly average the excitation cross-section over the electron velocity distribution, but be careful - if you do that you must take into account the fact that a minimum energy $(1/2)mv^2 = E_{jk}$ is required to excite the transition. Including this as a lower limit on the range of velocities considered gives the factor of e^{-E_{jk}/k_BT} that we deduced from thermal equlibrium.

In an HII region, we expect $U_\nu \sim B_\nu(T \approx 10^4 \text{ K})(R_\star/r)^2$ where the dilution factor $(R_\star/r)^2 \sim (10^{11} \text{ cm}/10^{18} \text{ cm})^2 \sim 10^{-14}$, which implies that the $\bar{J}$ terms will be small. Then,

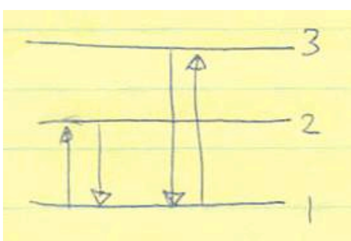
$$\frac{n_2}{n_1} \approx \frac{g_2}{g_1} e^{-E_{12}/k_B T} \frac{1}{1 + A_{21}/n_e \gamma_{21}}. \tag{6.96}$$

When collisions dominate, $n_2/n_1 \to (g_2/g_1)e^{-E_{12}/k_B T}$ which is the correct thermal equilibrium ratio. However, when the spontaneous emission rate is much greater than the collision deexcitation rate, $A_{21} \gg n_e \gamma_{21}$, then

$$\frac{n_2}{n_1} \to \frac{g_2}{g_1} e^{-E_{12}/k_B T} \frac{n_e \gamma_{21}}{A_{21}}. \tag{6.97}$$

For an optical transition (e.g. Lyα), $A_{21} \sim 10^8 \text{ s}^{-1}$, and $\gamma_{12} \sim 10^{-5}/T^{1/2} \sim 10^{-6}$ then $n_e \gamma_{21}/A_{21} \sim 10^{-14} n_e \ll 1$, so $n_2 \ll n_1$. The much faster decay rate due to spontaneous emission compared to the collision rate means that it is an excellent approximation to assume that the upper levels are not populated.

Next, consider the three level system



and neglect induced radiative transitions and also neglect 3 to 2 transitions. The ratio of intensities in the $3 \to 1$ to $2 \to 1$ transitions is

$$\frac{I_{31}}{I_{21}} = \frac{n_3 A_{31} h\nu_{31}}{n_2 A_{21} h\nu_{21}} = \frac{g_3 A_{31} \nu_{31}}{g_2 A_{21} \nu_{21}} \left[\frac{1 + A_{21}/n_e \gamma_{21}}{1 + A_{31}/n_e \gamma_{31}}\right] e^{-E_{23}/k_B T}. \tag{6.98}$$

If n_e is large enough, then

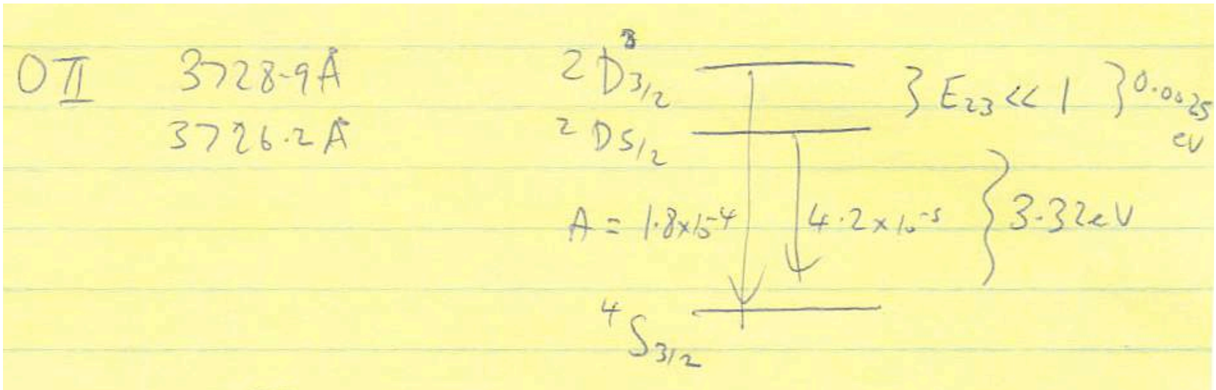
$$\frac{I_{31}}{I_{21}} = \frac{g_3 A_{31} \nu_{31}}{g_2 A_{21} \nu_{21}} e^{-E_{23}/k_B T}, \tag{6.99}$$

so that collisions maintain a thermal population of levels. However, if n_e is small, then

$$\frac{I_{31}}{I_{21}} = \frac{g_3}{g_2} e^{-E_{23}/k_B T} \frac{\nu_{31}}{\nu_{21}} \frac{\gamma_{31}}{\gamma_{21}} = \frac{\nu_{31}}{\nu_{21}} \frac{\gamma_{13}}{\gamma_{12}} \tag{6.100}$$

or in words every collisional excitation produces a photon.

This gives a method for determining the density of ionized gas. A famous example is the OII doublet

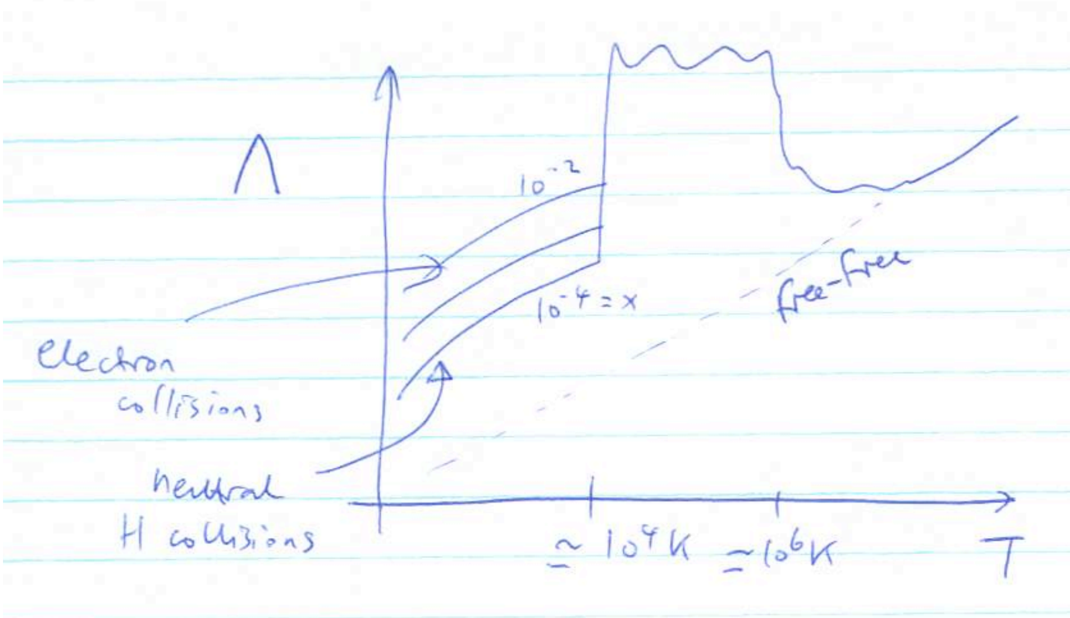


For $n_e < 10^2$ cm^{-3} the expected intensity ratio is $I_{31}/I_{21} = 2/3$ (the ratio of the collision strengths $\Omega(1,2)$ and $\Omega(1,3)$), whereas for $n_e > 10^4$ cm^{-3} the expected ratio is 2.9 (an additional factor of the ratio of A's). Notice that the A's for these transitions are very small $\sim 10^{-4}$ s corresponding to lifetimes of hours! This is because these transitions are forbidden under electric dipole selection rules. They are known as *forbidden transitions* and the lines as *forbidden lines*. It is not possible to see these transitions in the lab because collisional deexcitation always dominates. Nebulae are very bright in these forbidden lines as they are the mechanism by which decay to the ground state occurs. Bowen (1928) showed that the bright emission lines of nebulae, originally attributed to a hypothetical element "nebullium", were in fact forbidden transitions of this kind.

In general we can define a critical electron density $n_{cr} = \sum_{i<k} A_{ki} / \sum_{i \neq k} \gamma_{ki}$ above which collisional excitation/deexcitation gives rise to a thermal population of energy levels, and below which radiative deexcitation dominates and the upper levels have a much smaller population than in LTE.

6.15. Application: The cooling function

Line emission is crucial for understanding the cooling function of gas below $\sim 10^6$ K. For higher temperatures, the gas is ionized and thermal bremsstrahlung is the main cooling mechanism, for 10^4 K $\lesssim T \lesssim 10^6$ K, electron collisions excite electronic levels of neutral or ionized components, while for $T \lesssim 10^4$ K electron or neutral collisions (depending on the ionization fraction) excite fine structure levels of the ground state. The general shape of the cooling function Λ looks like

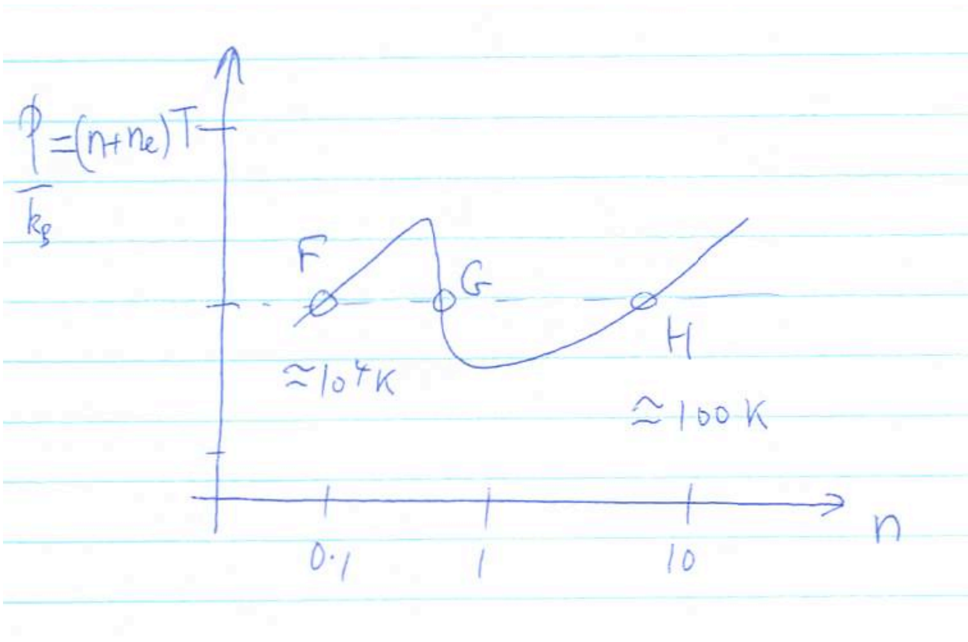


where the quantity Λ plotted has units erg cm^3 s^{-1} so the cooling rate per unit volume is $n_H^2\Lambda$ (see Dalgarno & McCray 1972; Sutherland & Dopita 1993 for detailed calculations). The cooling rate is set by the collision rate which we saw earlier could be written as $\gamma_{12} \approx 10^{-5}$ cm^2 s^{-1} $\Omega(1,2)/\sqrt{T}$. For the fine structure lines, we can take an excitation temperature $T_e \sim 100$ K (or photon energy $\approx 10^{-2}$ eV), giving

$$\Lambda \sim Y_i \gamma_{12} k_B T_e \sim 10^{-24} \text{ erg cm}^3 \text{ s}^{-1} \left(\frac{10^4 \text{ K}}{T}\right)^{1/2} \Omega(1,2) \left(\frac{T_e}{100 \text{ K}}\right) \left(\frac{Y_i}{10^{-3}}\right) \qquad (6.101)$$

where Y_i is the abundance of the metal ion in question. This is about the right order of magnitude. Above 10^4 K the important lines are electronic levels which have $k_B T_e \sim 1$ eV, and so we expect a jump in Λ of order 100 as we cross from below to above 10^4 K, in good agreement with the detailed calculations.

The non-monotonic nature of the cooling function means that in general there are multiple steady-state solutions in which heating balances cooling, which gives rise to different *phases* of the ISM, for example stable phases of hot low density gas and cold high density gas can exist in pressure equilibrium. The classical papers are Field (1965) on thermal stability, Field, Goldsmith and Habing (1969) on the 2 phase ISM, and McKee and Ostriker (1977) on the three-phase model of the ISM. To get the basic idea, write the balance between heating and cooling as $H = n\Lambda$ where H is the heating rate per ion from for example cosmic ray collisions. Because Λ is non-monotonic, multiple solutions are possible. Field et al. (1969) plotted the pressure against number density for the stable solutions. At a given pressure, there are three possible solutions F, G, and H.



However, phase G is *thermally unstable*. To see this, perturb the equilibrium $T \to T + \delta T$, and let's assume the heating rate H is a constant. Then the response of the gas will be

$$\frac{d\delta T}{dt} \propto -n\delta T\frac{\partial \Lambda}{\partial T} \tag{6.102}$$

which implies that the perturbed gas cools back to the equilibrium state if $\partial\Lambda/\partial T > 0$ (stable) but will undergo a thermal runaway if $\partial\Lambda/\partial T < 0$ (unstable). Therefore the phases F and H are stable, but G is not.

6.16. Absorption and emission by dust

We close with a few words about dust. Dust consists of small particles, ranging from large molecules to μm size particles. The dominant composition is graphite which shows an emission feature at 2175Å, silicates (e.g. $(MgFe)_2SiO_4$) which are responsible for features at 9.7 μm and 18 μm due to Si$-$O bending and stretching modes, and polycyclic aromatic hydrocarbons (PAHs) such as napthalene ($C_{10}H_8$ with two benzene rings). The typical dust to gas mass ratio is $\sim 10^{-2}$. The typical linear size distribution is $n(a)da \propto a^{-3.5}da$ so that the mass is in the large grains, but the area (important for extinction) is in the small grains. Typical dust temperatures are a few tens of K, giving thermal emission in the IR. Dust is extremely important: it is the dominant opacity source for non-ionizing photons, dust grains lock up a substantial fraction of heavy elements, there is important surface chemistry e.g. molecule formation on the surface of dust grains, and dust plays an important role in energy balance. Here, we focus on how to understand the typical dust absorption cross-section, and the dust temperature.

The dust optical depth is generally written as

$$\tau = L \int_{a_-}^{a_+} n(a) da \, \sigma_d(a) Q_{\rm ext}(\lambda, a) L \tag{6.103}$$

where L is the path length, $\sigma_d = \pi a^2$ is the geometric cross-section for a sphere, and "ext" refers to extinction which can be due to absorption or scattering (when the dust lies along the line of sight to an object scattering removes photons from the beam), so that

$$Q_{\rm ext}(\lambda, a) = Q_{\rm abs}(\lambda, a) + Q_{\rm sca}(\lambda, a). \tag{6.104}$$

In astronomy, the extinction in magnitudes at a given wavelength $A_\lambda = -2.5 \log_{10}(e^{\tau_\lambda}) = 1.086\tau_\lambda$. Commonly dust extinction will be measured in A_V the visual extinction. This is related to the hydrogen column,

$$N_H \approx 5.9 \times 10^{21} \ {\rm cm}^{-2} \ {\rm mag}^{-1} \ E_{B-V} \tag{6.105}$$

where $E_{B-V} = A_B - A_V$ is the difference between the B and V band extinctions. The slope of the extinction curve near V band is $R_V = A_V/E_{B-V}$ with R_V commonly taken to be 3.1 (can be as large as 5 for lines of sight into dense clouds).

For calculations of the optical properties of grains, cross-sections and dust temperatures see Draine and Lee (1984). To calculate the absorption and scattering cross-sections from first principles requires understanding the optical properties of the grains, in other words the refractive index n or dielectric constant $\epsilon = n^2$. The dielectric constant is complex in general, and the solutions for scattering and absorption by small particles are known as *Mie theory.* For spherical particles, and in the long-wavelength limit $x = 2\pi a/\lambda \ll 1$, the cross-sections are

$$Q_{\rm sca} = \frac{8}{3} x^4 \left| \frac{\epsilon - 1}{\epsilon + 2} \right|^2 \propto \frac{1}{\lambda^4} \tag{6.106}$$

(Rayleigh scattering) and

$$Q_{\rm abs} = 4x \ {\rm Im} \left(\frac{\epsilon - 1}{\epsilon + 2} \right) \propto \frac{1}{\lambda}. \tag{6.107}$$

For short wavelength photons $x \gg 1$, $Q_{\rm sca} \rightarrow 1$ and $Q_{\rm abs} \rightarrow 1$. We've written the wavelength scalings here for constant ϵ. In fact, ϵ depends on frequency in general which changes the scaling. The extinction cross-section is usually written

$$Q_{\rm ext} = Q_0 \left(\frac{\lambda_0}{\lambda} \right)^\beta \tag{6.108}$$

with $\beta = 1$–2. For graphite and silicates the long wavelength emission has $\beta \approx 2$.

We can get the cross-sections in equations (6.106) and (6.107) from a straightforward calculation along the lines of the classical derivation of the Thomson cross-section. A dielectric sphere in a uniform electric field has an induced dipole moment $\vec{p} = \alpha\vec{E}$, where the polarizability $\alpha = 4\pi\epsilon_0 a^3(\epsilon - 1/\epsilon + 2)$ (SI units). Plugging this into Larmors formula $P = (1/2)(\omega^4 p^2/6\pi\epsilon_0 c^3)$ and dividing by the incoming flux $c\epsilon_0 E^2/2$ gives the scattering cross-section in equation (6.106). The absorption cross-section is obtained by calculating the work done $\vec{J}\cdot\vec{E}$, where $\vec{J}$ is the polarization current

$$\frac{\partial\vec{P}}{\partial t} = \frac{i\omega\vec{p}}{4\pi a^3/3}. \tag{6.109}$$

The power dissipated is

$$\left(\vec{J}\cdot\vec{E}\right)\left(\frac{4}{3}\pi a^3\right) = i\omega\vec{p}\cdot\vec{E} = i\omega\alpha E^2 \tag{6.110}$$

or taking a time average

$$\frac{\omega}{2}\mathrm{Im}(\alpha)E^2. \tag{6.111}$$

Again, dividing by the incoming flux in the wave gives the cross-section, which agrees with equation (6.107).

The equilibrium dust temperature is given by balancing heating with cooling. The heating rate is

$$4\pi a^2 \int \pi J_\lambda Q_{\rm abs}(\lambda, a)d\lambda. \tag{6.112}$$

For short wavelength UV and optical photons which are responsible for most of the heating, $Q_{\rm abs} \sim 1$ in which case we can write the heating rate as $4\pi^2 a^2 J_{UV}$ where J_{UV} is the mean intensity of the photons with wavelengths shorter than typical dust sizes. The cooling or the dust emissivity we can get from Kirchoff's law, giving

$$4\pi a^2 \int \pi B_\lambda(T_d) Q_{\rm abs}(\lambda, a)d\lambda \tag{6.113}$$

where T_d is the dust temperature. The dust temperature is therefore given by

$$J_{UV} = \int B_\lambda(T_d) Q_{\rm abs}(\lambda, a)d\lambda. \tag{6.114}$$

For typical dust temperatures $T_d \approx 20$ K, we need the absorption cross-section near $\lambda = 0.29\ {\rm cm}/T \approx 200\ \mu{\rm m}$. Writing $Q_{\rm abs} = Q_0(\lambda_0/\lambda)^\beta = (2\pi a/\lambda)(\lambda_0/\lambda)^{\beta-1}$ gives

$$J_{UV} = \int \frac{2\pi a}{\lambda}\left(\frac{\lambda_0}{\lambda}\right)^{\beta-1} \frac{2h\nu^3}{c^2}\frac{1}{e^{h\nu/k_BT}-1}\frac{d\nu}{d\lambda}d\lambda = \frac{4\pi a \lambda_0^{\beta-1}h}{c^{2+\beta}}\left(\frac{k_BT_d}{h}\right)^{4+\beta}\int\frac{x^{3+\beta}\,dx}{e^x-1} \tag{6.115}$$

or $T_d \propto (J_{UV}/a)^{1/(4+\beta)}$. For 0.1 μm size grains, the temperatures are $\approx$ 20K.

An interesting situation arises for small grains (Purcell 1976). A 100Å grain at 20 K has a thermal content of $\sim$ 1 eV. Partly this is due to the $(T/\Theta_D)^3$ suppression of the specific heat at low temperatures, where the Debye temperature Θ_D is typically hundreds of K. The temperature of these grains is time-dependent since absorption of a photon significantly changes the thermal energy, followed by rapid cooling. The observed emission of small grains can therefore be at significantly higher temperature, e.g. 50K (see Purcell 1976).

Summary and Further Reading

Here are the main ideas and results that we covered in this part of the course:

- Saha equation. Ionization of hydrogen at $T \approx 10^4$ K. Application to stellar spectral types.
- Line profiles. Natural line width Λ. Collisional broadening. Doppler broadening $\Delta\nu/\nu \approx \langle v\rangle/c$. Voigt function. Definition of equivalent width. Curve of growth.
- Hydrogen-like atoms. $\chi = (1/2)Z^2\alpha^2 m_e c^2$, $a_0 = (\hbar/m_e c)\alpha^{-1}Z^{-1}$, $E_{nm} = \chi(n^{-2}-m^{-2})$.
- Calculation of radiative transitions. Fermi's Golden Rule. Hamiltonian for a charged particle in an electromagnetic field. Origin of selection rules in the spatial integrals.
- Bound-bound transitions. Einstein A coefficients: $A \sim (mc^2/\hbar)\alpha^5$. Ly$\alpha$ has $A = 0.6\times 10^9$ s^{-1}. Bound-bound absorption cross-section: $\sigma_{12} = f(\pi e^2/m_e c)\phi(\nu)$ or $\approx (\lambda^2/8\pi)(A_{21}/\Delta\nu)$, where $\Delta\nu$ is the linewidth.
- Bound-free (photoionization) cross-section

  $$\sigma_{bf} = \alpha\frac{\pi a_0^2}{Z^2}\left(\frac{\omega_n}{\omega}\right)^3\frac{64}{3\sqrt{3}}n g_{bf}$$

  Absorption edges. $\kappa_{bf} \propto \rho T^{-3.5}$. Milne relations. Recombination coefficient $\langle\sigma_{fb}v\rangle \sim 10^{-14}$ cm^3 s^{-1}. Application to HII regions. The Stromgen sphere. Case A and case B. Application to X-ray absorption by the ISM.
- Collisional excitation and dexcitation. Cross-section $\sigma_{jk} = (\pi/g_j)(\hbar/m_e v)^2\Omega(j,k)$. Averaging over a Maxwell-Boltzmann distribution of velocities gives the deexcitation rate coefficient

  $$\gamma_{kj} = \frac{8.6\times 10^{-6}\ \Omega(j,k)}{g_k T^{1/2}}\ \mathrm{cm^3\ s^{-1}}.$$

The excitation rate coefficient is then $\gamma_{jk} = (g_k/g_j)e^{-E_{jk}/k_BT}\gamma_{kj}$. The critical density $n_{e,c} = A_{21}/\gamma_{21}$. Application: the use of forbidden line ratios as a probe of density. The importance of line emission in the cooling function.

- Dust. Sizes, composition, dust to gas ratio, temperatures, importance. The basic idea of scattering and absorption by small particles. For $\lambda \gg a$, $\sigma \propto 1/\lambda^4$ (Rayleigh scattering, constant ϵ). Generally, $Q_{\rm ext} \propto \lambda^{-\beta}$ with $\beta = 1$–2.

Reading

- RL chapter 10. The calculation of radiative transitions is covered at the end of most introductory quantum books, e.g. the books by Gasiorowicz and Townsend. Shu also has a lot of detail on this.
- Radial integrals of hydrogen-like wavefunctions: Gordon (1929) Ann Phys **2**, 1031.
- Osterbrock "Astrophysical Gaseous Nebulae" is the classic book to look up cross-sections and transition rates etc. A classic but somewhat dated book on ISM physics is by Spitzer. Recombination coefficients: Cillié 1932, MNRAS, 92, 820. Seaton 1959, MNRAS, 119, 81.
- X-ray absorption by the ISM: Wilms, Allen, & McCray (2000), Morrison & McCammon (1983).
- Cooling function: Dalgarno & McCray (1972), Sutherland & Dopita (1993). Thermal stability and multi-phase interstellar medium: Field (1965), Field, Goldsmith & Habing (1969), McKee & Ostriker (1977).
- Dust: Mathis (1990 ARAA), Draine & Lee (1984 ApJ).

PHYS 642 Problem Sets and Solutions

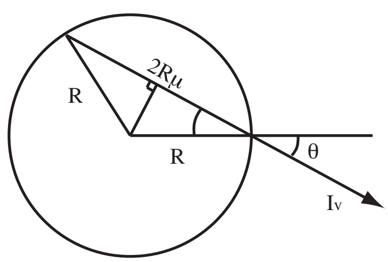
7. Problem Set 1 with Solutions

1. (a) **Question.** A sphere of material with radius R has constant emissivity j_ν, absorption coefficient α_ν, and source function S_ν. Calculate I_ν at the surface of the sphere and show that the flux is

$$F_\nu = \pi S_\nu \left[1 - \frac{2}{\tau^2}\left(1 - e^{-\tau}\right) + \frac{2e^{-\tau}}{\tau}\right], \tag{6.116}$$

where $\tau = 2\alpha_\nu R$. Sketch the angular dependence of I_ν for different values of τ. Check that I_ν and F_ν make sense in the limits $\tau \ll 1$ and $\tau \gg 1$.

Solution: At a given point on the surface, a ray emerging at angle θ with respect to the normal to the surface has traversed a distance $2R\cos\theta = 2R\mu$.



Defining the optical depth $\tau_\nu = 2\alpha_\nu R$, the general solution to the radiative transfer equation for constant source function S_ν gives

$$I_\nu(\mu) = S_\nu\left(1 - e^{-\tau_\nu \mu}\right) \tag{6.117}$$

The flux is given by integrating over outgoing rays $F = 2\pi \int_0^1 d\mu\, \mu I_\nu(\mu)$

$$F_\nu = 2\pi S_\nu \int_0^1 d\mu\, \mu(1 - e^{\tau_\nu \mu}) = \pi S_\nu \left(1 - \frac{2}{\tau_\nu^2} + 2e^{-\tau_\nu}\frac{(1+\tau_\nu)}{\tau_\nu^2}\right). \tag{6.118}$$

The large and small τ limits are $\tau \gg 1$ $F_\nu \to \pi S_\nu$ as expected for an optically thick source. For $\tau \ll 1$, $F_\nu \to 2\pi S_\nu \tau_\nu/3$. The luminosity is $4\pi R^2 F = (4\pi j_\nu)(4\pi R^3/3)$ as expected for an optically thin source, since all photons leave.

(b) **Question.** Now consider a large area slab of the same material with thickness H. Defining $\tau = \alpha_\nu H$ for this case, show that the flux at the surface is

$$F_\nu = \pi S_\nu \left[1 - 2E_3(\tau)\right], \tag{6.119}$$

where

$$E_n(\tau) = \int_1^\infty dx\; x^{-n} e^{-\tau x}. \tag{6.120}$$

Sketch the angular dependence of I_ν for different τ. Check that the $\tau \ll 1$ and $\tau \gg 1$ limits for I_ν and F_ν make sense.

Solution. This is similar to the previous question, but now the path length of the emerging ray increases with angle from the normal to the surface, $H/\cos\theta = H/\mu$. Writing $\tau_\nu = \alpha_\nu H$, the solution to the radiative transfer equation is

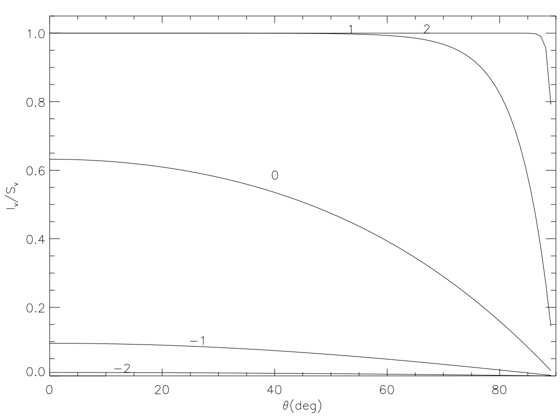
$$I_\nu(\mu) = S_\nu \left(1 - e^{-\tau_\nu/\mu}\right). \tag{6.121}$$

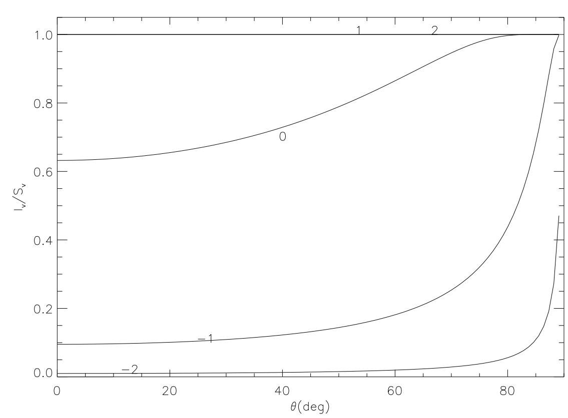
The flux is

$$F_\nu = 2\pi S_\nu \int_0^1 d\mu\, \mu \left(1 - e^{-\tau_\nu/\mu}\right) = 2\pi S_\nu \int_1^\infty \frac{dx}{x^3}\left(1 - e^{-\tau_\nu x}\right) = \pi S_\nu (1 - 2E_3(\tau_\nu)). \tag{6.122}$$

When $\tau \gg 1$, $E_3(\tau) \to e^{-\tau}/\tau \to 0$ and $F_\nu \to \pi S_\nu$, the correct limit for an optically thick medium. When $\tau \ll 1$, $2E_3(\tau) \approx 1 - 2x$, giving $F_\nu = 2\pi S_\nu \tau_\nu = (1/2)(4\pi j_\nu)H$. The factor of 1/2 comes in because there are two sides to the slab.

The following plots show the angular distributions in the two cases, labelled by $\log_{10}\tau$. The radiation from the sphere is concentrated in the forward direction, whereas for the slab the intensity is largest for rays travelling parallel to the surface.





(c) **Question.** In class, we derived the flux at the surface of a grey atmosphere,

$$F_\nu = 2\pi \int_0^\infty B_\nu(\tau')\, E_2(\tau')\, d\tau', \tag{6.123}$$

where B_ν is a function of τ through the dependence of temperature on τ. Show that in the isothermal limit, this result reduces to equation (2).

Solution. This question is poorly worded. What I had in mind was to replace the upper limit on the integration in equation (6.123) by a finite value τ instead of integrating all the way to ∞. For an isothermal atmosphere, B_ν is a constant and can be taken outside the integral. Then use the result $E_2(x) = -dE_3(x)/dx$ to do the integral,

$$F_\nu = -2\pi B_\nu \int_0^\tau \frac{dE_3(\tau')}{d\tau'} d\tau' = -2\pi B_\nu \left(E_3(\tau) - E_3(0)\right) = \pi B_\nu (1 - 2E_3(\tau)) \tag{6.124}$$

(since $E_3(0) = 1/2$). This answer agrees with part (b).

(d) **Question.** Plot the normalized flux $\nu F_\nu / \sigma T_{\rm eff}^4$ against normalized frequency $h\nu/kT_{\rm eff}$ for both blackbody and grey atmospheres. For the grey atmosphere, assume a temperature profile given by the Eddington approximation, $T^4 = T_{\rm eff}^4 (3/4)(\tau + 2/3)$, and numerically evaluate the integral in equation (6.123). How do the grey and blackbody spectra differ, and why?

Solution. First normalize the Planck spectrum as suggested, which gives

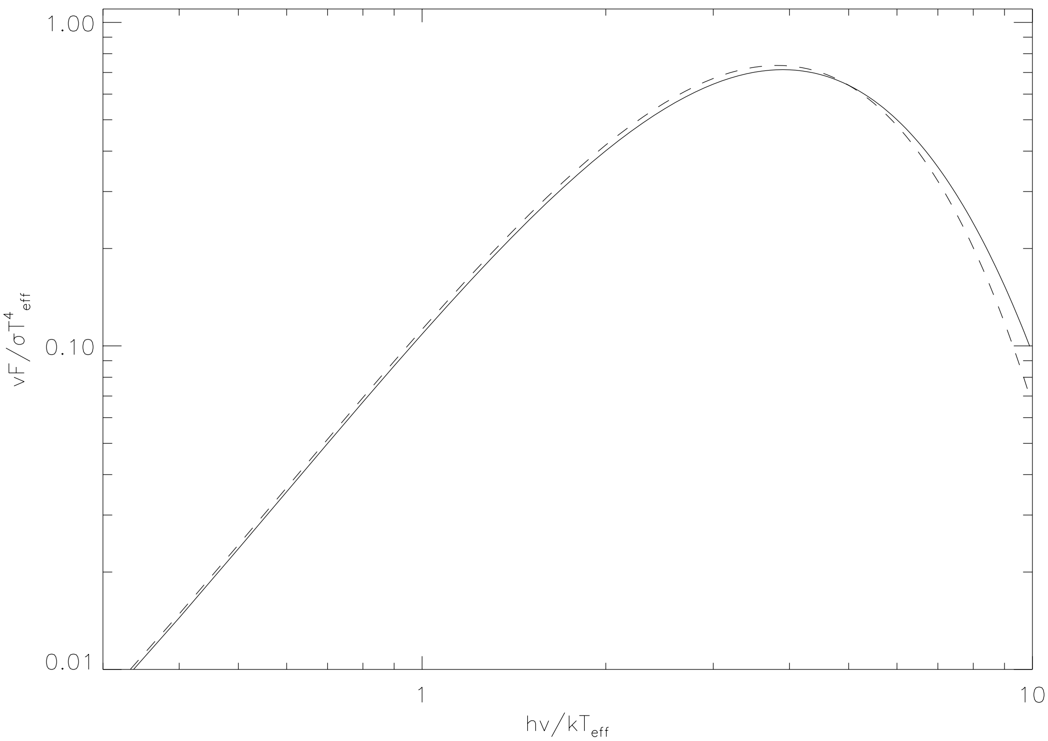
$$\frac{\nu \pi B_\nu}{\sigma T_{\rm eff}^4} = \frac{15}{\pi^4} \frac{x^4}{e^{x T_{\rm eff}/T} - 1}, \tag{6.125}$$

where $x = h\nu / k_B T_{\rm eff}$. The normalized grey atmosphere flux is then given by equation (6.123)

$$\frac{\nu F_\nu}{\sigma T_{\rm eff}^4}(x) = \frac{30}{\pi^4} \int_0^\infty \frac{x^4}{e^{x T_{\rm eff}(\tau')/T} - 1} E_2(\tau') d\tau' \tag{6.126}$$

with the temperature profile specified as a function of optical depth. In the Eddington approximation, we found $(T/T_{\rm eff})^4 = (3/4)(\tau + 2/3)$.

The grey atmosphere spectrum is plotted in the Figure (solid curve), compared with a blackbody spectrum with $T = T_{\rm eff}$ (dashed curve).



We can see that the grey atmosphere is *harder* than the blackbody, in other words it is brighter at high photon frequencies and fainter at low photon frequencies. As the integral in equation (6.123) shows, the spectrum is a result of summing blackbody spectra over depth in the atmosphere, each at a temperature given by the local temperature, and each weighted by the optical depth, with emission at large optical depth being exponentially suppressed.

One way to try to understand what's going on is to look at the low and high frequency limits. For low frequencies below the peak, $x \ll 1$, the ratio of grey atmosphere flux to blackbody flux is

$$\frac{F_\nu}{\pi B_\nu(T)} \approx \int_0^\infty \left(\frac{3}{4}\right)^{1/4} \left(\tau' + \frac{2}{3}\right)^{1/4} 2E_2(\tau')d\tau'. \tag{6.127}$$

Compared to the isothermal case at $T = T_{\rm eff}$ which would give the blackbody spectrum, the integrand is weighted towards lower optical depths and therefore smaller temperatures, which results in less emission at these low frequencies. (The mean brightness temperature is smaller than $T_{\rm eff}$.)

The high frequency limit ($x \gg 1$) is

$$\frac{F_\nu}{\pi B_\nu} \approx \int_0^\infty \exp\left(x\left[1 - \frac{1}{(3/4)^{1/4}(\tau' + 2/3)^{1/4}}\right]\right) 2E_2(\tau')d\tau'. \tag{6.128}$$

The grey atmosphere becomes increasingly brighter compared to the blackbody as x increases.

[The following properties of $E_n(x)$ are useful in this question:

$$E_n(x) \to e^{-x}/x \qquad x \to \infty$$

$$E_1(x) \to \ln(1/x) \qquad x \to 0$$

$$(n-1)E_n(x) = e^{-x} - xE_{n-1}(x)$$

$$dE_n(x)/dx = -E_{n-1}(x).$$

If you have time, you might like to try and prove these results.]

8. Problem Set 2 with solutions

1. **More on the bremsstrahlung derivation.**

(a) **Question.** In class, we calculated the radiation spectrum from a single collision of an electron and ion using only the perpendicular component of the acceleration. Repeat this calculation for the parallel component. You should find that the spectrum from the parallel component depends on the modified Bessel function K_0 rather than K_1. Plot the total radiation spectrum and the contribution from the two components.

Solution. We follow the same procedure as in the notes, but now take the parallel component of the acceleration. The integral is

$$\dot{u}_{\parallel}(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{i\omega t} dt \frac{Ze^2}{m}\frac{ut}{(b^2+u^2t^2)^{3/2}} \tag{6.129}$$

$$= \frac{1}{\sqrt{2\pi}}\frac{Ze^2}{mub}\int_{-\infty}^{\infty}\frac{dx\, xe^{ix\omega b/u}}{(1+x^2)^{3/2}}. \tag{6.130}$$

The trick is to now integrate by parts to get the integral into a form that looks like the integral representation of a modified Bessel function.

$$\dot{u}_{\parallel}(\omega) = \frac{i}{\sqrt{2\pi}}\frac{Ze^2}{mub}\frac{\omega b}{u}\int_{-\infty}^{\infty}\frac{dx\, e^{ix\omega b/u}}{(1+x^2)^{1/2}} \tag{6.131}$$

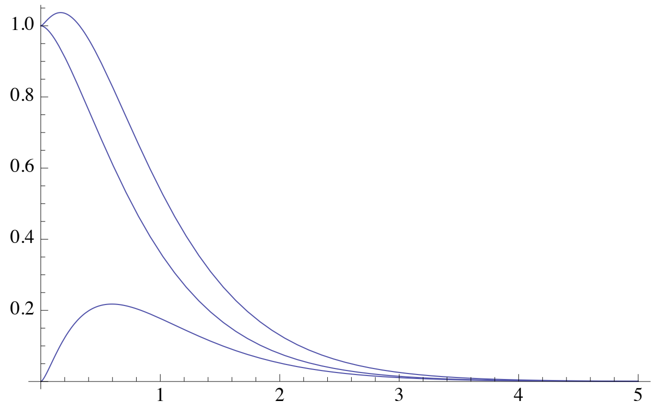
$$= \sqrt{\frac{2}{\pi}}\frac{Ze^2}{mub} iyK_0(y) \tag{6.132}$$

where $y = \omega b/u$. This is the same as the perpendicular acceleration, but with $yK_1(y)$ replaced by $iyK_0(y)$. The factor of i indicates that the two components of acceleration are $\pi/2$ out of phase, which makes sense since the perpendicular acceleration is maximum at the distance of closest approach, but the parallel component of acceleration is zero there.

Again following the notes, the total spectrum is proportional to $|\dot{u}_{\perp}|^2 + |\dot{u}_{\parallel}|^2 \propto y^2(K_0(y)^2 + K_1(y)^2)$. Here is a plot of each component and the total. Neglecting the parallel acceleration gives a $\approx 20\%$ reduction in the emissivity.

```
In[5]:= Show[Plot[(BesselK[0, x]^2 + BesselK[1, x]^2) x^2, {x, 0, 5}],
  Plot[BesselK[1, x]^2 x^2, {x, 0, 5}], Plot[BesselK[0, x]^2 x^2, {x, 0, 5}]]
```

Out[5]=



(b) **Question.** In class we discussed the different choices for $b_{\rm min}$ and $b_{\rm max}$. Use these to roughly check the analytic expressions given in Fig. 5.2 of RL (for the lower half of this figure, or $h\nu < k_BT$). Approximate the thermal averaging by replacing v with the thermal velocity. You won't get the ζ factor [optional: what is this? Maybe look up the Novikov and Thorne article]. Longair states (Vol. 1, p77) that for radio wavelengths,

$$\frac{\pi}{\sqrt{3}}g_{ff} = \frac{1}{2}\left[\ln\left(\frac{8(k_BT)^3}{\pi^2 m_e e^4 \nu^2 Z^2}\right) - \sqrt{\gamma}\right] \tag{6.133}$$

is the appropriate Gaunt factor to use, whereas for X-ray wavelengths

$$\frac{\pi}{\sqrt{3}}g_{ff} = \ln\left(\frac{k_BT}{h\nu}\right). \tag{6.134}$$

Does this make sense? (Don't worry about order unity factors, e.g. ignore the $\sqrt{\gamma}$ factor, which comes from a more careful derivation of the Gaunt factor. The constant $\gamma = 0.577...$ is Euler's constant.)

Solution. The idea here is to write

$$g_{ff} = \frac{\sqrt{3}}{\pi}\ln\left(\frac{b_{\rm max}}{b_{\rm min}}\right) \tag{6.135}$$

and use the different approximations from the notes for $b_{\rm max}$ and $b_{\rm min}$. First $b_{\rm max} = v/\omega$ (photon discreteness). For $b_{\rm min}$, we noted that there were two possible choices, "classical" $b_{\rm min} = 2Ze^2/mv^2$ or "quantum" $b_{\rm min} = \hbar/m_ev$, with the quantum result being applicable at high electron energies exceeding $Z^2\,{\rm Ry}$ and the classical result at lower energies. The final ingredient we need is to recognize that after the thermal averaging over the electron velocity distribution, the electron velocity will be replaced by $v \approx (3k_BT/m_e)^{1/2}$ (I've put a 3 in there because the mean value of $(1/2)mv^2$ for a classical gas is $(3/2)k_BT$, but in the Gaunt factor that 3 will be some other number that depends on the weighting during the averaging, so we shouldn't trust the prefactor in our results. Let's just see if we can get the scalings right.)

Now let's check the figure. In the lower left, the electron energy $\approx k_BT$ is small compared to Z^2Ry so we need the classical $b_{\rm min}$. Therefore $b_{\rm max}/b_{\rm min} = mv^3/2Ze^2\omega$. Next replace v with the thermal velocity and use the result that Ry$= \alpha^2 m_e c^2$ where $\alpha = 1/137 = e^2/\hbar c$ is the fine structure constant. I get

$$g_{ff} \approx \frac{\sqrt{3}}{\pi} \ln\left[\frac{3^{3/2}}{2}\frac{k_BT}{h\nu}\left(\frac{k_BT}{Z^2{\rm Ry}}\right)^{1/2}\right] \tag{6.136}$$

which agrees with the Figure apart from the numerical prefactor. *Note that the Figure has a typo compared to the original version in the Novikov and Thorne article: the square root should be on the last term inside the log, as we have derived here, and not on the whole log.*

On the lower right, we use the quantum estimate for $b_{\rm min}$. This time I get

$$g_{ff} \approx \frac{\sqrt{3}}{\pi} \ln\left[3\frac{k_BT}{h\nu}\right]. \tag{6.137}$$

Again we agree up to the prefactor.

Now looking at the Longair formulas, we see that the radio wavelength formula is using the classical $b_{\rm min}$ whereas the X-ray formula is using the quantum $b_{\rm min}$. What this seems to be saying is that if I have gas emitting thermal bremsstrahling in radio, then $k_BT \ll Z^2$Ry and the classical result applies. For X-ray gas, $k_BT \gg Z^2$Ry putting us in the quantum regime (X-ray photon energies $\sim$keV's).

2. **Thermal bremsstrahlung from cluster gas.** The X-ray emission from hot gas in galaxy clusters is powered by thermal Bremsstrahlung. Look at the paper by Jeltema et al. (2001, ApJ 562, 124).

(a) **Question.** First derive equation (1) in the paper. To do this, assume that $k_BT = GM/R$ (as we did in class to get the temperature at the center of the Sun). The critical density today is given by $\rho_c = 3H_0^2/8\pi G$, and increases with redshift z proportional to $(1+z)^3$. The parameter $h = H_0/100$ km s^{-1} Mpc measures the Hubble constant H_0 as measured locally. Use the mass-temperature relation and the fitted temperature for this cluster to estimate the mass of the cluster in dark matter.

Solution. First, estimate the radius R from $k_BT \sim GMm_p/R$ (the formula in the question has a missing m_p). It's good to check the number that comes out to see if it makes sense. I get

$$R = 4.5 \text{ Mpc} \left(\frac{M}{10^{15}\ M_\odot}\right)\left(\frac{k_B T}{10 \text{ keV}}\right)^{-1}. \tag{6.138}$$

The density is $\rho = 200\rho_0$ which gives $n = \rho/m_p = 2.3 \times 10^{-3}\ \text{cm}^{-3}\ h^2(1+z)^3$.

The simplest estimate is to take a constant density sphere with radius R, which gives a mass

$$M = \left(\frac{3}{4\pi\rho}\right)^{1/2}\left(\frac{k_B T}{G m_p}\right)^{3/2} = 2.1 \times 10^{14}\ M_\odot \left(\frac{k_B T}{10 \text{ keV}}\right)^{3/2} h^{-1}(1+z)^{-3/2}. \tag{6.139}$$

The scalings agree with equation (1) of Jeltema et al, but the prefactor is too low by almost an order of magnitude.

For the cluster in that paper, $z = 0.83$ and $T = 10.4$ keV. Using the prefactor from the paper, the mass derived is $6.4 \times 10^{14}\ M_\odot h^{-1}$.

(b) **Question.** Now calculate the total X-ray luminosity expected from the cluster. Assume that the mass in gas is a fraction f of the total cluster mass. Compare your prediction with the observed luminosity. What do you deduce about the fraction of mass in gas?

Solution. The thermal bremsstrahlung emissivity is

$$\epsilon_{ff} = 1.4 \times 10^{-27} \text{ erg cm}^{-3}\text{ s}^{-1}\ (T/\text{K})^{1/2}(n/\text{cm}^{-3})^2 \tag{6.140}$$

where for simplicity we set the Gaunt factor to unity, $n_e = n_i$, and $Z = 1$. Using the number density n and radius R from part (a) gives an emissivity per gram of $\epsilon_{ff}/\rho = 0.020 \text{ erg g}^{-1}\text{ s}^{-1}$, which for a mass $6.4 \times 10^{14}\ M_\odot h^{-1}$ is a total luminosity

$$L_X = 2.6 \times 10^{44} \text{ erg s}^{-1} \left(\frac{f}{0.1}\right)^2\left(\frac{T}{10^8 \text{ K}}\right)^2 h(1+z)^{3/2}. \tag{6.141}$$

The observed bolometric luminosity is 1.2×10^{45} erg s$^{-1}h^{-2}$ which requires $f \approx 20\%$.

(c) **Question.** What relation do you predict between the X-ray luminosity of a cluster and its temperature? In fact, the observed correlation is different. What are some of the ideas to explain the discrepancy? [For example, see Holden et al. (2002, AJ 124, 33) and other papers in the literature. The simple relation was originally derived by Kaiser (1986).]

Solution. The predicted relation is $L_X \propto T^2$. In fact the observed relation is steeper than this, closer to T^3.

3. **Free-free absorption**.

(a) **Question.** Compare the free-free opacity with the Thomson scattering opacity for conditions at the center of the Sun. Which form of opacity do you expect to dominate in very massive stars and in low mass stars (compared to the solar mass)? [Assume that radius R is proportional to mass M for all stars].

Solution. At the center of the Sun, $\rho = 150$ g cm^{-3} and $T = 1.5 \times 10^7$ K. Using the expression for $\kappa_{ff} \propto \rho T^{-7/2}$ from the notes, and setting the composition factors and Gaunt factor to unity, I get $\kappa_{ff} \approx 0.9$ cm^2 g^{-1}. This is the same order of magnitude as the Thomson scattering opacity, $\kappa_{es} = 0.2(1 + X)$ cm^2 g^{-1} (where X is the hydrogen mass fraction). So for the Sun, these opacity sources are comparable to each other.

To estimate the scaling with stellar mass, note that hydrostatic balance (or virial theorem) suggests $k_B T \sim GMm_p/R$ or $T \propto M/R$ or in other words a roughly constant temperature if $M \propto R$. The density $\rho \propto M/R^3 \propto M^{-2}$ decreases with mass. Therefore, we predict that κ_{ff} decreases with mass $\propto M^{-2}$, whereas κ_{es} remains roughly constant, so that electron scattering will dominate for $M > M_\odot$ and free-free absorption for $M < M_\odot$. (This is a very rough argument — given the sensitivity of κ_{ff} to T one might worry that a better estimate of the temperature scaling with mass could change the answer. Our conclusion is however correct and agrees with stellar models.)

You may recall that the Sun is a switching point in mass not only for opacity but also for the type of nuclear burning. Hydrogen burns by the pp-chain for $M \lesssim M_\odot$ and by the CNO cycle for $M \gtrsim M_\odot$. The fact that both opacity and nuclear burning sequence change at $M \approx M_\odot$ is a coincidence.

(b) **Question.** Calculate the scaling of luminosity with mass for two cases: (1) a star in which the opacity is set by Thomson scattering and (2) in which the opacity is from free-free absorption. [Use the radiative diffusion equation, and approximate all derivatives by ratios as we did in class. Concentrate on the scaling, not on the prefactor.] Use your results to plot a prediction for the luminosity-mass relation for main sequence stars.

Solution. This is a classic question in back of the envelope stellar structure. The idea is to write the thermal diffusion equation

$$L = -4\pi r^2 \frac{4acT^3}{3\kappa\rho} \frac{dT}{dr} \tag{6.142}$$

which gives $L \propto RT^4/\kappa\rho \propto M^3/\kappa$ (where we take $\rho \propto M/R^3$ and $T \propto M/R$). For constant opacity, which applies for electron scattering, we find $L \propto M^3$, our prediction for stars more massive than the Sun.

For free-free, $\kappa \propto \rho T^{-7/2} \propto (M/R^3)(M/R)^{-7/2}$ which is $\propto M^{-2}$ if we assume $M \propto R$. Therefore, we predict the steeper dependence $L \propto M^5$ for $M < M_\odot$.

4. **Thermal bremsstrahlung spectrum including self-absorption.** Consider a spherical HII region with radius 0.5 pc, uniform temperature 8000 K, and uniform density $n_e = 1000\ \mathrm{cm}^{-3}$. The distance to the HII region is 500 pc. Assume the gas is pure hydrogen, and is in LTE.

(a) **Question.** Show that for radio observations, $h\nu$ is safely much smaller than $k_B T$. In this case, we are always in the Rayleigh-Jeans part of the spectrum.

Solution. For $T = 8000$ K, the frequency corresponding to $h\nu = k_B T$ is $\sim 10^{15}$ Hz. This is well above radio frequency (~GHz).

(b) **Question**. A useful quantity, especially when observing in the Rayleigh-Jeans part of the spectrum, is the brightness temperature T_B, defined by

$$T_b = \frac{c^2}{2\nu^2 k_B} I_\nu. \tag{6.143}$$

Show that the general solution of the radiative transfer equation can be written

$$T_b = T_b(0)e^{-\tau_\nu} + T(1 - e^{-\tau_\nu}). \tag{6.144}$$

If $T_b(0) = 0$, what are the optically thick and thin limits of equation (6)?

Solution. This is just a straightforward rewrite of the general solution to the radiative transfer equation, using the fact that $I_\nu \propto T_b$ and $B_\nu \propto T$ in the Rayleigh-Jeans part of the spectrum. The limits are $T_b \to T$ for $\tau \gg 1$, and $T_b \approx (1 - \tau)T_b(0) + \tau T$ for $\tau \ll 1$.

(c) **Question.** Calculate the optical depth to free-free absorption across the HII region. Remember that $h\nu \ll k_B T$ allowing you to expand the stimulated emission correction factor. Above what frequency does the HII region become optically thin?

Solution. From the notes, the free-free absorption coefficient in the Rayleigh Jeans part of the spectrum is $\alpha_\nu^{ff} = 0.018\ \mathrm{cm}^{-1}\ (T/\mathrm{K})^{-3/2} Z^2 (n_e n_i/\mathrm{cm}^{-6})(\nu/Hz)^{-2}$. The optical depth is $L\alpha_\nu$ where L is the size of the region, here 0.5 pc. This gives unity optical depth for $\nu = \nu_0 = 0.2$ GHz.

(d) **Question.** Calculate the radio spectrum you expect, and plot F_ν against ν. For F_ν, use units of Janskys (where 1 Jy $= 10^{-23}$ erg cm^{-2} s^{-1} Hz^{-1}). Plot also the brightness temperature as a function of frequency.

Solution. The rough picture is as follows. For $\nu < \nu_0$, the HII region is optically thick, the free-free emission is self-absorbed. The spectrum will then be a blackbody in the Rayleigh-Jeans limit $I \propto \nu^2$. For $\nu > \nu_0$ the spectrum will be that of optically thin free-free emission, that is a roughly flat spectrum in frequency (up to an exponential cutoff at $h\nu \approx k_B T$ but this is well out of the radio band).

To calculate this in detail, you can use the result from HW1 question 1(a) which tells you the flux at the surface of a sphere as a function of τ, which you know from part (c).

9. Problem Set 3 with solutions

Inverse Compton spectrum for single scattering: Monte Carlo calculation.

Question. A simple way to calculate the spectrum for inverse Compton from a single electron is to use a Monte Carlo numerical method. Assume that all of the "seed" photons have the same energy. First, choose a random photon direction in the lab frame. Make sure you use the correct probability distribution for $\cos\theta$: note that the incoming flux of photons at angle θ to the direction of the electron's motion is $\propto 1 - \beta\cos\theta$. Calculate the energy of this photon in the electron rest frame. Choose a random scattering angle assuming that the scattering is isotropic and elastic in the rest frame. Transform back into the lab frame to find the final photon energy. Repeat this calculation many times to build up a distribution of final photon energies. Check that this distribution agrees with the analytic result given in RL equation (7.24).

Solution. This part is quite straightforward. The trickiest part is to make sure that you use the right probability distribution when choosing the direction of the incoming photon. If the electron was at rest and the photon distribution was isotropic, then $\mu = \cos\theta$ would be uniformly distributed between -1 and 1. (The probability of being at a particular θ and ϕ is $d\Omega/4\pi = \sin\theta d\theta d\phi = d\phi d\mu$). However, for a moving electron, we need to choose μ proportional to the flux of photons in direction μ, i.e. we need to choose μ from the probability distribution $P(\mu)d\mu = (1 - \beta\mu)d\mu$. One way to do this is to define the variable $y = -(1 - \beta\mu)^2/2\beta$ which gives $dy = d\mu(1 - \beta\mu)$. Choosing y from a uniform distribution between $(1 - \beta)^2/2\beta$ and $(1 + \beta)^2/2\beta$ and then obtaining μ from $\mu = (1 - \sqrt{2\beta y})/\beta$ gives μ distributed in the desired way. (You can read more about this "transformation method" in Numerical Recipes section 7.2).

The procedure is then to first choose a value of $\mu = \cos\theta$ for the incoming photon, then use this to get the energy of the photon in the rest frame $\epsilon'_f = \epsilon'_i = \epsilon_i\gamma(1 - \beta\mu)$ (elastic scattering).

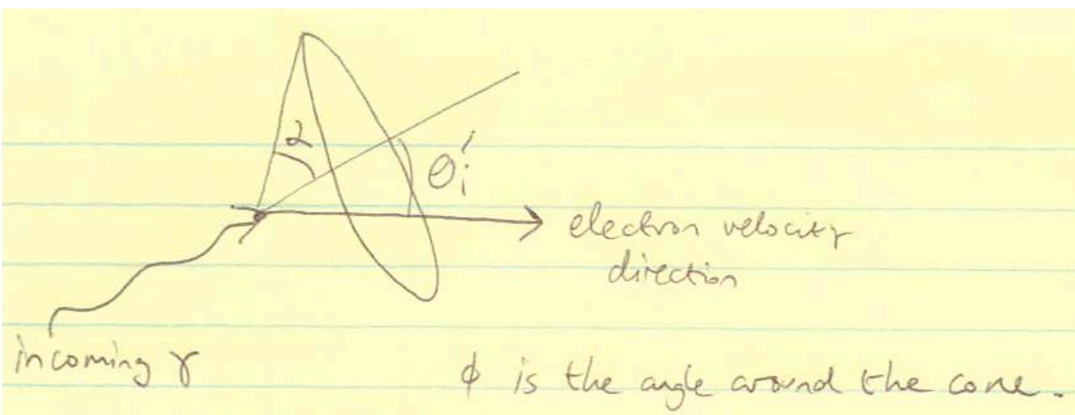
Because we assume in this part that the scattering is isotropic, the outgoing photon direction is uniformly distributed on the sky for any incoming photon direction. So we don't need to keep track of what the incoming photon direction is in the rest frame, we just need to pick an outgoing photon direction uniform on the sky, or in other words choose $\mu' = \cos\theta'_f$ uniformly distributed between -1 and 1. Then the final photon energy in the lab frame is $\epsilon_f = \epsilon'_f\gamma(1 + \beta\mu')$.

You can then compare a histogram of ϵ_f values with the analytic formulas from equation (7.24) of Rybicki and Lightman.

Question. Now repeat this calculation using the angular distribution appropriate for Thomson scattering of an unpolarized beam. Compare your answer with RL equation (7.27) in the limit $\gamma \gg 1$.

Solution. This part is more difficult because we must now keep track of the various angles in the rest frame. For a scattering angle α, the Thompson cross-section is $\propto 1+\cos^2\alpha$. This means that the angle α should be chosen so that $y = \cos\alpha$ has a distribution $P(y)dy = (1+y^2)dy$. One way to do this is the "rejection method" (Numerical Recipes section 7.3): choose two uniformly distribution variables x and y where x is in the range 0 to 2 and y is in the range -1 to 1. If x is smaller than $1+y^2$ then use the value of y, otherwise reject these values and try again.

The outgoing photon lies somewhere on a cone with opening angle α centered on the incoming photon direction. We must also choose a value of the angle around the cone ϕ (uniformly distributed between 0 and 2π).



Then some trigonometry gives

$$\mu' = \cos\theta_f' = \cos\alpha\cos\theta_i' - \sin\theta_i'\sin\alpha\cos\phi. \tag{6.145}$$

Two limits are $\phi = 0$ or π means that the outgoing photon is in the same plane as the incoming photon direction and the electron velocity; in that case $\theta_f' = \alpha \pm \theta_i'$.

Then $\epsilon_f = \epsilon_f'\gamma(1+\beta\mu')$ as before. A histogram of the ϵ_f values agrees well with Rybicki and Lightman's formula in the limit $\gamma \gg 1$.

I've put my IDL code *compton.pro* on the website which you can try. To run it type *compton, gamma, n* where *gamma* is the gamma factor of the electron and n is the number of photons to calculate. The routine *compton_iso* is the same but for isotropic scattering (part a). These routines automatically display a histogram and the Rybicki and Lightman formulas to compare against.

```
pro compton, gamma,n

  beta=sqrt(1d0-1d0/gamma^2)
  print, 'beta=', beta
  seed=120L

  ; choose the incoming angle at random in the lab frame
  ; We need to choose mu=cos theta from the probability distribution
  ; Prob(mu) d mu = (1-beta mu) dmu
  ; (i.e. incident flux is propto 1-beta mu)
  ; Use the substitution method
  y1=(1d0-beta)^2/(2d0*beta)
  y2=(1d0+beta)^2/(2d0*beta)
  y=y1+(y2-y1)*randomu(seed,n)
  cos_theta=(1d0-sqrt(2d0*beta*y))/beta
  ; this is the photon energy in the rest frame
  ef_p=gamma*(1d0-beta*cos_theta)
  ; the photon direction in the rest frame
  cos_theta_p=(cos_theta-beta)/(1d0-beta*cos_theta)

  ; now choose the scattering angle
  ; isotropic
  ; cos_alpha_p=-1d0+2d0*randomu(seed,n)
  ; Thomson  Prob(mu) = 1+mu^2 dmu  where mu=cos alpha
  ; use a rejection method to select from the Prob distribution
  cos_alpha_p=dindgen(n)
  for i=1L,n-1 do begin
     x=-1d0+2d0*randomu(seed)
     y=2d0*randomu(seed)
     while (y ge 1d0+x^2) do begin
        x=-1d0+2d0*randomu(seed)
        y=2d0*randomu(seed)
     endwhile
     cos_alpha_p[i]=x
  endfor
  ; phi is uniformly distributed
  cos_phi_p=cos(randomu(seed,n)*2d0*!dpi)
  ; now calculate the outgoing angle
  sin_theta_p=sqrt(1d0-cos_theta_p^2)
  sin_alpha_p=sqrt(1d0-cos_alpha_p^2)
```

```
  ; outgoing cos theta
  cos_theta_p2=cos_theta_p*cos_alpha_p-sin_alpha_p*sin_theta_p*cos_phi_p

  ; calculate the photon energy in the lab frame
  ef=ef_p*gamma*(1d0+beta*cos_theta_p2)
  ; and plot the distribution
  hist_plot_linear, ef,/normalize

  ; plot the analytic result from RL for isotropic scattering
  y=dindgen(100)*0.01*(1.0-(1.0-beta)/(1.0+beta))+(1.0-beta)/(1.0+beta)
  ef=(1+beta)*y-(1-beta)
  oplot, y,ef/(2.0*beta)
  y=dindgen(100)*0.01*4.0*gamma^2+1.0
  ef=(1+beta)-(1-beta)*y
  oplot, y,ef/(2.0*beta)

  ; plot the analytic result for Thomson scattering when gamma>>1
  y=(dindgen(100)+1.0)*0.01
  ef=2*y*alog(y)+y+1-2*y^2
  oplot, y*4.0*gamma^2,ef/ef[0]


end

pro compton_iso, gamma,n

  beta=sqrt(1d0-1d0/gamma^2)
  print, 'beta=', beta
  seed=120L

  ; choose the incoming angle at random in the lab frame
  y1=(1d0-beta)^2/(2d0*beta)
  y2=(1d0+beta)^2/(2d0*beta)
  y=y1+(y2-y1)*randomu(seed,n)
  cos_theta=(1d0-sqrt(2d0*beta*y))/beta

  ; this is the photon energy in the rest frame
  ef_p=gamma*(1d0-beta*cos_theta)
```

```
  ; now choose the outgoing angle in the rest frame
  cos_theta_p=-1d0+2d0*randomu(seed,n)

  ; calculate the photon energy in the lab frame
  ef=ef_p*gamma*(1d0+beta*cos_theta_p)

  hist_plot_linear, ef,/normalize

  ; plot the analytic result from RL
  y=dindgen(100)*0.01*(1.0-(1.0-beta)/(1.0+beta))+(1.0-beta)/(1.0+beta)
  ef=(1+beta)*y-(1-beta)
  oplot, y,ef/(2.0*beta)
  y=dindgen(100)*0.01*4.0*gamma^2+1.0
  ef=(1+beta)-(1-beta)*y
  oplot, y,ef/(2.0*beta)

end


pro hist_plot_linear, data, min=min_value, max=max_value, binsize=binsize, \
  normalize=normalize, fill=fill, errstyle=errstyle,errcolor=errcolor,\
   errorbars=errorbars, _extra=extra_keywords

if (n_params() ne 1) then message, 'Usage: hist_plot, data'
if (n_elements(data) eq 0) then message, 'data is undefined'

if (n_elements(min_value) eq 0) then min_value=min(data)
if (n_elements(max_value) eq 0) then max_value=max(data)
if (n_elements(binsize) eq 0) then binsize=(max_value-min_value)*0.01
binsize=binsize > ((max_value-min_value)*1.0e-5)

hist=histogram(float(data),binsize=binsize,min=min_value,max=max_value)
hist=[hist,0L]
nhist=n_elements(hist)

histerr=sqrt(float(hist))

histmax=float(max(hist))
;if keyword_set(normalize) then hist=hist/float(n_elements(data))
if keyword_set(normalize) then hist=float(hist)/histmax
```

```
if keyword_set(normalize) then histerr=float(histerr)/histmax

bins=lindgen(nhist)*binsize+min_value

x=fltarr(2*nhist)
x[2*lindgen(nhist)]=bins
x[2*lindgen(nhist)+1]=bins
y=fltarr(2*nhist)
y[2*lindgen(nhist)]=hist
y[2*lindgen(nhist)+1]=hist
y=shift(y,1)

plot, x,y ,_extra=extra_keywords
if keyword_set(errorbars) then oploterror, bins+0.5*binsize, hist, histerr,/nohat \
  psym=3,errstyle=errstyle,errcolor=errcolor

if keyword_set(fill) then polyfill, [x,x[0]],[y,y[0]],_extra=extra_keywords
end
```

10. Problem Set 4 with solutions

1. **Polarization of synchrotron radiation.**

(a) **Question**. Show that a single electron radiates 7 times as much power polarized perpendicular to the projected magnetic field direction than parallel to it.

Solution. Taking the ratio of $P_\perp(\omega)$ and $P_\parallel(\omega)$ using the expressions from class, integrating each of them over frequency, we get

$$\frac{\int dx(F(x)+G(x))}{\int dx(F(x)-G(x))}.$$

The integrals are (using the identities in the notes or in Rybicki and Lightman)

$$\int dx\, F(x) = \Gamma\left(\frac{7}{3}\right)\Gamma\left(\frac{2}{3}\right) \qquad \int dx\, G(x) = \Gamma\left(\frac{4}{3}\right)\Gamma\left(\frac{2}{3}\right),$$

and using the identity $\Gamma(n+1) = n\Gamma(n)$, the ratio simplifies to 7 as required.

(b) **Question**. Show that the degree of polarization $\Pi = (P_\perp - P_\parallel)/(P_\perp + P_\parallel) = (p+1)/(p+7/3)$ for a power law distribution of electrons $N(\gamma)d\gamma = \gamma^{-p}d\gamma$.

Solution. This time we need to integrate over the electron energy distribution. Since $P_\perp - P_\parallel \propto G(x)$ and $P_\perp + P_\parallel \propto F(x)$ with the same constant of proportionality, we just need to integrate

$$\frac{\int G(x)\gamma^{-p}d\gamma}{\int F(x)\gamma^{-p}d\gamma}.$$

As we do several times in the notes, change variables in the integral from γ to $x \propto \gamma^{-2}$. This gives

$$\frac{\int G(x)x^{(p-3)/2}dx}{\int F(x)x^{(p-3)/2}dx}$$

which can be evaluated using the identities giving integrals of $F(x)$ and $G(x)$. The result is $(p+1)/(p+7/3)$ as required. Note that we integrated over γ here but not over frequency, so this result is true frequency by frequency.

2. **Synchrotron energetics.**

(a) **Question.** For a power law distribution of electrons $N(\gamma)d\gamma = \gamma^{-p}d\gamma$ between γ_1 and γ_2, show that the total energy density in the electrons is approximately

$$U_e = \left(\frac{p-1}{p-2}\right)\gamma_1 n_e m_e c^2, \tag{6.146}$$

where n_e is the total number density of electrons and we have assumed that $\gamma_2 \gg \gamma_1$ and $p > 2$.

Solution. Write $N(\gamma) = N(\gamma_1)(\gamma/\gamma_1)^{-p}$. The normalization $N(\gamma_1)$ we can get from

$$n_e = \int_{\gamma_1}^{\gamma_2} N(\gamma_1) \left(\frac{\gamma}{\gamma_1}\right)^{-p} d\gamma$$

which gives

$$N(\gamma)d\gamma = \frac{n_e(p-1)}{\gamma_1} \left(\frac{\gamma}{\gamma_1}\right)^{-p} d\gamma.$$

Then

$$U_e = \int_{\gamma_1}^{\gamma_2} d\gamma\, N(\gamma)\gamma m_e c^2 = n_e m_e c^2 \gamma_1 \frac{p-1}{p-2}.$$

(b) **Question.** Show that for $p = 2.5$, the cooling time due to synchrotron or inverse Compton can be written

$$t_{\rm cool} \approx \frac{10^{10}\ {\rm yrs}}{\sqrt{\gamma_1\gamma_2}} \left(\frac{U}{10^{-10}\ {\rm erg\ cm^{-3}}}\right)^{-1} \tag{6.147}$$

where $U = U_B = B^2/8\pi$ for synchrotron, or $U = U_\gamma$ the photon energy density for inverse Compton.

Solution. The total power radiated is

$$P = \frac{4}{3}\sigma_T c U \int_{\gamma_1}^{\gamma_2} d\gamma\, N(\gamma)\gamma^2,$$

where we assume the electrons are relativistic ($\beta = 1$), and where $U = U_B$ for synchrotron and $U = U_\gamma$ for inverse Compton. The integral gives

$$P = \frac{4}{3} n_e \sigma_T c U \left(\frac{p-1}{3-p}\right) \frac{\gamma_2^3}{\gamma_1} \left(\frac{\gamma_2}{\gamma_1}\right)^{-p} = \frac{4}{3} n_e \sigma_T c U \left(\frac{p-1}{3-p}\right) \gamma_2^{1/2} \gamma_1^{3/2},$$

where we use the fact that $p = 2.5$ which means that the upper limit dominates the integral.

The cooling time is

$$t_{\rm cool} = \frac{U_e}{P} = \frac{3}{4}\frac{m_e c}{\sigma_T}\frac{1}{\sqrt{\gamma_1\gamma_2}}\frac{1}{U} \approx \frac{10^{10}\ {\rm yrs}}{\sqrt{\gamma_1\gamma_2}} \left(\frac{U}{10^{-10}\ {\rm erg\ cm^{-3}}}\right)^{-1}.$$

(c) **Question.** Calculate the cooling time for electrons due to inverse Compton scattering of CMB photons. Show that for large enough redshift, inverse Compton cooling from

CMB photons is significant even if the electrons are non-relativistic (take $T_{CMB} \propto 1+z$, and age of universe $t \propto (1+z)^{-3/2}$).

Solution. First calculate the energy density in CMB photons. The CMB temperature is $T_\gamma = T_0(1+z)$ where $T_0 = 2.7$ K is the temperature today. The energy density is $U_\gamma = aT_\gamma^4 = 4.0 \times 10^{-13}$ erg cm^{-3} $(1+z)^4$.

For non-relativistic electrons, the energy density is $U_e = (3/2)n_e k_B T_e$ and the inverse Compton power is $P = (4k_BT_e/m_ec^2)n_e\sigma_T c U_\gamma$, giving a cooling time

$$t_{\rm cool} = \frac{U_e}{P} = \frac{3}{8}\frac{m_e c}{\sigma_T}\frac{1}{U_\gamma}.$$

Take the age of the universe at redshift z as $t_{\rm age} = t_0(1+z)^{-3/2}$ where $t_0 \approx 13$ Gyrs is the current age.

Then setting $t_{\rm cool} = t_{\rm age}$ gives

$$1+z = \left[\frac{3}{8}\frac{m_e c}{\sigma_T t_0 a T_0^4}\right]^{2/5}$$

which gives $t_{\rm cool} = t_{\rm age}$ for $z \approx 5$. At higher redshift the cooling time becomes shorter than the age of the universe, implying that this is a significant source of cooling for hot electrons. At lower redshift, the cooling time becomes much longer than the timescale on which the universe is evolving.

(d) **Question.** An object is observed to have a total synchrotron luminosity L. Show that the total energy of the system is minimized when $U_e \approx U_B$, i.e. the electrons and magnetic field are close to equipartition.

Solution. The synchrotron luminosity is

$$L = \frac{4}{3}n_e\sigma_T c\langle\gamma^2\rangle U_B$$

where $\langle\gamma^2\rangle$ is an average of γ^2 over the electron distribution. If we write $U_e^2 \propto \langle\gamma^2\rangle$ and $U_B = U_{\rm tot} - U_e$ then $L \propto U_e^2(U_{\rm tot} - U_e)$ which is a maximum for $U_e = (2/3)U_{\rm tot}$, i.e. approximately equipartition. The result then follows because if we moved U_e away from equipartition, a greater $U_{\rm tot}$ would be required to match the observed luminosity.

(e) **Question.** The radio lobes of the radio galaxy Cyg A are roughly 50 kpc across and have a total luminosity $\approx 10^{45}$ erg s^{-1} observed at radio frequencies of several GHz. Assuming a typical magnetic field $B \approx 10^{-4}$ G what else can you deduce about the physical conditions of the plasma? What is the cooling time of the electrons?

Solution. Since we know the magnetic field, the first quantity we can calculate is the γ factor of electrons that radiate at several GHz in a 10^{-4} G magnetic field. The non-relativistic cyclotron frequency is $f_c = eB/2\pi m_e c = 280$ Hz $(B/10^{-4}$ G$)$. We need to boost this by a factor of 10^7 to get it to a few GHz, which implies that $\gamma \sim \sqrt{10^7} \sim 3000$.

With this γ factor, the power radiated per electron is $(4/3)\gamma^2\sigma_T c U_B \approx 10^{-16}$ erg s^{-1} which implies that we're looking at $\approx 10^{61}$ electrons. The number density is $10^{61}/(50\text{ kpc})^3 \approx 10^{-8}$ cm^{-3}.

The energy density in the magnetic field is $B^2/8\pi = 4\times10^{-10}$ erg cm^{-3}. Using the values of n_e and γ that we inferred already, the electron energy density is $n_e\gamma m_e c^2 = 3\times 10^{-11}$ erg cm^{-3}, about an order of magnitude smaller than U_B.

The total energy in the electrons is $\sim (50\text{ kpc})^2(3\times10^{-11}\text{ erg cm}^{-3}) \approx 10^{59}$ erg. Therefore the cooling time is $\sim 10^{59}$ erg$/10^{45}$ erg s^{-1} $\sim 3\times10^6$ years. The total energy in the magnetic field is ten times larger.

(f) **Question.** Consider a source of synchrotron radiation which is self-absorbed. Write the luminosity as roughly $L = \nu_p L_\nu$, where ν_p is the frequency at which the spectrum peaks. Also take the brightness temperature in the self-absorbed part of the spectrum to be given by $k_B T_b \approx \gamma m_e c^2$. Calculate the value of T_b at which the energy density in the synchrotron photons is equal to U_B (you should find $T_b \approx 10^{12}$ K for $\nu_p = 1$ GHz). What would happen if the brightness temperature exceeded this value (this is known as the "inverse Compton catastrophe"; see e.g. Readhead 1994).

Solution. The energy density in the synchrotron photons is roughly $U_\gamma \approx \nu_p I_{\nu_p}/c \approx 2\nu_p^3 k_B T_b/c^3$. The radiation at ν_p comes from electrons with a γ that satisfies $2\pi\nu_p = \gamma^2 eB/m_e c \approx eB(k_BT_b)^2/m_e^3c^5$. Therefore the magnetic energy density is

$$U_B = \frac{B^2}{8\pi} = \frac{\pi}{2}\left(\frac{m_e^3c^5\nu_p}{e(k_BT_b)^2}\right)^2.$$

Now set $U_B = U_\gamma$ and solve for T_b:

$$T_b = \frac{1}{k_B}\left(\frac{\pi}{4}\frac{m_e^6c^{13}}{e^2\nu_p}\right)^{1/5} = 3\times10^{12}\text{ K }\left(\frac{\nu_p}{1\text{ GHz}}\right)^{-1/5}.$$

The argument is that if the brightness temperature exceeded this value, $U_\gamma > U_B$ which would lead to catastrophic inverse Compton cooling of the electrons by the same photons that they created by synchrotron emission. See for example Readhead (1994) for discussion of whether this limit is actually the correct explanation for the upper limit of observed brightness temperatures.

11. Problem Set 5 with solutions

1. **Spectral lines and curve of growth.**

(a) **Question**. Plot the Voigt profile for different values of natural to Doppler width.

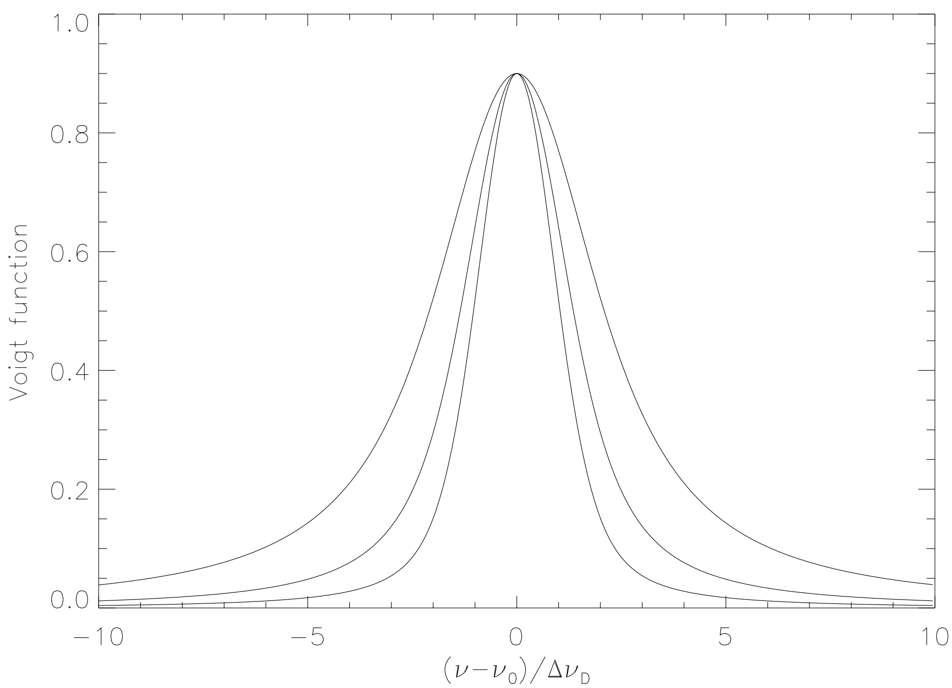
Solution. The Voigt profile is

$$\phi(\nu) = \frac{1}{\sqrt{\pi}\Delta\nu_D} H\left(a, \frac{\nu - \nu_0}{\Delta\nu_D}\right)$$

where $\Delta\nu_D = \nu_0(2k_BT/mc^2)^{1/2} \sim \nu_0\langle v\rangle/c$ is the Doppler width, $a = \Gamma/4\pi\Delta\nu_D$ compares the Doppler and natural linewidths, and

$$H(a,u) = \frac{a}{\pi}\int_{-\infty}^{\infty} \frac{e^{-y^2}dy}{a^2 + (u-y)^2}.$$

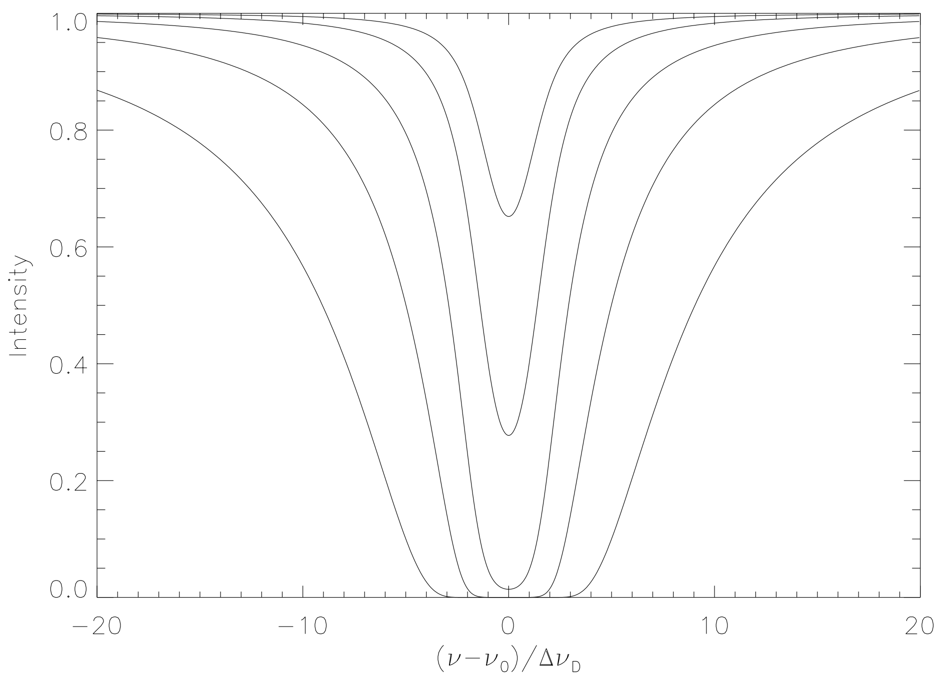
Here is a plot of this function normalized to the same peak value, for $a = 0.5$, 1, and 2.



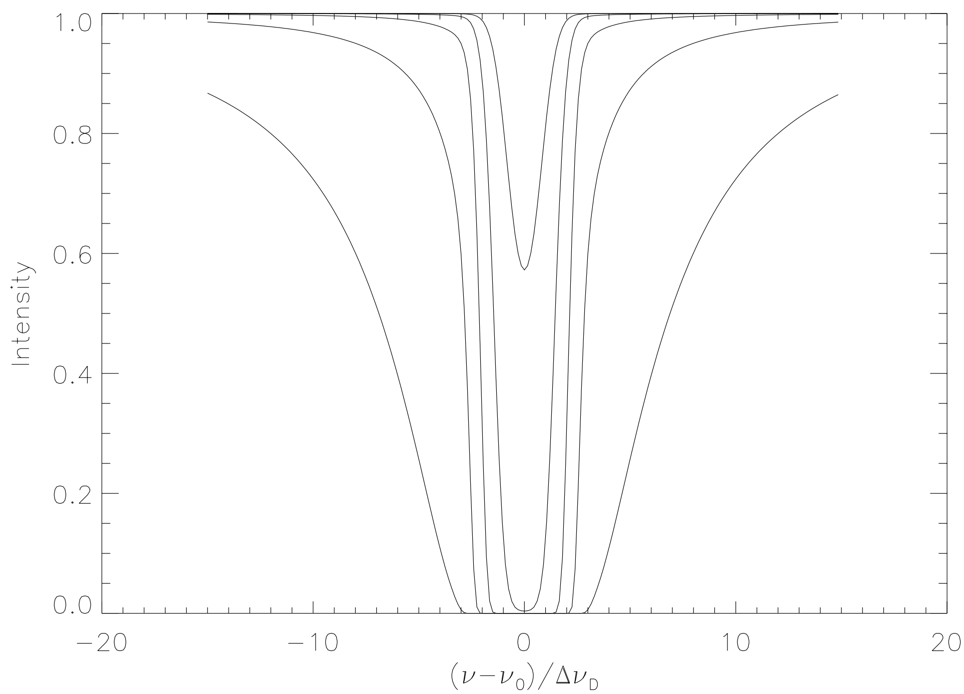
(b) **Question**. By integrating the Voigt profile over frequency, calculate the curve of growth and confirm the scalings given in the notes, EW$\propto N$, $\propto \sqrt{\ln N}$ and $\propto \sqrt{N}$.

Solution. The idea here is to write the observed intensity as $\propto \exp(-\tau_\nu)$ where the optical depth is $\tau_\nu = N\sigma_\nu$ with the column of absorbers N and the shape of the cross-section determined by the Voigt function, $\sigma_\nu \propto H(a,(\nu - \nu_0)/\Delta\nu_D)$.

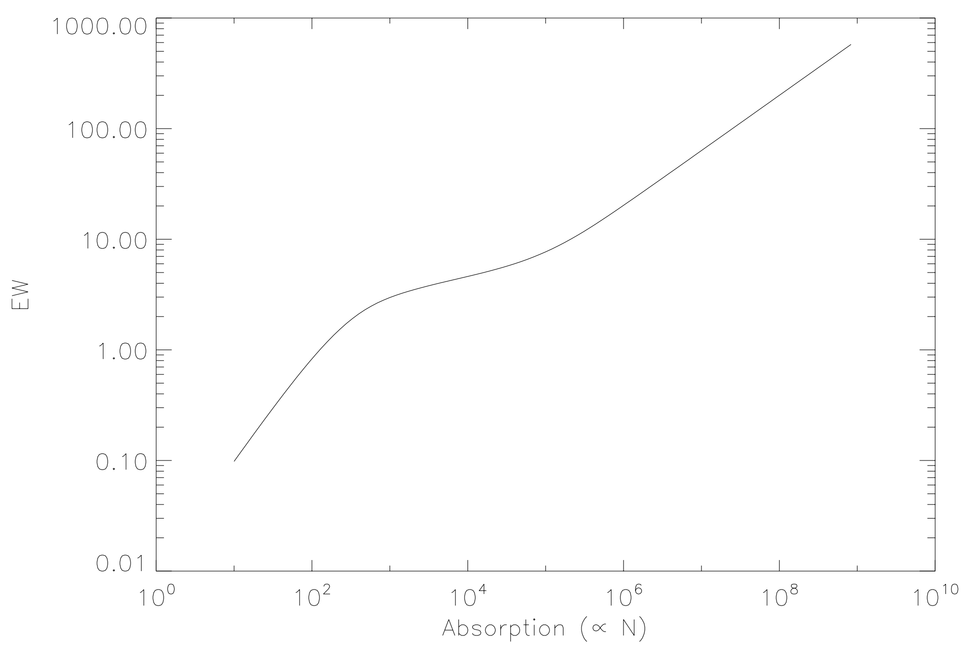
Here is a plot of $e^{-\tau_\nu}$ for increasing column of absorbers by a factor of 3 each time. I've set $a = 1$.



Next I show a similar plot but now for $a = 0.01$ (Doppler width much greater than natural width) with steps of ten in the absorbing column. You can see here the different regimes - initial linear growth as the Gaussian profile becomes deeper, once it saturates the total area doesn't change too much (logarithmic part) until the damping wings start to absorb. You will only see the logarithmic part for a less than one, otherwise there is just a transition from the linear to square root scaling.



Finally, the next plot shows the curve of growth as a function of absorption column (arbitrary units), obtained by integrating the area under the continuum in the previous plot. You can clearly see the linear, almost flat, and square root scalings.



2. **Radiative transitions**

Question. Using Fermi's Golden Rule as discussed in class, show that the $2p \to 1s$ transition rate for hydrogen is

$$A_{21} = \left(\frac{2}{3}\right)^8 \alpha^5 \frac{m_e c^2}{\hbar}.$$

Solution. This calculation involves assembling the various pieces that go into Fermi's Golden Rule that are discussed in the lecture notes in sections 6.4 to 6.6. (This is a standard textbook example which you can find towards the end of non-relativistic quantum mechanics books.) We have

$$A_{21} = \frac{2\pi}{\hbar} \int \frac{4\pi p^2 dp}{h^3} \delta(pc - E_{21}) \left|M_{fi}\right|^2$$

where M_{fi} is the matrix element connecting the initial and final states, $E_{21} = (3/8)\alpha^2 m_e c^2$ is the energy of the transition (the energy difference between the $n = 1$ and $n = 2$ states of hydrogen), and the delta function ensures the outgoing photon takes away the transition energy. We've also assumed that the outgoing photon is isotropically distributed, which is okay because we will sum over all m for the initial $2p$ state - ie. the atom initially has no preferred direction. We can use the delta function to do the integral, which gives (using photon frequency rather than momentum)

$$A_{21} = \frac{2\pi}{\hbar} \frac{4\pi(\hbar\omega)^2}{h^3 c^3} \left|M_{fi}\right|^2.$$

In section 6.4, we show that in the dipole approximation we can write

$$\left|M_{fi}\right|^2 = 2\left(\frac{e}{m_e c}\right)^2 \frac{2\pi\hbar c^2}{\omega} m^2\omega^2 \left|\langle \mathrm{f}|\vec{\epsilon}\cdot\vec{r}|\mathrm{i}\rangle\right|^2$$

where the last factor is the square of the overlap integral between the initial $2p$ and final $1s$ state, and the initial factor of 2 counts the two photon polarizations. The radial part of the overlap integral is given in the notes as

$$\int_0^\infty R^\star_{10}(r) R_{21}(r) r^3 dr = \frac{24}{\sqrt{6}}\left(\frac{2}{3}\right)^5 a_0,$$

which just leaves the angular part. Before we do that let's put everything we have so far together:

$$\frac{2\pi}{\hbar}\frac{4\pi(\hbar\omega)^2}{h^3c^3} 2\left(\frac{e}{m_e c}\right)^2 \frac{2\pi\hbar c^2}{\omega} m^2\omega^2 \left[\frac{24}{\sqrt{6}}\left(\frac{2}{3}\right)^5 a_0\right]^2 \left|M_\Omega\right|^2$$

where M_Ω is the angular part of the overlap integral. Simplifying gives

$$A_{21} = \frac{2^8}{3^6}\alpha^5 \frac{m_e c^2}{\hbar} \left|M_\Omega\right|^2.$$

From the notes, the angular integral is

$$\int d\Omega \; Y_{00}^{\star} \vec{\epsilon} \cdot \vec{e}_r Y_{1,m}$$

which depends on the initial m of the $2p$ state and again from the notes we can write

$$\vec{\epsilon} \cdot \vec{e}_r = \sqrt{\frac{4\pi}{3}} \left(\epsilon_z Y_{1,0} + \frac{-\epsilon_x + i\epsilon_y}{\sqrt{2}} Y_{1,1} + \frac{\epsilon_x + i\epsilon_y}{\sqrt{2}} Y_{1,-1} \right). \tag{6.148}$$

In principle we are dealing with an integral of the product of three Y_{lm}'s. However $Y_{00} = 1/\sqrt{4\pi}$ is a constant, and so pulls out of the integral. The orthogonality of the Y_{lm}'s then means that each choice of m picks out of the terms in equation (6.148). Summing over the three terms gives $(\epsilon_x^2 + \epsilon_y^2 + \epsilon_z^2)/3 = 1/3$. We then divide by another factor of three to average over the m values. The final rate is

$$A_{21} = \left(\frac{2}{3}\right)^8 \alpha^5 \frac{m_e c^2}{\hbar}.$$

3. **HII regions**

(a) **Question**. In the interior of a typical HII region with electron density $n_e \approx 100$ cm^{-3}, what is the lifetime of a proton to recombination, and of a hydrogen atom to photoionization ? Estimate the ionization fraction.

Solution. The recombination rate per proton is

$$n_e \langle \sigma_{fb} v \rangle \approx 10^{-11} \text{ s}^{-1} \left(\frac{n_e}{100 \text{ cm}^{-3}} \right) \left(\frac{\langle \sigma_{fb} v \rangle}{10^{-13} \text{ cm}^3 \text{ s}^{-1}} \right)$$

where we use a typical value for the recombination coefficient from section 6.9 of the notes. The photoionization rate per proton is

$$\frac{\dot{N}_\gamma}{4\pi r^2} \sigma_{bf} \sim 10^{-5} \text{ s}^{-1} \left(\frac{1 \text{ pc}}{r} \right)^2 \left(\frac{\dot{N}_\gamma}{10^{50} \text{ s}^{-1}} \right) \left(\frac{\sigma_{bf}}{10^{-17} \text{ cm}^2} \right)$$

where we again take a typical value for the photoionization cross-section near threshold and $\dot{N}_\gamma$ is the photon luminosity of the central star.

Therefore we estimate that a proton will recombine once every 10^{11} s $\approx$ 3000 years, whereas a hydrogen atom is ionized about once a day. This tells you that in steady-state there must be many more protons than neutral hydrogen atoms. The neutral fraction must be 1 day divided by 3000 years or $\sim 10^{-6}$. (So the ionization fraction is 1 minus 10^{-6}). This number compares well with values in Table 2.2 of Osterbrock.

This also gives you a way to understand the size of the HII region, since the star produces 10^{50} ionizing photons per second, or 10^{61} ionizing photons in the recombination time of 10^{11} s. Therefore we need 10^{61} protons to be able to keep absorbing the ionizing photons in steady-state. For $n \approx 100$ cm^{-3}, the volume is $10^{61}/100 \sim 10^{59}$ cm^3, or a radius of about 10pc.

We could also calculate the time between collisions. The rate per atom is $n_e\gamma_{kj}$ with $\gamma_{kj} \approx 10^{-7}$ cm^3 s^{-1} $\Omega(j,k)T_4^{-1/2}/g_k$. Therefore an atom undergoes a collision about once a day. As we discussed in class, collisional deexcitation is therefore only important for transitions with small A values $A \sim 10^{-5}$ s^{-1} such as forbidden transitions (see next question).

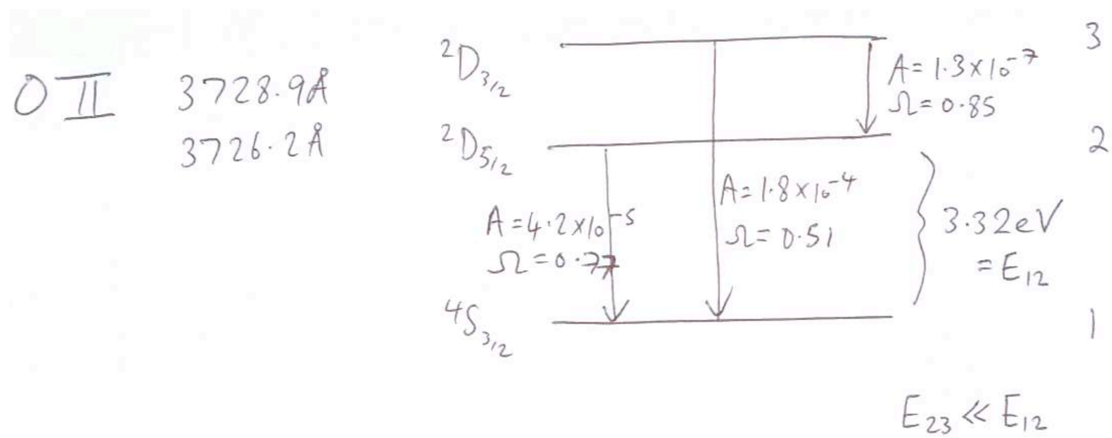
(b) **Question**. Calculate the mean free path of an ionizing photon in neutral hydrogen. What does your result say about the thickness of the transition zone between HII and HI at the edge of an HII region?

Solution. The typical photoionization cross-section of 10^{-17} cm^2 implies that we need a column of hydrogen $N_H \approx 10^{17}$ cm^{-2} for optical depth unity. The mean free path is therefore ≈ 0.03pc $(n_H/\mathrm{cm}^{-3})^{-1}$. This sets the scale for the transition between ionized and neutral gas at the edge of the HII region. The transition is very sharp.

4. **Line diagnostics**

Question. Calculate and plot the intensity ratio of the OII 3728.9Å and 3726.2Å emission lines discussed in class as a function of n_e, with and without taking collisional transitions between the two upper levels into account, and comment on whether it is important to include the transitions between the upper two levels when using the line ratio to determine electron density. (One place to find decay rates and collision strengths for the $^2D_{3/2},^2D_{5/2}$ and $^4S_{3/2}$ levels of OII is Seaton and Osterbrock 1957).

Solution. Here is a summary of the atomic data that comes from Seaton & Osterbrock (1957):



We'll refer to the three levels as 1, 2, and 3. The degeneracies are $g_3 = 4$, $g_2 = 6$, $g_1 = 4$ based on the values of $J = 3/2, 5/2$ and $3/2$ respectively. In class, we derived the expression for the intensity ratio of the two lines ignoring transitions between levels 2 and 3. The result is

$$\frac{I_{31}}{I_{21}} = \frac{g_3 A_{31}\nu_{31}}{g_2 A_{21}\nu_{21}} e^{-E_{23}/k_BT} \left[\frac{1 + A_{21}/n_e\gamma_{21}}{1 + A_{31}/n_e\gamma_{31}}\right]. \tag{6.149}$$

Now define the critical densities $n_{c,2} = A_{21}/\gamma_{21}$ and $n_{c,3} = A_{31}/\gamma_{31}$. The deexcitation rate coefficient is from the notes

$$\gamma_{kj} = 8.6 \times 10^{-6}\ \mathrm{cm}^3\ \mathrm{s}^{-1}\ \frac{\Omega(j,k)}{g_k T^{1/2}}. \tag{6.150}$$

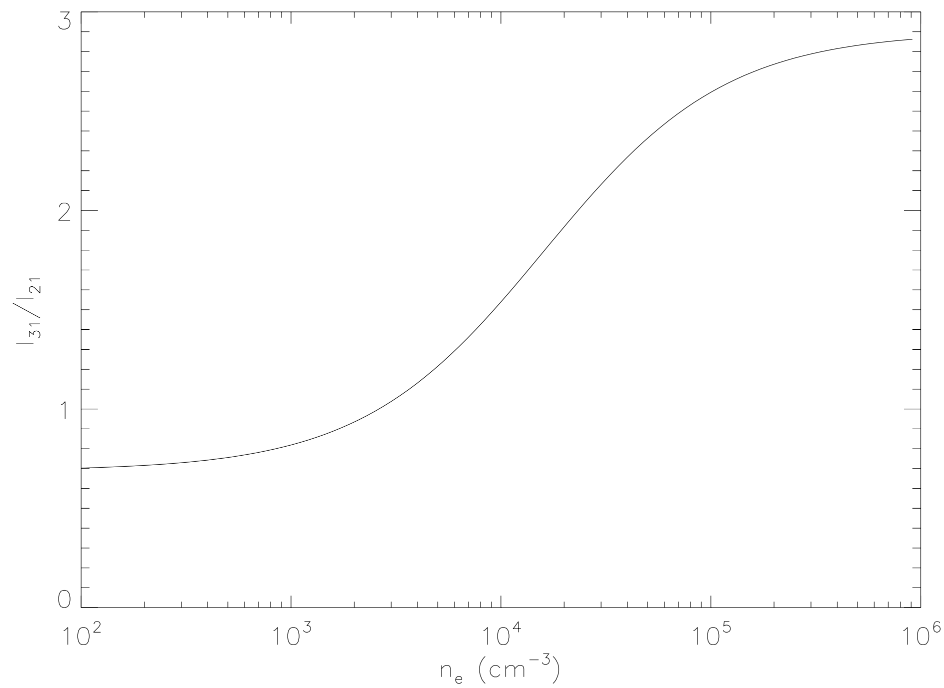
This gives

$$n_{c,2} = 3.8 \times 10^3\ \mathrm{cm}^{-3} T_4^{1/2} \qquad n_{c,3} = 1.6 \times 10^4\ \mathrm{cm}^{-3}\ T_4^{1/2}.$$

To get the prefactor in equation (6.149), note that $E_{23} \ll k_BT$ for typical HII region temperatures, so we can set the exponential to unity, and also $\nu_{31} \approx \nu_{21}$ because $E_{23} \ll E_{12}$. Therefore the prefactor is $g_3A_{31}/g_2A_{21} = 2.9$, giving

$$\frac{I_{31}}{I_{21}} = 2.9 \left[\frac{1 + 3.8 \times 10^3\ \mathrm{cm}^{-3} T_4^{1/2}/n_e}{1 + 1.6 \times 10^4\ \mathrm{cm}^{-3}\ T_4^{1/2}/n_e}\right].$$

The high density limit is 2.9, the low density limit is given by $\Omega(1,3)/\Omega(1,2) = 0.51/0.77 = 0.66$. Here is a plot of I_{31}/I_{21} against n_e for $T_4 = 1$:



Now let's put in collision-induced transitions between levels 2 and 3. We can safely ignore the spontaneous emission from 3 to 2 because $A_{32}/A_{31} \approx 10^{-3}$ so only one in a thousand radiative decays from level 3 goes to level 2. In equilibrium, the level populations are given by

$$n_1(n_e\gamma_{12} + n_e\gamma_{13}) = n_2(n_e\gamma_{21} + A_{21}) + n_3(n_e\gamma_{31} + A_{31}) \qquad (6.151)$$

$$n_2(A_{21} + n_e\gamma_{21} + n_e\gamma_{23}) = n_3(A_{32} + n_e\gamma_{32}) + n_1(n_e\gamma_{12}) \qquad (6.152)$$

$$n_3(A_{31} + A_{32} + n_e\gamma_{32} + n_e\gamma_{31}) = n_1 n_e\gamma_{13} + n_2 n_e\gamma_{23} \qquad (6.153)$$

where we write an equation for each level, with the rate of transitions out of that level on the left, and into that level on the right. Any two of these equations can be solved by eliminating n_1 to get an equation for n_3/n_2 (the third equation is redundant since all transitions are occurring within this set of levels). This gives

$$\frac{n_3}{n_2} = \frac{\gamma_{21}\gamma_{13} + \gamma_{23}\gamma_{12} + \gamma_{23}\gamma_{13} + \gamma_{13}A_{21}/n_e}{\gamma_{12}\gamma_{31} + \gamma_{32}\gamma_{12} + \gamma_{13}\gamma_{32} + \gamma_{12}A_{31}/n_e}. \qquad (6.154)$$

The next step is to realize that

$$\frac{\gamma_{21}\gamma_{13} + \gamma_{23}\gamma_{12} + \gamma_{23}\gamma_{13}}{\gamma_{12}\gamma_{31} + \gamma_{32}\gamma_{12} + \gamma_{13}\gamma_{32}} = \frac{g_3}{g_2} e^{-E_{23}/k_B T}$$

which you can show using the relation between γ_{jk} and γ_{kj} or simply by considering the limit of equation (6.154) in which the collisional terms dominate.

The line intensity ratio can now be written

$$\frac{I_{31}}{I_{21}} = \frac{g_3 A_{31} \nu_{31}}{g_2 A_{21} \nu_{21}} e^{-E_{23}/k_B T} \left[\frac{1 + n_{c,2}/n_e}{1 + n_{c,3}/n_e} \right] \tag{6.155}$$

as before (compare eq. [6.149]), but with critical densities

$$n_{c,2} = \frac{A_{21}}{\gamma_{21}} \left[1 + \frac{\gamma_{23}}{\gamma_{21}} \left(1 + \frac{\gamma_{12}}{\gamma_{13}} \right) \right]^{-1}$$

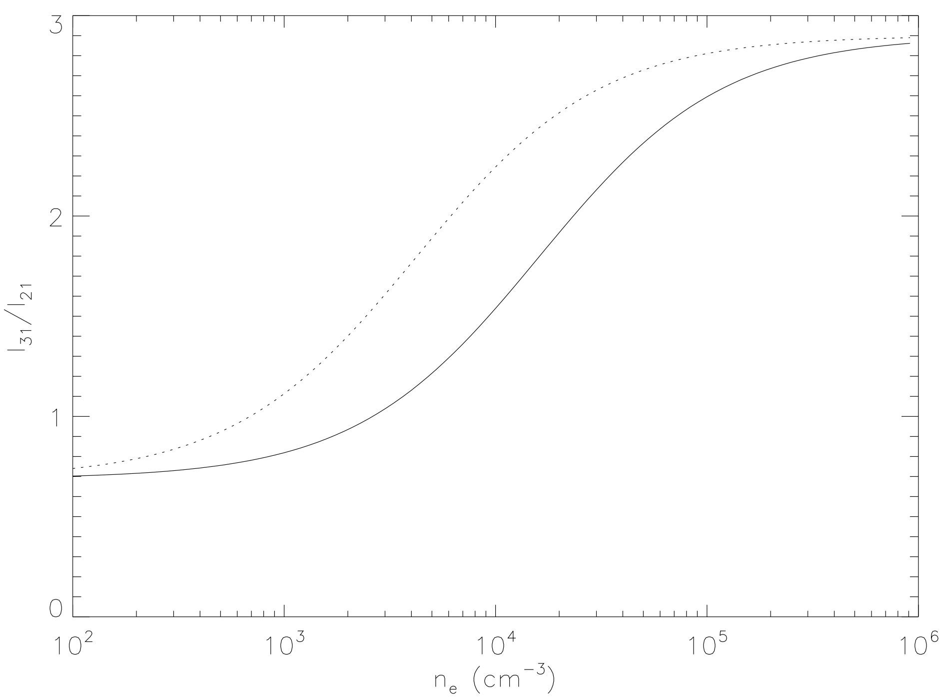
$$n_{c,3} = \frac{A_{31}}{\gamma_{31}} \left[1 + \frac{\gamma_{32}}{\gamma_{31}} \left(1 + \frac{\gamma_{13}}{\gamma_{12}} \right) \right]^{-1} .$$

We just need to calculate the factors in square brackets and modify our previous values for the critical density. Using the relation between γ_{jk} and γ_{kj}, and using the expression for the deexcitation rate constant γ_{kj} in equation (6.150), we find that both these factors are equal to

$$1 + \Omega(3,2) \left[\frac{1}{\Omega(3,1)} + \frac{1}{\Omega(2,1)} \right] = 3.8$$

where we've set $e^{-E_{32}/k_B T} \approx 1$.

Here is the plot (dotted curve is with collisional transitions between 3 and 2; solid curve is no transitions between 3 and 2):



Including the 3 to 2 transitions reduces the inferred density for a given measured line ratio by about a factor of 4. Note that the low and high density limits are unchanged: at high densities we must still have the Boltzmann occupation of levels, and at low densities because we haven't changed the radiative decays: both levels decay straight to the ground state. Another way to look at this is that at a fixed n_e, including the 3 to 2 transitions increases I_{31}/I_{21}. Since it helps the levels reach a thermal population.

Figure 5.8 of Osterbrock's book shows the inverse ratio I_{21}/I_{31} as a function of n_e. They include also excitation of the higher energy ^{2}P levels which can cascade down and populate our levels 1,2 and 3. This shifts the curve to lower densities by about another order of magnitude, so that the transition occurs for n_e between 10^2 and 10^3 cm^{-3}.