PHYS 643 Computational Exercise 2: Steepening

You should email me a write-up of your results and the code that you used to do the calculation and make plots before the class on October 23.

Overview. The goal of this exercise is to write a 1D hydro code and use it to demonstrate the steepening of a sound wave.

Algorithm. The lecture notes on hydrodynamics from the University of Heidelberg give a simple algorithm that you can use to solve the 1D hydro equations (in Chapter 5). First, the fluid equations are written in flux-conservative form, with conserved quantities

$$f_1 = \rho$$

$$f_2 = \rho v$$

(the mass and momentum densities) and it is assumed that $P = c_s^2 \rho$ with constant sound speed c_s. The equations to solve are then

$$\frac{\partial f_1}{\partial t} + \frac{\partial}{\partial x}(v f_1) = 0$$

$$\frac{\partial f_2}{\partial t} + \frac{\partial}{\partial x}(v f_2) = -\frac{\partial P}{\partial x}.$$

These are in flux-conservative form with the pressure gradient acting as a source term for the momentum density f_2. Note that given f_2 and f_1, the velocity at the grid centre can be obtained from the ratio f_2/f_1.

The algorithm has two steps:

1. Use donor-cell advection to update f_1 and f_2. To calculate the velocity at the cell boundaries, you can take an average of the velocity at the cell centres

$$v_{j+\frac{1}{2}} = \frac{1}{2}\left(v_j + v_{j+1}\right).$$

2. Add an additional update to the value of f_2 from step 1 to take into account the source term. You can do this by writing

$$\frac{\partial P}{\partial x} = c_s^2 \frac{\partial \rho}{\partial x}$$

and using a first order difference to calculate the density gradient in terms of the new values of f_1 you found in step 1.

Questions

1. Choose an initial condition that has a sinusoidal variation in density and/or velocity. Check that for small amplitudes, the wave propagates as expected.

2. Do you see steepening at larger wave amplitudes?
3. How large a timestep can you take and still be numerically stable?
4. Do you form a shock in your simulation? What sets its thickness?

Possible extensions

- An interesting extension is to add the energy equation to your code. We need a third quantity
$$f_3 = \rho e_{\rm tot}$$
where e is the specific total energy (sum of kinetic and internal energy). The energy equation is
$$\frac{\partial f_3}{\partial t} + \frac{\partial}{\partial x}\left(v f_3\right) = -\frac{\partial}{\partial x}\left(vP\right).$$
As for momentum, we can solve this by advecting f_3 in step 1 and then updating f_3 using the source term on the right hand side in step 2. The pressure is
$$P = (\gamma - 1)\rho e$$
where
$$e = e_{\rm tot} - \frac{v^2}{2}$$
is the thermal energy.

  With the energy equation included, you will be able to validate the shock jump conditions, e.g. check that the compression factor is $(\gamma+1)/(\gamma-1)$ for a strong shock.
- As a next step, you could modify your code to work in spherical symmetry (adding appropriate r^2 factors), and then model the Sedov-Taylor blast wave.
- Another area to investigate is to use a higher order approximation for the advection step (section 4.3 in the Heidelberg notes) and see how it improves the results.