PHYS 643 Week 6: Inflows and Outflows

Outflows and inflows are important in astrophysics in systems with winds (star, galaxies, planets, accretion disks), jets (e.g. from accreting black holes) and accretion onto a central object (e.g. in compact binaries, star or planet formation, black hole growth). These notes discuss some different examples. We start with the simplest case of a spherically-symmetric wind from or accretion flow onto a point mass. We then consider some more complex examples that break spherical symmetry: stellar wind from a magnetized star and jets.

These flows have some common features that you should look out for. One is that there are conserved quantities that we can use to learn about the flow without necessarily solving for the detailed structure. Another feature is the existence of critical points at which the flow speed equals a wave speed, for example a sonic point at which the flow transitions from subsonic to supersonic.

Bondi accretion and Parker wind

Consider spherically-symmetric, steady, radial flow either onto or away from a point mass. The continuity equation is $\nabla \cdot \rho \mathbf{v} = 0$, which in spherical coordinates is

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2 \rho v\right) = 0,$$

which shows that $r^2 \rho v$ is constant throughout the flow. It is convenient to write this in terms of the mass loss rate or accretion rate

$$\dot{M} = 4\pi r^2 \rho v$$

(units of g s^{-1}). The momentum equation is

$$\rho v \frac{dv}{dr} = -\frac{dP}{dr} - \rho \frac{GM}{r^2}, \tag{1}$$

where we assume that the gravitational acceleration is dominated by the mass of the central star, ie. there is negligible mass in the flow itself.

As usual to simplify things we can make an assumption about the relation between P and ρ so that we don't have to worry about the energy equation. For an isothermal gas, $P = c_s^2 \rho$ with c_s constant, giving

$$v \frac{dv}{dr} = -c_s^2 \frac{d \ln \rho}{dr} - \frac{GM}{r^2}. \tag{2}$$

From the continuity equation,

$$\frac{d \ln \rho}{dr} = -\frac{2}{r} - \frac{d \ln v}{dr}$$

and so, eliminating the density gradient from the momentum equation,

$$\left(v - \frac{c_s^2}{v}\right) \frac{dv}{dr} = \frac{2c_s^2}{r} - \frac{GM}{r^2}.$$

Defining the sonic radius

$$r_s = \frac{GM}{2c_s^2}$$

we can rewrite this as

$$\left(1 - \frac{v^2}{c_s^2}\right)\frac{d\ln v}{d\ln r} = 2\left(\frac{r_s}{r} - 1\right).$$

This shows that if the flow makes a transition from subsonic to supersonic or vice-versa, that must happen at $r = r_s$ in order for the velocity gradient to remain finite. This is known as the *sonic point*.

The velocity and density profiles $v(r)$ and $\rho(r)$ can be obtained by integrating the continuity and momentum equations. (In fact, in the case of an isothermal flow, this integration can be done analytically). Two solutions are possible which go through the sonic point: (i) subsonic flow close to the star $v < c_s$ with v increasing outwards, becoming supersonic at $r > r_s$, or (ii) subsonic flow beyond the sonic point $r > r_s$ and v increasing inwards, becoming supersonic at $r < r_s$. Option (i) corresponds to a wind, discussed by Parker (1958) in the context of the Sun, whereas option (ii) with $v < 0$ corresponds to accretion of mass onto the central star, first discussed by Bondi (1952).

The mass accretion rate or mass loss rate can be written in terms of the sound speed (or equivalently temperature) by evaluating equation (1) at the sonic point where $v = c_s$ and $\rho = \rho_s$:

$$\dot{M} = 4\pi r_s^2 c_s \rho_s = \pi \frac{(GM)^2}{c_s^3}\rho_s.$$

To find ρ_s for a given situation, we map from the boundary as appropriate. Integrating equation (2) gives

$$\frac{1}{2}v^2 + c_s^2 \ln\rho - \frac{GM}{r} = B = \text{constant}.$$

For the accretion case, $v \to 0$ and $GM/r \to 0$ at large distance, so $B = c_s^2 \ln\rho_\infty$. At the sonic point therefore

$$\frac{1}{2}c_s^2 + c_s^2 \ln\rho_s - \frac{GM}{r_s} = c_s^2 \ln\rho_\infty,$$

or $\rho_s = \rho_\infty e^{3/2}$. This gives the *Bondi accretion rate*

$$\dot{M} = \pi e^{3/2}\frac{(GM)^2}{c_s^3}\rho_\infty,$$

the accretion rate onto a point mass[1] placed in a gas with sound speed c_s and density ρ_∞ (assuming the flow is isothermal). For the case of a wind, we apply the boundary conditions at the stellar surface $r = R$ where we take $v = 0$ and $\rho = \rho_\star$, giving

$$B = c_s^2 \ln\rho_\star - \frac{GM}{R}.$$

[1]Hoyle & Lyttleton (1939) derived a similar formula but they considered accretion by a star moving through the interstellar medium (they were interested in whether accretion could power stellar luminosities). Their result is $\dot{M} \sim \rho_\infty (GM)^2/v_\star^3$, the same scalings but with $c_s \to v_\star$. The geometry of the flow is quite different in that case, with the incoming matter being gravitationally-focussed behind the star (in the star's frame) and then falling in.

Therefore

$$\ln\left(\frac{\rho_s}{\rho_\star}\right) = \frac{3}{2} - \frac{GM}{Rc_s^2} = \frac{3}{2} - \frac{2r_s}{R} < 0, \tag{3}$$

and the *mass loss rate in the wind* is

$$\dot{M} = \pi e^{3/2} \frac{(GM)^2}{c_s^3} \rho_\star e^{-GM/Rc_s^2}.$$

Magnetized stellar wind and angular momentum loss

Magnetic fields can play an important role in stellar winds from rotating stars. In particular, through magnetic forces in the azimuthal direction, the magnetic field determines the angular momentum loss rate in the wind, and therefore how quickly the star spins down.

Weber & Davis (1967) made a simple model in which they considered only the equatorial plane and assumed that the wind had "combed out" the magnetic field, so that the magnetic field lies in the equatorial plane

$$\mathbf{B} = B_r(r)\hat{\mathbf{e}}_r + B_\phi(r)\hat{\mathbf{e}}_\phi.$$

Everything is assumed to be axisymmetric and so only depends on r, and steady. Because the magnetic field has to be divergence free,

$$\nabla \cdot \mathbf{B} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 B_r\right) = 0,$$

$r^2 B_r$ must be constant, i.e. $B_r \propto 1/r^2$. The velocity is

$$\mathbf{v} = v_r(r)\hat{\mathbf{e}}_r + v_\phi(r)\hat{\mathbf{e}}_\phi.$$

As in the Parker wind, mass continuity tells us that $r^2 \rho v_r = \dot{M}/4\pi$ is constant. With $\mathbf{v}$ and $\mathbf{B}$ being in the equatorial plane, and $\partial/\partial\phi = 0$, the only possible non-zero component of the induction equation is

$$\frac{\partial B_\phi}{\partial t} = -c(\nabla \times \mathbf{E})_\phi = -\frac{1}{r}\frac{d}{dr}\left[r\left(v_r B_\phi - v_\phi B_r\right)\right].$$

In a steady state, we must therefore have

$$r\left(v_r B_\phi - v_\phi B_r\right) = \text{constant} = -R^2 \Omega B_r(R) = -r^2 \Omega B_r$$

where Ω is the spin of the star. This tells us that

$$\frac{B_\phi}{B_r} = \frac{v_\phi - \Omega r}{v_r} \tag{4}$$

so that the flow is along the magnetic field lines everywhere in the rotating frame. There is a steady pattern in the rotating frame.

To focus on the angular momentum, let's look at the ϕ component of the momentum equation. This is

$$\rho v_r \frac{1}{r}\frac{d}{dr}\left(r v_\phi\right) = \frac{1}{c}\left(\mathbf{J}\times\mathbf{B}\right)_\phi = \frac{B_r}{4\pi r}\frac{d}{dr}\left(r B_\phi\right),$$

where we use

$$\mathbf{J} = \frac{c}{4\pi}\nabla\times\mathbf{B} = -\frac{c}{4\pi}\frac{1}{r}\frac{d}{dr}\left(r B_\phi\right)\hat{\mathbf{e}}_\theta.$$

But $\rho v_r \propto 1/r^2$ and $B_r \propto 1/r^2$, and so we can integrate this:

$$r v_\phi - \frac{B_r}{4\pi\rho v_r} r B_\phi = \text{constant} = L. \tag{5}$$

We call the constant L because we see from the first term that this is the angular momentum per unit mass. If the magnetic field was not present, rV_ϕ would be constant, but the magnetic torques cause this to change across the flow.

The Alfven velocity $v_A^2 = B_r^2/4\pi\rho$ can be used to define a radial Alfven Mach number

$$M_A = \frac{v_r}{v_A} = \frac{\sqrt{4\pi\rho}\, v_r}{B_r}.$$

Equation (5) becomes

$$r v_\phi - \frac{1}{M_A^2} r v_r \frac{B_\phi}{B_r} = L. \tag{6}$$

We see that since $B_r \propto \rho v_r$, $M_A^2 \propto v_r/B_r \propto 1/\rho$, so the Alfven Mach number increases through the flow.

Together, equations (4) and (6) can be used to solve for v_ϕ:

$$r v_\phi = \frac{L M_A^2 - r^2\Omega}{M_A^2 - 1}.$$

Close to the star, $M_A \ll 1$, $v_\phi \approx r\Omega$ which corresponds to rigid rotation at the stellar spin frequency. Far from the star, $M_A \gg 1$ and $r v_\phi = L$, so the flow has constant angular momentum per unit mass. What's happening is that close to the star the magnetic field is strong enough to keep the fluid moving rigidly with the star; far from the star the magnetic torques are no longer important and so the fluid moves outwards with constant angular momentum. The transition occurs at a particular radius, the *Alfven radius* $r_A = (L/\Omega)^{1/2}$ at which $M_A = 1$.

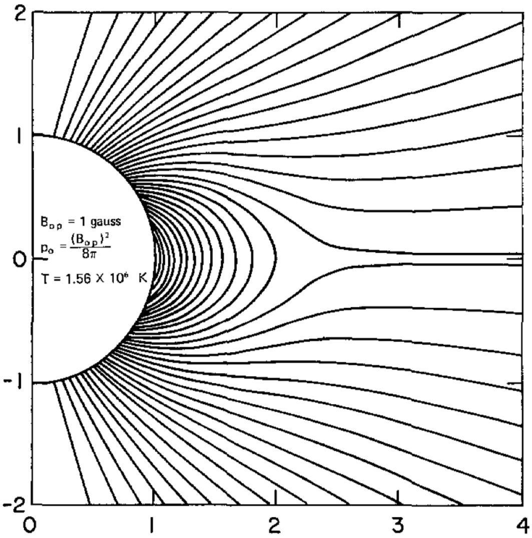
The angular momentum loss in the wind is therefore

$$\dot{M} r_A^2 \Omega$$

which can be much greater than $\dot{M} R_\star^2 \Omega$ which would be the angular momentum loss rate for the Parker wind. The magnetic field keeps the plasma rotating rigidly out to $r = r_A$ which gives a larger "lever arm" for the torque.

In their paper, Weber & Davis go on to look at the radial structure of the wind. As well as the Alfven point (at $\sim 30 R_\odot$ in their model), there is also a sonic point as in the spherical solution (located much closer to the Sun at a few solar radii). (In fact, there are multiple critical points where the flow velocity matches one of the wave speeds as they discuss in detail in the paper).

The plot below is taken from Pneuman & Kopp (1973) which was an early paper doing a multi-D model of the wind, with the stellar field assumed to be a dipole. We see the same ideas apply: a closed zone close to the star where the magnetic torques dominate, and open field lines further out where the field becomes flow-aligned.

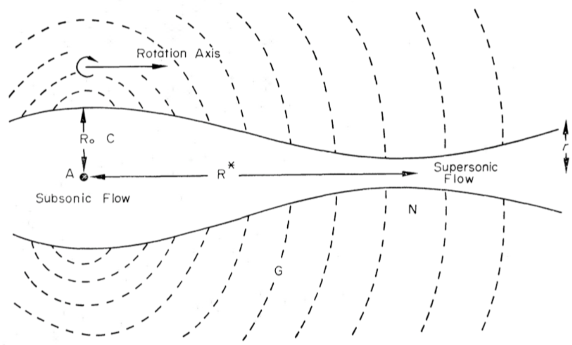


Jets as nozzles

Next, let's discuss an example which is more one-dimensional: the collimation of a jet. Active galaxies in particular show jets that remain remarkably collimated over huge scales (much larger than the size of the host galaxy). An early idea discussed by Blandford & Rees (1974) is that the pressure of the gas around the source could act to collimate the jet and achieve a supersonic outflow.

We showed earlier that for a 1D isentropic flow, the mass flux increases with velocity for subsonic flows (incompressible flow) but decreases with velocity for supersonic flows (compressible flow; see eq. [4] of the Week 4 notes). One place where this comes up is in designing a nozzle through which gas can flow and become supersonic. If a sonic transition is to occur with a steady flow, the product of area and mass flux must be constant. This means that the nozzle must be designed to have a decreasing area at first while the flow is subsonic, but then increase again later so that the flow can continue to accelerate. This kind of nozzle is known as a *de Laval nozzle*.

Blandford & Rees proposed that a similar effect is happening in radio galaxies, except the area of the nozzle is not specified in advance but rather that the confining pressure from the external gas sets the area of the flow. The figure below from their paper shows the overall idea:



For a 1D isentropic flow, the momentum equation can be written

$$\frac{d}{dx}\left(\frac{1}{2}v^2 + \frac{c_s^2}{\gamma - 1}\right) = 0$$

where $c_s^2 = \gamma P/\rho$ is the adiabatic sound speed and $P \propto \rho^\gamma$. The Bernoulli constant

$$B = \frac{1}{2}v^2 + \frac{c_s^2}{\gamma - 1}$$

is a constant of the flow. If there is some pressure and density P_0 and ρ_0 at which $v = 0$ (in the nozzle context, this is the pressure and density in the container; for the jet it's the pressure and density at the base of the flow) then keeping B constant implies

$$\frac{v^2}{c_s^2} = \frac{2}{\gamma - 1}\left[1 - \left(\frac{P}{P_0}\right)^{(\gamma-1)/\gamma}\right]$$

which gives the velocity as a function of pressure. For a large enough pressure drop in the external gas, the flow will make a transition to supersonic flow. Blandford & Rees made essentially this argument, although they used relativistic equations since the flow speed is a significant fraction of c for the radio jets.

References

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