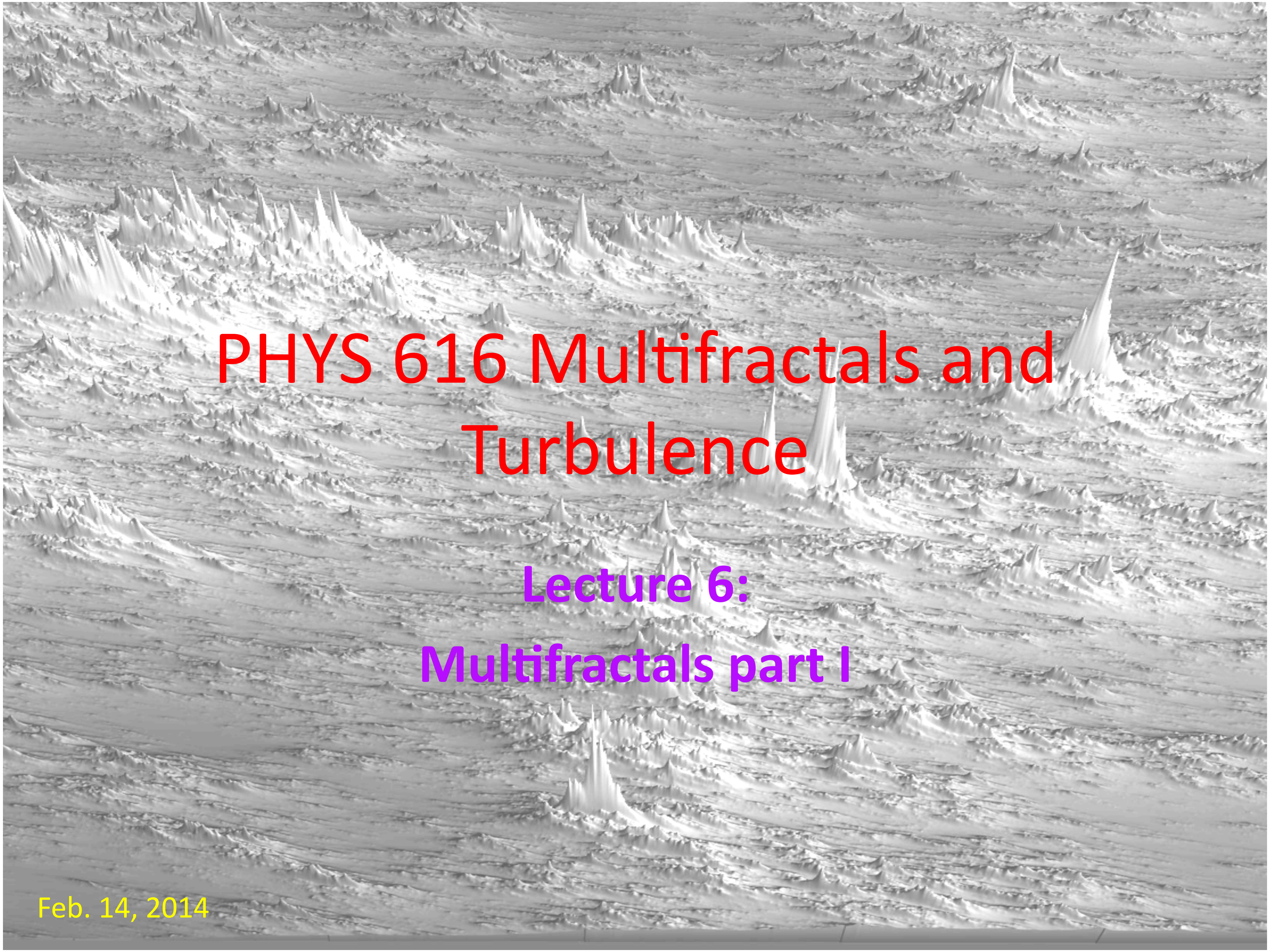


PHYS 616 Multifractals and Turbulence

Lecture 6: Multifractals part I

Feb. 14, 2014

Recap alpha model

Alpha model

The α model is a two state (binomial) process with $\mu\varepsilon$ = either $\lambda_0^{\gamma+}$ or $\lambda_0^{\gamma-}$ where $\gamma_+>0$ corresponds to a boost ($\mu\varepsilon>1$) and γ_- to a decrease ($\mu\varepsilon<1$) . As in the β model, the corresponding probabilities can be written λ_0^{-c} and 1-λ_0^{-c} respectively where $c>0$ is a parameter (it corresponds to the maximum codimension of the process. Formally:

$$\Pr\left(\mu\varepsilon = \lambda_0^{\gamma_+}\right) = \lambda_0^{-c}$$

$$\Pr\left(\mu\varepsilon = \lambda_0^{\gamma_-}\right) = 1 - \lambda_0^{-c}$$

Although the α model apparently involves three parameters (γ_+, γ_-, c), due to the conservation constraint:

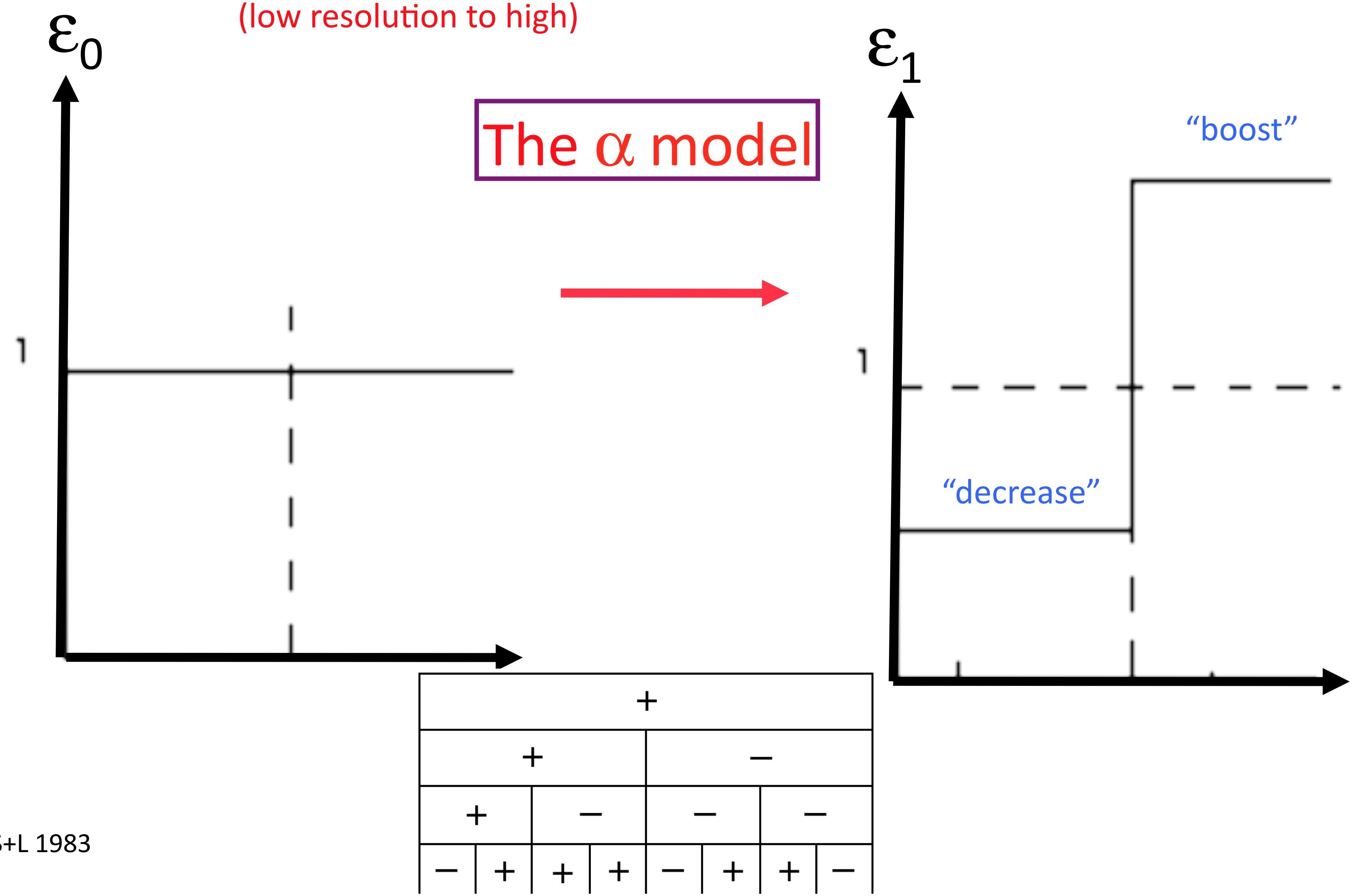
$$\langle \mu\varepsilon \rangle = \lambda_0^{-c}\lambda_0^{\gamma_+} + \left(1 - \lambda_0^{-c}\right)\lambda_0^{\gamma_-} = 1$$

only two can be freely chosen.

We can see that the β model is recovered in the limit $\gamma_+ \longrightarrow c$ which is the same as $\gamma_- \longrightarrow -\infty$

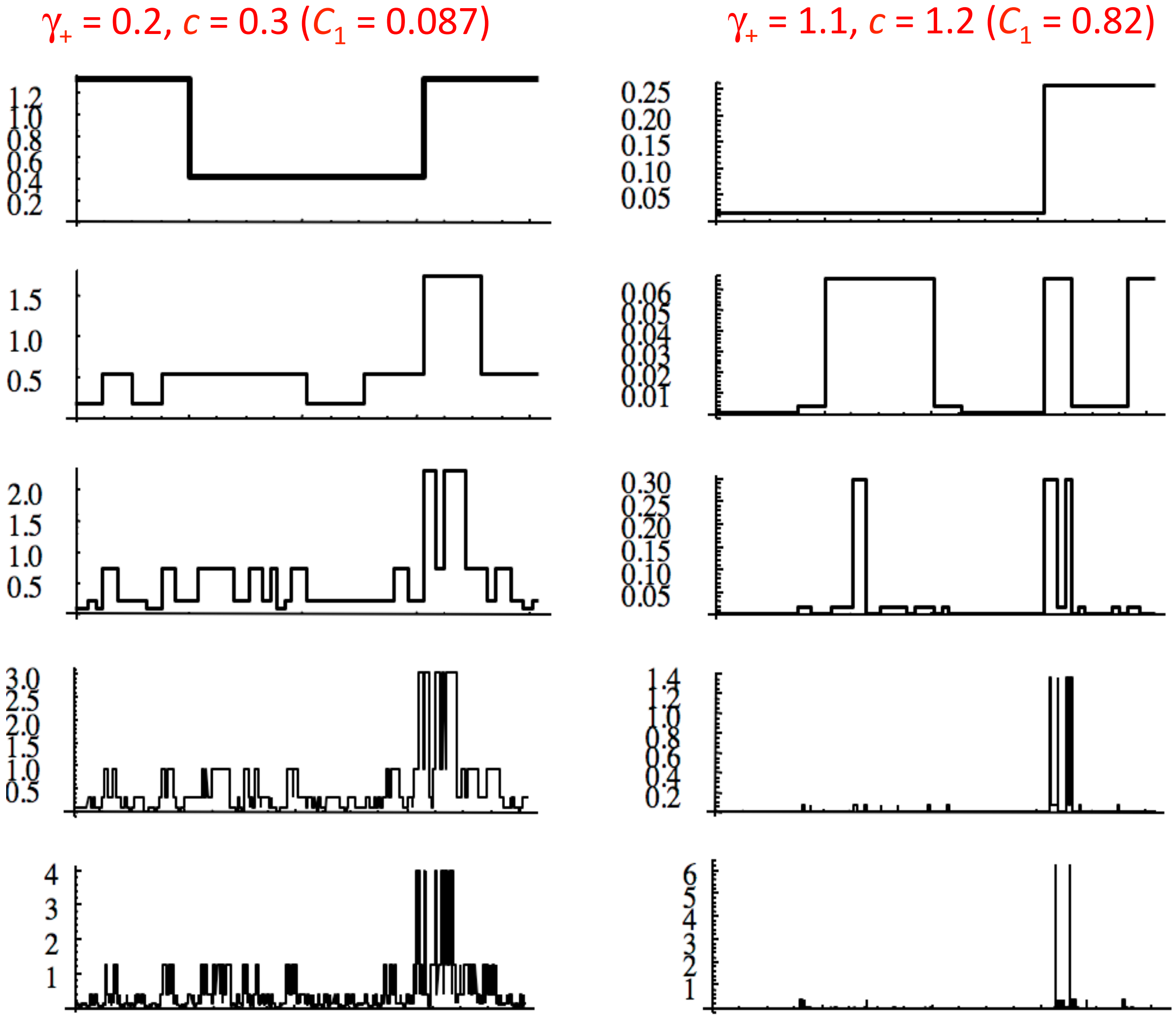
Cascades and Multifractals

Simulations: **multiplicative** introduction of small scale details (low resolution to high)



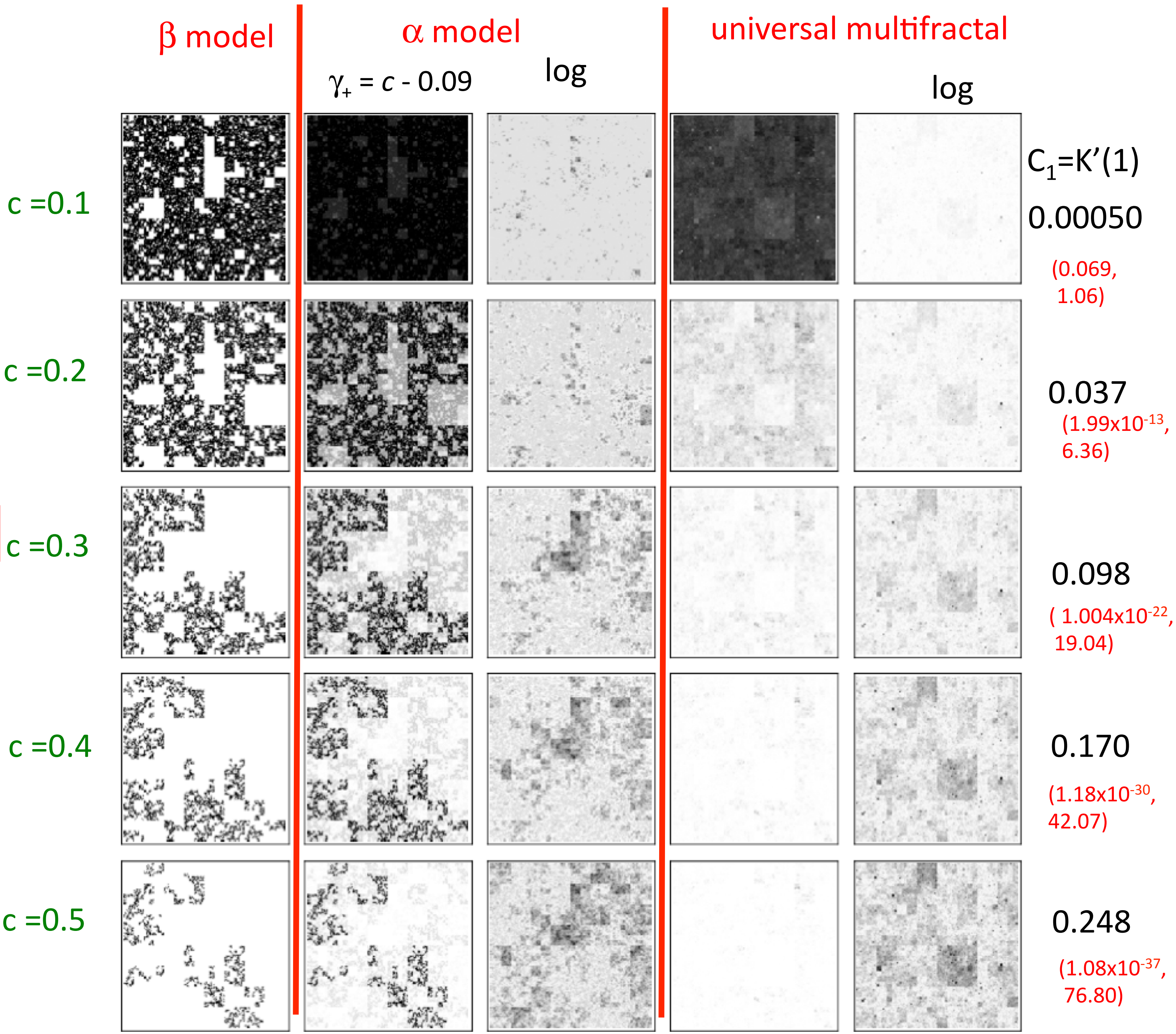
S+L 1983

Alpha model

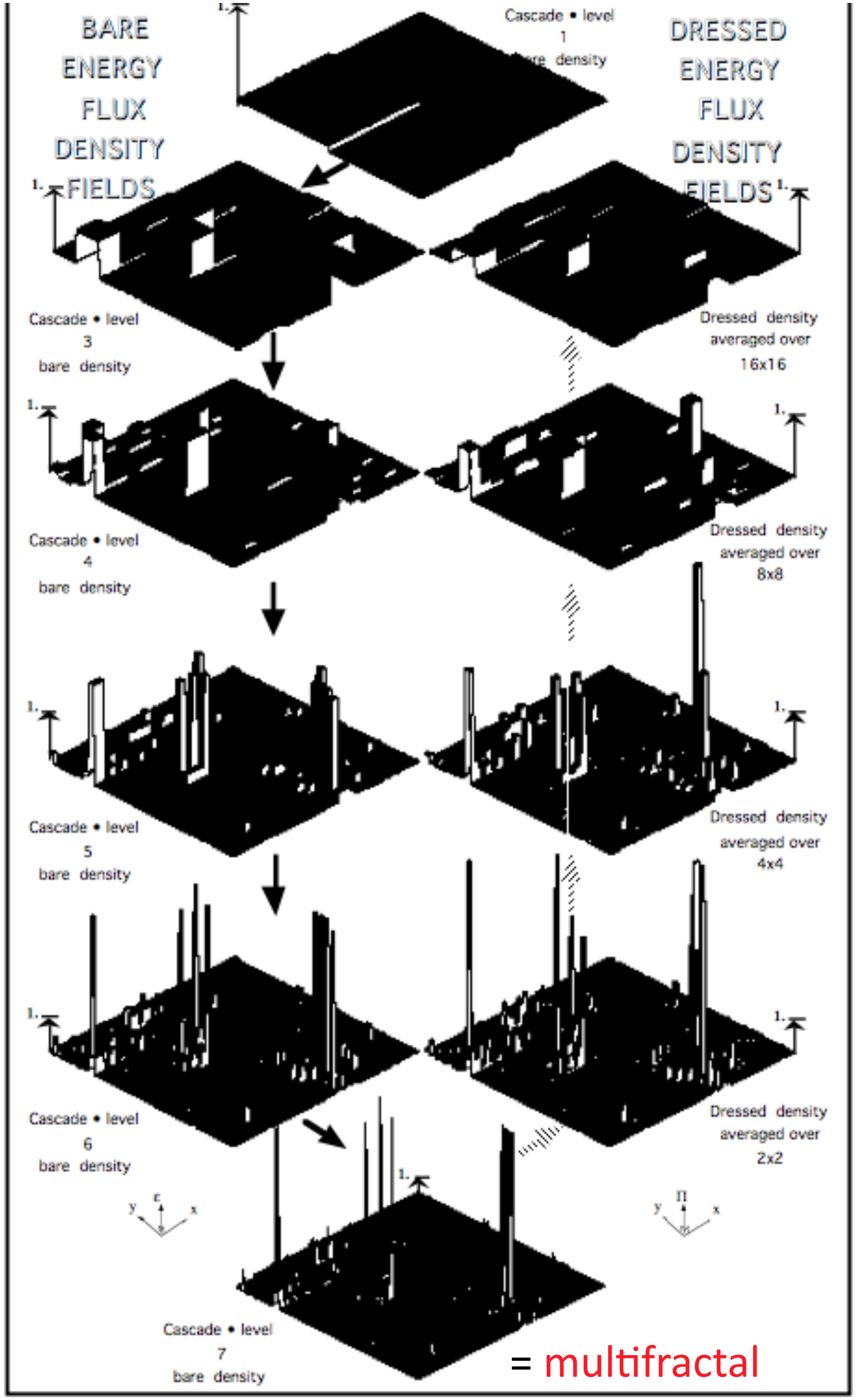


From top to bottom every second cascade step is shown (a factor of λ_0^2) is shown, 10 steps in all, the total range of scales is $2^{10} = 1024$). Notice the changing vertical scales

2-D Alpha model



Multiplicative Cascades: α model



= multifractal

End of recap

p model: microcanonical conservation

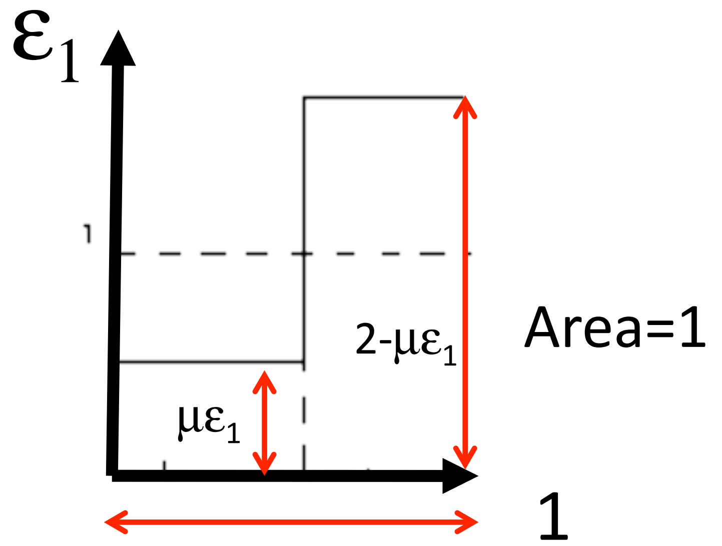
Canonical conservation (ensemble only): $\langle \mu\varepsilon \rangle = 1$ Example: α model

"Microcanonical" conservation = at each cascade step, each realization.

To understand this, take the example of the 1-D α model with λ_0 = 2, it can be transformed into the microcanonical "*p*" model by requiring at each step that the two random multipliers $\mu\varepsilon_1$, $\mu\varepsilon_2$, needed to yield a daughter from a parent eddy have exactly an average of 1 so that $\mu\varepsilon_2 = 2 - \mu\varepsilon_1$. In other words, at each step there is exactly one γ_+ and one γ_- satisfying (with λ_0 = 2). The only choice in the *p* model is thus whether γ_+ is on the right or left hand side of each interval.

Microcanoncial conservation in d dimensions

$$\frac{1}{\lambda_0^d}\sum_{i=1}^{\lambda_0^d}\mu\varepsilon_i = 1$$



Microcanonical versus Canonical conservation

Due to the microcanonical constraint there are subtle correlations between multipliers $\mu\varepsilon$.

However, it turns out that the most important difference between the α and *p* models is in the largest events that they can generate.

Whereas we have pointed out that in the α model, any $\gamma_+ \geq 0$ is possible, in the *p* model, the requirement that the multipliers are ≥ 0 (so that γ_- is real), implies (in 1-D) an upper limit $\gamma_+ \leq 1$.

More generally, the microcanonical constraint is:

$$\frac{1}{\lambda_0^d} \sum_{i=1}^{\lambda_0^d} \mu\varepsilon_i = 1$$

Since $\mu\varepsilon \geq 0$, the most extreme microcanonical model is that in which all the multipliers are zero except for a single one, whose value is thus λ_0^d.

This implies that in *d* dimensions, $\gamma_+ = d$ is the most extreme microcanonical model possible.

General cascade statistics

Characterize the statistics of $\mu\varepsilon$ by K(q): $\langle \mu\varepsilon^q \rangle = \lambda_0^{K(q)}$

An important and trivial consequence of the independence of the cascade steps (and of the corresponding weights $\mu\varepsilon$), is that $K(q)$ is scale invariant, i.e. independent of the number n of steps:

$$\langle \varepsilon_n^q \rangle = \left\langle \prod_{j=1}^{n} \mu\varepsilon_j^q \right\rangle = \prod_{j=1}^{n} \langle \mu\varepsilon_j^q \rangle = \langle \mu\varepsilon^q \rangle^n = \lambda_0^{nK(q)}$$

with respect to the overall scale ratio λ since the cascade started:

$$\lambda = \lambda_0^n$$

We can now write the general expression for the statistical properties after a total scale range λ:

$$\langle \varepsilon_\lambda^q \rangle = \lambda^{K(q)}$$

This is the basic formula for cascade statistics. As indicated above, this specification of the statistics of $\mu\varepsilon$, and (also of ε_λ) via their statistical moments is equivalent to their specification by their probabilities.

Characterizing the statistics of $\mu\varepsilon$: K(q)

We have seen that in order to respect the scale by scale conservation of the mean (its independence of *n*), we require the canonical conservation $\langle \mu\varepsilon \rangle = 1$, this ensures that

$$\langle \varepsilon_n \rangle = 1$$

The overall characterization of the statistical properties is conveniently made with the help of the "moment scaling exponent" *K*(*q*) which can be defined by the statistics of the distribution of random weights $\mu\varepsilon$:

$$K(q) = Log_{\lambda_0} \langle \mu\varepsilon^q \rangle = Log \langle \mu\varepsilon^q \rangle / Log\lambda_0$$

Introducing the (random) cascade "generator" Γ, the logarithm of the multiplier:

$$\Gamma = Log_{\lambda_0}(\mu\varepsilon)$$

K(*q*) is is the (Laplace, base λ_0) second characteristic function ("cumulant generating function") of Γ:

$$K(q) = \log_{\lambda_0} \langle e^{q\Gamma} \rangle$$

Properties of the Moment scaling exponent $K(q)$

1) In order to see the general shape of the $K(q)$ function, we may first note that conservation from one scale to another requires $K(1) = 0$. $\langle \varepsilon_\lambda \rangle = 1 = \lambda^0$

1) In addition, because any positive number raised to the zero power is one, we have <1> = 1, hence $K(0) = 0$.
2)
3) Finally, a basic property of second characteristic functions is that $K(q)$ must be convex, i.e. $K''(q)>0$; this can be shown directly by doubly differentiating $K(q) = \log\langle e^{q\Gamma}\rangle/\log\lambda$.

We therefore conclude that the typical $K(q)$ looks something like the next slide which shows the $K(q)$ for the α model and the universal multifractal models in the fourth and fifth columns of the earlier example. The models are tangent to each other at q =1 because the derivatives at q = 1 were deliberately chosen to be equal to each other. This value: $C_1 = K'(1)$ tangent at the mean

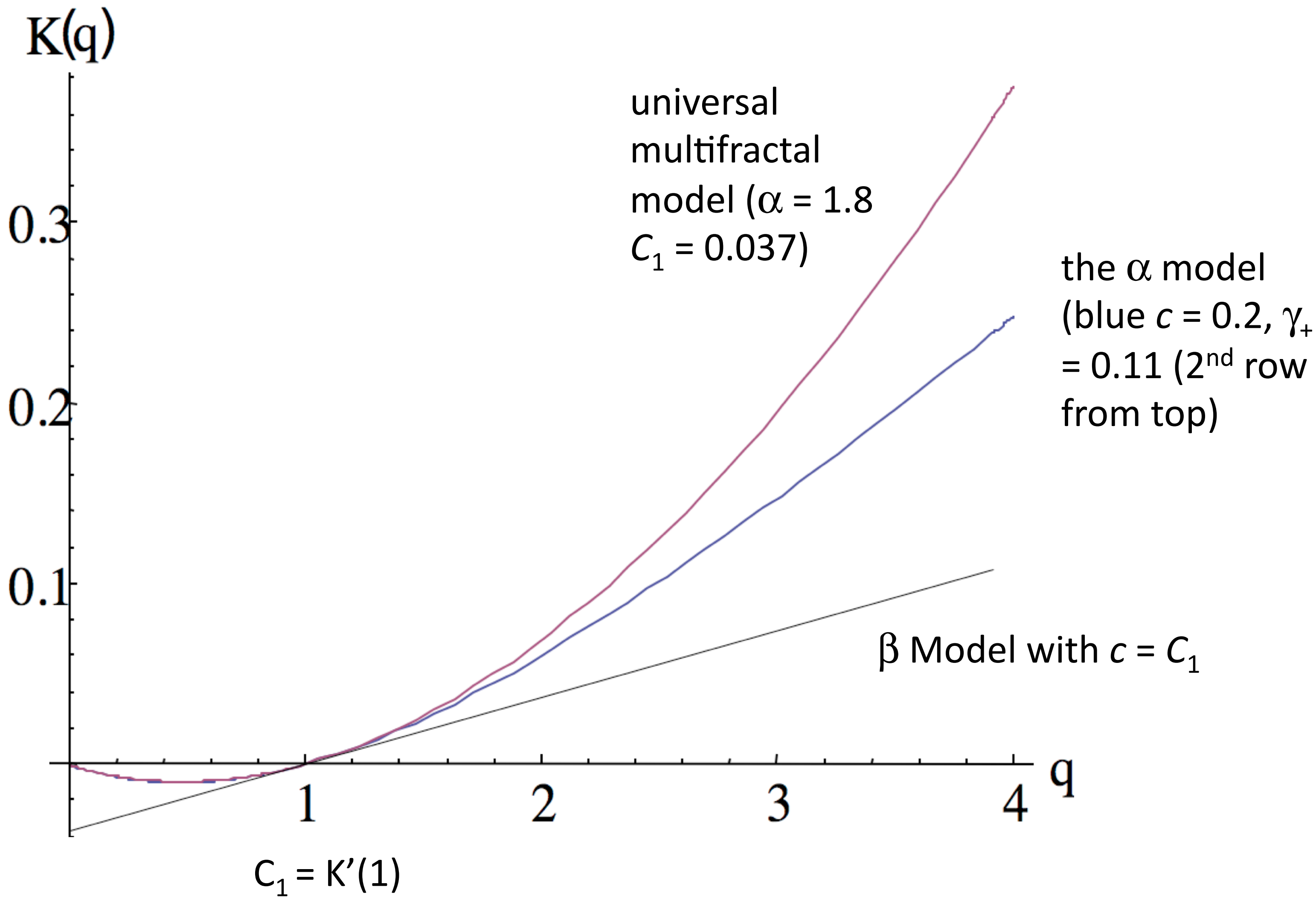
"the codimension of the mean"; it is a basic characterization of the variability near the mean

We can already use this idea to give a "local" (in q space) definition of the "degree of multifractality" α:

$$\alpha = K''(1)/K'(1)$$

Curvature near the mean

Comparison of the K(q) for examples



Properties of α model

We can make some explicit calculations for the a model. For example, we have $K_\alpha(q)$:

$$\left\langle \varepsilon_{\lambda_0}^{q} \right\rangle = \lambda_0^{K_\alpha(q)}$$

Hence:

$$K_\alpha(q) = \log\left[\lambda_0^{q\gamma_+ - c} + \lambda_0^{q\gamma_-}\left(1-\lambda_0^{-c}\right)\right] / \log\lambda_0$$

$$K'_\alpha(q) = \frac{\gamma_+\lambda_0^{q\gamma_+ - c} + \gamma_-\lambda_0^{q\gamma_-}\left(1-\lambda_0^{-c}\right)}{\lambda_0^{q\gamma_+ - c} + \lambda_0^{q\gamma_-}\left(1-\lambda_0^{-c}\right)}$$

Hence:

$$C_1 = K'_\alpha(1) = \gamma_+\lambda_0^{\gamma_+ - c} + \gamma_-\lambda_0^{\gamma_-}\left(1-\lambda_0^{-c}\right)$$

Also, we may note that for the α model, there are low and high *q* asymptotes whose slopes are:

$$\lim_{q\to\pm\infty} K'_\alpha(q) = \gamma_\pm$$

The β model limit

Considering now the special case when $\gamma_+ = c$, , we obtain the results for the β model:

$$K_\beta(q) = C_1(q-1)$$

The β model with corresponding C_1 can be said to provide a "monofractal" (on/off) approximation to the mean (q = 1) behaviour of the cascade, but obviously this approximation is only valid for $q \approx 1$, otherwise it is may be misleading. Note that $\lim_{q\to 0} K_\beta(q) = -C_1$ so that in the β model, C_1 is also the codimension of the nonzero regions, of the "support".

The convexity of *K*(*q*)

It turns out that an important property of *K*(*q*) is that it is convex; *K*''>0, we have already appealed to this property in this chapter, and it is exploited systematically in ch. 5. In this appendix, we derive this convexity property. Consider:

$$\left\langle \varepsilon_\lambda^q \right\rangle = \lambda^{K(q)} = \int \varepsilon_\lambda^q p(\varepsilon_\lambda) d\varepsilon_\lambda$$

where $p(\varepsilon_\lambda)$ is the probability density of ε_λ. Then

$$\frac{\partial \left\langle \varepsilon_\lambda^q \right\rangle}{\partial q} = \left\langle \varepsilon_\lambda^q \log \varepsilon_\lambda \right\rangle; \qquad \frac{\partial^2 \left\langle \varepsilon_\lambda^q \right\rangle}{\partial q^2} = \left\langle \varepsilon_\lambda^q \left(\log \varepsilon_\lambda\right)^2 \right\rangle$$

Differentiating $K(q)\ln\lambda = \ln\left\langle \varepsilon_\lambda^q \right\rangle$, and using the above we obtain:

$$\log\lambda \, \frac{\partial^2 K(q)}{\partial q^2} = \frac{1}{\left\langle \varepsilon_\lambda^q \right\rangle^2}\left(\left\langle \varepsilon_\lambda^q \right\rangle \left\langle \varepsilon_\lambda^q \left(\log \varepsilon_\lambda\right)^2 \right\rangle - \left\langle \varepsilon_\lambda^q \log \varepsilon_\lambda \right\rangle^2 \right)$$

To determine the sign of the term in parentheses, we can apply the Schwartz inequality:

$$\left(\int f^2 \, dx\right)\left(\int g^2 \, dx\right) \geq \left(\int fg \, dx\right)^2$$

with $f = \varepsilon_\lambda^{q/2}\left[p(\varepsilon_\lambda)\right]^{1/2}$, $g = \varepsilon_\lambda^{q/2}\left[p(\varepsilon_\lambda)\right]^{1/2}\log\varepsilon_\lambda$ and $dx = d\varepsilon_\lambda$, we obtain:

$$\left\langle \varepsilon_\lambda^q \right\rangle \left\langle \varepsilon_\lambda^q \left(\log \varepsilon_\lambda\right)^2 \right\rangle \geq \left\langle \varepsilon_\lambda^q \log \varepsilon_\lambda \right\rangle^2$$

from which it follows that

$$\log\lambda \, \frac{\partial^2 K(q)}{\partial q^2} > 0$$

and since $\lambda > 1$, we have *K*''(*q*)>0, *i.e.*, *K*(*q*) is convex (this corresponds to a basic result in probability theory, that second characteristic functions are always convex).

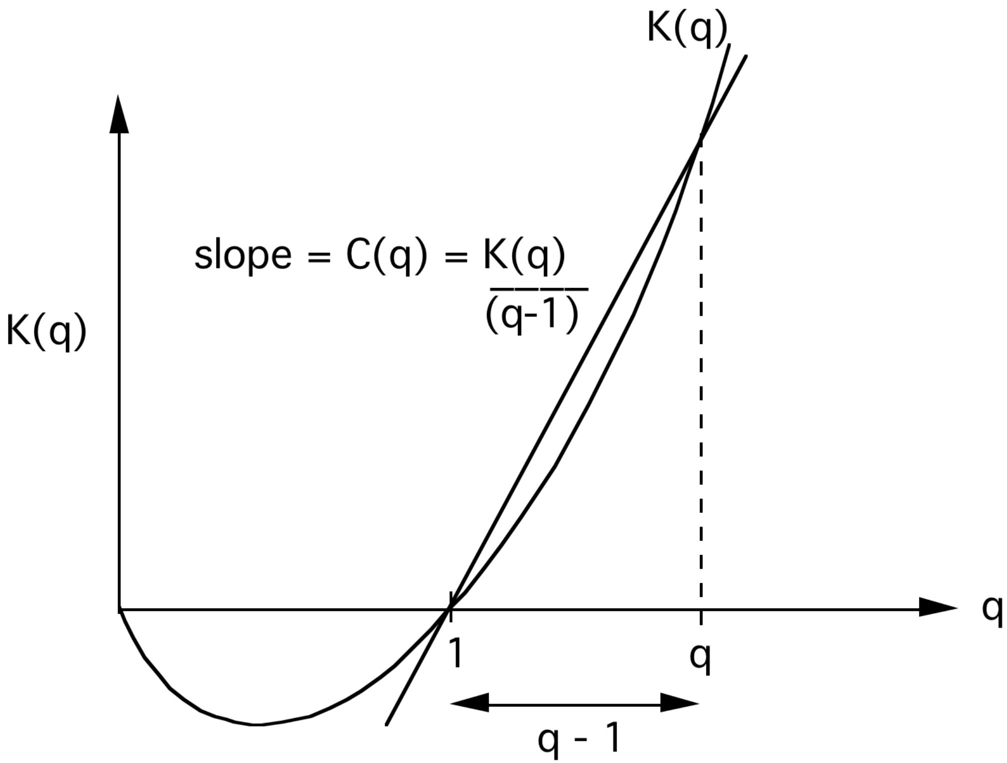
The Dual Codimension Function *C*(*q*)

Functions that will be useful in later analysis are the codimension and dimension functions *C*(*q*), *D*(*q*):

$$C(q)=\frac{K(q)}{q-1};\quad D(q)=d-C(q)$$

By the graphical construction (at right) it is clear that *C*(*q*) is the slope of the chord between the points and (*q*, *K*(*q*)), and by the convexity of *K*(*q*), *C*(*q*) must be an increasing function of *q*: *C'*(*q*) ≥ 0 for all *q*, hence the dimension function *D'*(*q*) ≤ 0. Note that using l'Hopital's rule at *q* =1 we see that:

$$C(1) = K'(1) = C_1$$



Autocorrelations and spectra (1)

Due to the Wiener-Khintchin theorem, it suffices to determine the autocorrelation function $R(\Delta x)$ of the cascade, the spectrum is then obtained as its Fourier transform. We have (in 1-D):

$$E(k) = \int_{-\infty}^{\infty} R(\Delta x) e^{ik\Delta x}\, d\Delta x; \quad R(\Delta x) = \langle \varepsilon(x-\Delta x)\varepsilon(x) \rangle$$

From the Tauberian theorem if $R(\Delta x) \approx \Delta x^{-\delta}$, then $E(k) \approx k^{-\beta}$ with $\beta = 1+\delta$, hence we need only determine δ from the cascade. For the discrete cascade, we can follow the argument from (Yaglom, 1966). Consider a cascade in 1-D developed over n steps, for a total scale ratio λ_0^n. Consider next a lag Δx_m such that:

$$\lambda_0^{-(m+1)} < \Delta x_m < \lambda_0^{-m}$$

so that $\log\Delta x_m \approx -m\log\lambda_0$. The q^{th} order autocorrelation is thus:

$$\left\langle \varepsilon_n^q (x-\Delta x_m) \varepsilon_n^q (x) \right\rangle = \left\langle \prod_{i=1}^{n} \prod_{j=1}^{n} \left(\mu\varepsilon_i \mu\varepsilon_j \right)^q \right\rangle$$

where the index i refers to the multipliers at the point x- Δx_m and the j to those at the point x.

Autocorrelations and spectra (2)

The lag Δx_m is the typical size of the m^{th} level structures, so that if $m \geq n$, the two points will likely share all the multipliers and

$$\left\langle \varepsilon_n^q \left(x - \Delta x_m\right) \varepsilon_n^q \left(x\right) \right\rangle = \left\langle \varepsilon_n^{2q} \left(x\right) \right\rangle = \lambda_0^{nK(2q)}$$

Δx smaller than smallest cascade scale

If we consider now the case $m<n$ then, typically we will find that the multipliers at the points x, $x - \Delta x_m$ will be shared up to level m, but will be different for the levels $>m$. This implies:

$$\left\langle \varepsilon_n^q \left(x - \Delta x_m\right) \varepsilon_n^q \left(x\right) \right\rangle \approx \left\langle \mu\varepsilon^{2q} \right\rangle^m \left(\left\langle \mu\varepsilon^q \right\rangle^2 \right)^{(n-m)} = \lambda_0^{mK(2q)-2(n-m)K(q)}$$

Δx larger than smallest cascade scale

Using $\lambda_0^{-m} \approx \Delta x_m$ and $\lambda_0^n = \lambda$, we obtain: $\left\langle \varepsilon_\lambda^q \left(x - \Delta x\right) \varepsilon_\lambda^q \left(x\right) \right\rangle \approx \Delta x^{-(K(2q)-2K(q))} \lambda^{-2K(q)}$

Autocorrelations and spectra (3)

We have dropped the subscripts "*m*" on the Δx, and indicated the resolution of e directly by the total scale range λ rather than the number of steps *n*. Finally, the usual autocorrelation is obtained by taking $q = 1$; using the scale by scale conservation condition $K(1) = 0$ we obtain the particularly simple result:

$$\left\langle \varepsilon_\lambda (x-\Delta x)\varepsilon_\lambda (x)\right\rangle \approx \Delta x^{-K(2)}; \quad 1 \geq \Delta x > \lambda^{-1}$$

$$\left\langle \varepsilon_\lambda (x-\Delta x)\varepsilon_\lambda (x)\right\rangle \approx \lambda^{K(2)}; \quad \Delta x < \lambda^{-1}$$

The normalized autocorrelation function $R_\lambda(\Delta x)$ with the property $R_\lambda(0) = 1$ can be obtained by normalizing by the value at $\Delta x = 0$ to obtain:

$$R_\lambda(\Delta x) \approx (\lambda \Delta x)^{-K(2)}; \quad 1 \geq \Delta x > \lambda^{-1}$$

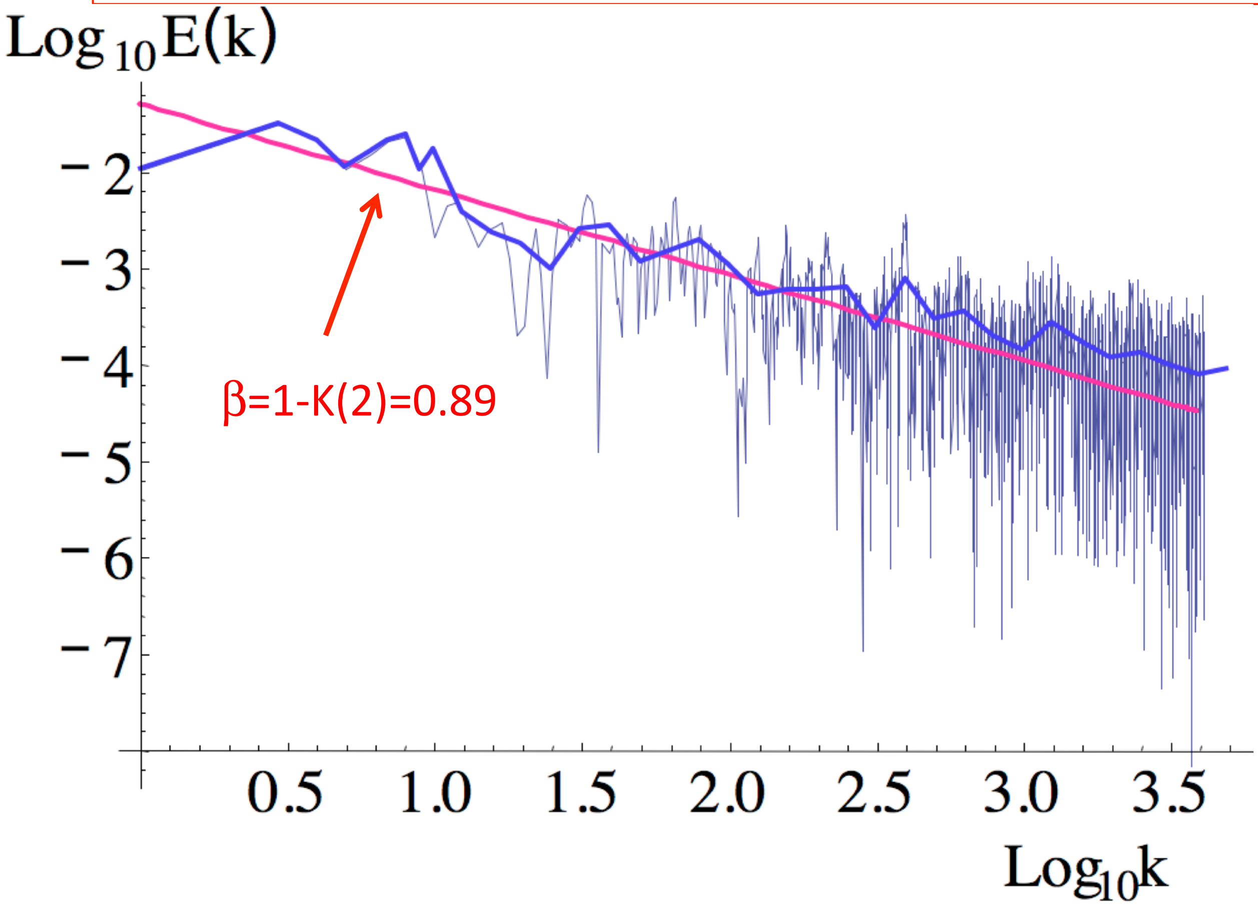
$$R_\lambda(\Delta x) \approx 1; \quad \Delta x < \lambda^{-1}$$

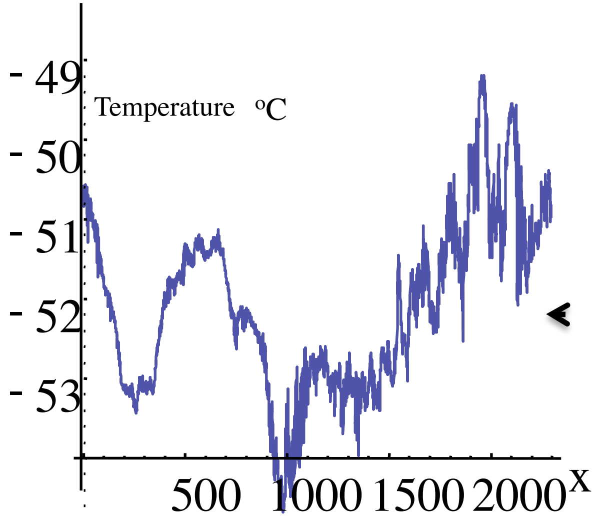
Hence, the spectrum:

$$E(k) = \int_{-\infty}^{\infty} R_\lambda(\Delta x) e^{ik\Delta x}\, d\Delta x \approx \lambda^{K(2)} k^{-\beta}; \quad \beta = 1 - K(2)$$

Since $K(1) = 0$, $K'(1)>0$ and $K''>0$, we have $K(2)>0$ we see that $\beta<1$.

Ex: Spectrum of temperature flux

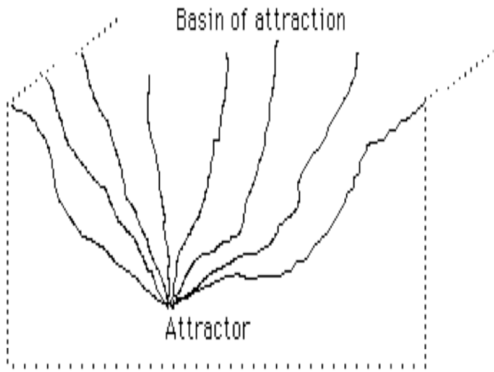




This is the spectrum (thin line) of the fluxes from the aircraft transect shown in at right with its average over logarithmically spaced bins (thick line) along with a reference line with slope -0.89 ($K(2) = 0.11$, the value for $C_1 = K'(1) = 0.06$, $\alpha = K''(1)/K'(1) = 1.8$).

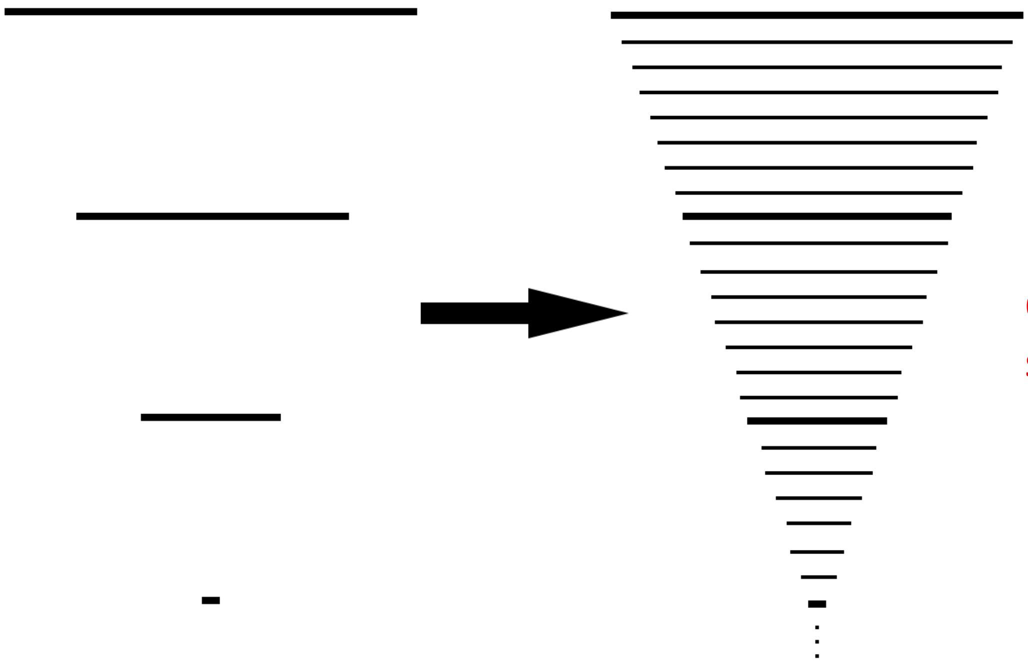
Universality: How many parameters for turbulence?

Answer	Date	References	Explanation	Parameters
1	1941	Kolmogorov (Homogeneous turbulence)	$\Delta v_\lambda \approx \overline{\varepsilon}^{1/3}\lambda^{-1/3}$	H=1/3
2	1962	Kolmogorov-Obukhov, (lognormal model)	$\langle \varepsilon_\lambda^q \rangle = \lambda^{K(q)}$ $K(q) = \frac{\mu}{2}(q^2 - q)$	H, μ
2	1964	Novikov-Stewart, Mandelbrot, Frisch et al, β model	$K(q) = C_1(q-1)$	H, C_1
∞	1974	(Mandelbrot, 1974)	$K(q)$	Any $K(q)$ convex with $K(0)=K(1)=0$



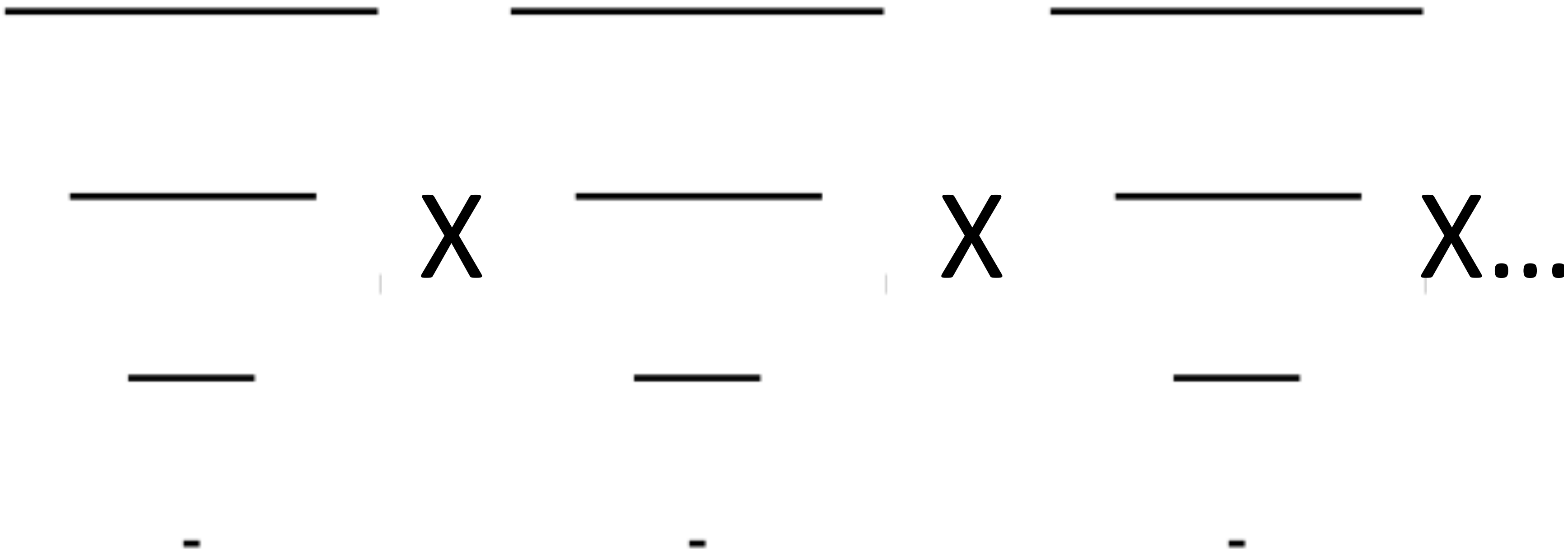
Routes to universality:
1) Densification of scales

Discrete in scale
(ex. β, α models)



Continuous in scale

Routes to universality: 2) “Mixing” of independent discrete cascades



ε_1 ε_2 ε_3

$$\varepsilon = \left(\prod_{i=1}^{N} \frac{\varepsilon_i}{a_N} \right)^{1/b_N}$$

N independent cascades, renormalized by a_N, b_N

For the generators $\varepsilon = e^{\Gamma}$

$$\Gamma = \frac{1}{b_N} \sum_{i=1}^{N} \left(\Gamma_i - \log a_N \right)$$

Normalized, centred sums

Universality in cascades:
a "multiplicative central limit theorem"

The problem is that the cascade requires a scale by scale conservation principle, otherwise there are no well defined small scale cascade limits, and it turns out that this normalization is in contradiction with the normalization required for central limit convergence (specifically the former requires $<\mu\varepsilon> = 1$ whereas the latter requires $<\Delta\Gamma> = 0$ where $\Delta\Gamma = \log\mu\varepsilon$, and due to the convexity of the logarithm function, we have necessarily $< \Delta\Gamma > = <\log\mu\varepsilon><0$ for any probability distribution of $\mu\varepsilon$ which is constrained such that $<\mu\varepsilon>=1$

Universality: case 1, Gaussian generators

If we assume that the $K_1(q)$ for a single cascade step for each of the interacting processes ε_i are analytic at $q = 0$, then we can make a Taylor expansion about the origin:

Single cascade step, ratio λ_1

$$\langle \varepsilon_1^q \rangle = \langle \mu\varepsilon^q \rangle = \lambda_1^{K_1(q)} \qquad K_1(q) = \sum_{i=1}^{\infty} A_i q^i$$

where the A_i are the expansion coefficients (the sum starts at $i = 1$ since $K(0) = 0$). In order to obtain an exactly log-normal cascade we may consider e which is the result of nonlinear (renormalized, multiplicative) interaction of N (generally non- lognormal) statistically independent discrete cascades with a total range of scale λ:

Analytic K(q) special case

$$\varepsilon = \left(\prod_{i=1}^{N} \frac{\varepsilon_i}{a_N} \right)^{1/b_N}$$

$$K_N(q) = \log_\lambda \langle \varepsilon_\lambda^q \rangle = \frac{Nq}{b_N}\left(A_1 - Log_\lambda a_N\right) + A_2 N \left(\frac{q}{b_N}\right)^2 + A_3 N \left(\frac{q}{b_N}\right)^3 + \ldots$$

here, i indexes the N independent cascade processes which interact (are multiplied together) and a_N, b_N are recentring and renormalizing constants which must be chosen so that the limit of many interacting processes ( ) is well defined. In the case of analytic $K(q)$ (which turns out to be exceptional!), we can choose to recentre (a_N) and renormalize (b_N) by:

$$b_N = N^{1/2}; \quad a_N = e^{A_1}$$

thus obtaining:

$$K_\infty(q) = \lim_{N\to\infty} K_N(q) = A_2 q^2$$

Independent of single cascade statistics $K_1(q)$

Gaussian generators (2)

i.e. the higher order terms disappear, thus is a pure quadratic function independent of *N*, it is the moment scaling function of a pure lognormal cascade:

$$\left\langle \varepsilon_\lambda^q \right\rangle = e^{A_2 q^2 \log\lambda} = \lambda^{A_2 q^2}$$

Once the central limit theorem convergence has been achieved ( ), one then considers the small scale limit; here we must normalize the pure log-normal process so that the small scale cascade limit is well behaved, this is easily performed by noting that an unnormalized ε may be normalized by so that so that we obtain:

Normalization

$$\varepsilon \rightarrow \frac{\varepsilon}{\langle\varepsilon\rangle} \qquad K(q) \rightarrow K(q) - qK(1)$$

$$K(q) = C_1\left(q^2 - q\right)$$

where we have used the notation C_1 for the constant A_2 since $K'(1) = C_1$

Universality, general case: Levy generators (1)

Indeed, more generally we must allow for the possibility of nonanalytic single cascade step $K_1(q)$ with the following small q expansion:

$$K_1(q) = A_\alpha q^\alpha + A_1 q + A_2 q^2 + O(q^3)$$

K(q) nonanalytic at origin

if the new nonanalytic term has α<2, then, repeating the above universality argument, with the choice:

$$b_N = N^{1/\alpha}; \quad a_N = e^{A_1}$$

we obtain:

$$K_\infty(q) = A_\alpha q^\alpha; \quad 0 \le \alpha < 2$$

for $\alpha \neq 1$. When $\alpha = 1$, the nonanalytic term must be taken as $q\log q$, see below. This K(q) corresponds to a random generator $\Gamma = \log\varepsilon$ that follows an “extremely asymmetric” Lévy distribution, sufficient for cascade processes.

Levy Generators (2)

The final normalization step needed for small scale convergence (analogous to the log-normal derivation: $K(q)\text{->}K(q)\text{-}qK(1)$) leads to:

$$K'(1) = A_\alpha(\alpha - 1) = C_1$$

$$K''(1) = A_\alpha \alpha(\alpha - 1) = \alpha K'(1)$$

Hence:

$$K(q) = \frac{C_1}{\alpha - 1}\left(q^\alpha - q\right); \quad 0 \le \alpha \le 2$$

(for α= 1, using l'Hôpital's rule for the limit α->1, we have $C_1 q \log q$).

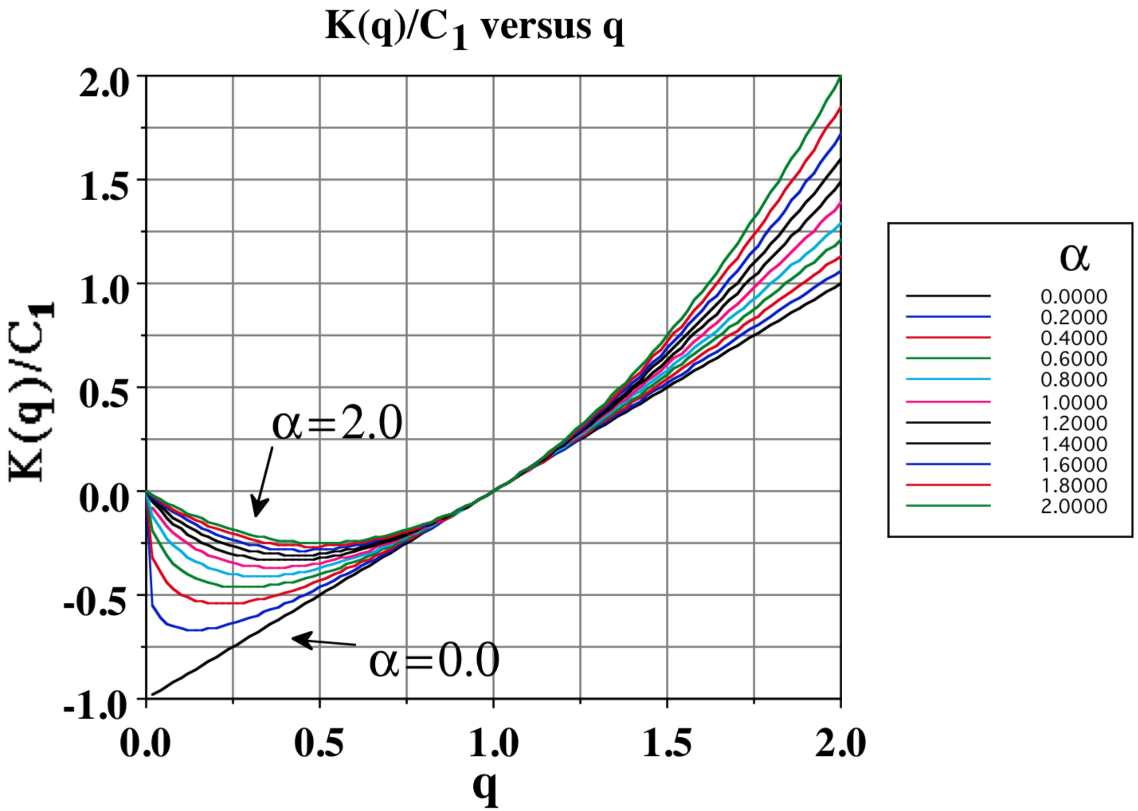
Once again, the constant has been written this way so that $K'(1) = C_1$. We may also check that the local (near the mean) curvature characterization:
$\alpha = K''(1)/K'(1)$

Note that when α<2, and q<0, then ; this is a consequence of the extreme Lévy tail on the negative (but not positive) fluctuations of logε. The possibility (even likelihood) of: $\left\langle \varepsilon_\lambda^q \right\rangle \to \infty$

for q<0 means that extreme caution should be used when analysing negative moments of empirical data.

K(q) for universal multifractals

$$K(q)/C_1 = (q^{\alpha} - q)/(\alpha - 1)$$



Universal $K(q)/C_1$ as a function of q, for different a values from 0 to 2 by increments of $\Delta\alpha = 0.2$.