

An Introduction to Stochastic Multifractal Fields

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Abstract

Fractal and multifractal concepts are introduced with the help of rain and turbulent phenomenology, as well as with the help of very simple toy models. A particular emphasis is placed on defining the adequate formalism to take into account in a straightforward manner the random nature of the fields, as well as its consequences. It is first shown that the notion of (statistical) codimension is much more convenient, and presumably much more fundamental than the notion of dimension, in order to characterize the (random) singularities of the fields. Within this formalism, rather generic features of stochastic multifractal processes are discussed: multifractal universality, finite sample size and second order phase multifractal transition, statistical divergences and first order phase multifractal transition. All of these features are well beyond the scope of deterministic-like multifractal formalism and have enormous practical importance. This is in particular the case for the extremes of the fields at large scale, e.g. the

climatological fluctuations of the geophysical fields. It is also shown that these results can be easily extended into a scaling anisotropic framework.

1 Introduction

Everyone has some rather intuitive notions of the intermittency of precipitation. They are based on a common sense and empirical knowledge: most of the time it does not rain furthermore when it rains its intensity can be extremely variable. Nevertheless, the corresponding adequate mathematical framework had been paradoxically rather elusive for a while and began to be elaborated only during the last 15 years. Indeed, this variability of precipitation, which occurs on a wide range of (space and time) scale and intensity, is well beyond the scope of classical approaches in Geophysics. A symptom of this problem corresponds to the fact that the rain rate r , which is the basic quantity of interest for precipitation, has strong scale dependence. Therefore, it has no self-consistent definition of a function $r(\underline{x}, t)$ of space coordinates \underline{x} and time t , contrary to an hypothesis which has been often taken as granted. Indeed, it should correspond to a density of rain per elementary space-time volume (in general per elementary horizontal surface $d\underline{x}$ and elementary time increment dt) and therefore should have a scale independent limit for small scales. In other words, contrary to classical assumptions the rain rate r does not correspond to a regular (mathematical) measure $dR(x, t)$ with respect to the (Lebesgue) volume measure. More precisely the rain rate cannot be defined as the density $r(\underline{x}, t)$ of the measure $dR(x, t)$ with respect to the Lebesgue measure, i.e. $dR(x, t) = r(x, t) dx dt$. We will show that stochastic multifractal fields offer a very convenient and operational framework to handle such stochastic (multi-) singular measures.

As a consequence, stochastic multifractal fields overcome the strong limitations of traditional approaches to studying extremely variable fields. These approaches are compelled to proceed to drastic scale truncations, transforming partial differential equations (PDE) into ordinary differential equations (ODE), arbitrarily hypothesizing regularity of the fields, and performing ad-hoc and unjustified parameterizations (in particular for non explicit scales). These various manipulations and mutilations violate a fundamental symmetry of nonlinear PDE's: scale

invariance. Even in spite of these (over) simplifying assumptions, the consequences of such choices are ultimately complex and unwieldy numerical codes. Often the relevance of such codes, remain highly questionable: increasingly, they are "tested" by making "intercomparisons" with other models! This is the case for rain field modeling, in particular due to the large difference between the explicit scales of the model and the observation scale.

The alternative approach that is discussed below is on the contrary based on a fundamental property of the nonlinear (e.g. Navier Stokes) equations: scale invariance. Indeed, the simplest way of understanding how extreme variability occurs over a very large range of scales is to suppose that the same type of elementary process acts at each relevant scale (from the large scale to the viscosity scale). At first, this began as a fractal approach, even before the word was coined, with Richardson's celebrated poem on self-similar cascades ([*Richardson*, 1922]). Then it evolved (after 1983), into a multifractal approach. The earliest scale invariant multifractal models, which we will review, are superficially quite simple phenomenological "toy models". Nevertheless, they yield exotic phenomena (exotic compared to conventional smooth mathematical descriptions of the real world...) and have highly nontrivial consequences! For example, as we will see later, simple cascade models already give rise to a fundamental difference between observables and truncated processes, and such a difference is a general property of the wide class of "hard" multifractal processes (which distinguish between "dressed" and "bare" properties respectively). These models produce hierarchies of self-organized random structures.

2 Fractal notions

2.1 Fractal dimension and counting occurrences

Fractal (geometrical) sets ([*Mandelbrot*, 1977; *Mandelbrot*, 1983]) provide the simplest nontrivial example of scale invariance. Unfortunately, we are usually much more interested in fields (with values at each point or at each neighborhood of points) and rarely interested in geometrical sets. However, over long time series, fractal dimensions can still be useful in "counting the occurrences of a given phenomenon"—as long as this question can properly be

posed. If this is the case and the phenomenon is scaling, then the number of occurrences ($N_A(l)$ at resolution scale l in space and/or time of a phenomenon occurring on a set A) follows a power law (here and below the sign \sim means equality within slowly varying and constant factors).

$$N_A(l) \sim \left(\frac{l}{L}\right)^{-D_F} \quad (1)$$

D_F is the (unique) fractal dimension, generally not an integer, and L is the (fixed) largest scale. For a very classical example, see Fig. 1, which illustrates the Cantor, set and its main properties.

For instance, let us consider the occurrences of rain: Fig. 2 displays the records of rain events during the last 45 years in Dedougou ([*Hubert and Carbonnel*, 1989]). These authors show that the occurrence of rainy days in intervals of duration T is fractal, have a dimension $D_F \approx 0.8$, which accounts for the fact that the rain events on the time axis form a Cantor-like set. Amusingly, the wet season is often considered to last 7 months per year, and $0.8 \approx \text{Log}(7)/\text{Log}(12)$. We recall that the standard Cantor set (see Fig.1) which is obtained by iteratively removing the (closed) middle section of the unit interval is of dimension $\text{Log}(2)/\text{Log}(3) \approx 0.63$.

2.2 Codimension and probability of events

A strong emphasis has been very unfortunately placed for years and years on fractal dimensions and especially their connections with the mathematically defined Hausdorff dimension: this connection suffers of many troubles, which are rather symptomatic of a fundamental problem. Indeed, it turns out that it is quite more rewarding ([*Schertzer and Lovejoy*, 1992]), at least quite less cumbersome, to use the notion of codimension as the fundamental notion, whereas usually the latter is introduced in a restrictive way (as discussed below) with the help of the former. Indeed for stochastic processes, one is not so much able to count events, but rather their frequency, especially when the latter is finite, whereas the former is not!

One must note that the notion of codimension is not restricted to stochastic processes, although it is definitely required for them! Indeed, the notion of fractal codimension can be defined both statistically and geometrically. While the geometrical definition is much more popular, we will demonstrate that the statistical definition is much more useful and general since not only it is already interesting for deterministic processes, but it is rather indispensable for stochastic processes, whereas dimension notions get into trouble.

2.2.1 Geometric definition of a fractal codimension:

Lest us recall the classical definition of the fractal codimension, i.e. its geometric definition, which we will show as being rather restrictive. Let $A \subset E$ (E being the embedding space with $\dim(E) = D$ and $\dim(A) = D_g(A)$ the (geometric) dimension of the set A , then the (geometric) codimension $C_g(A)$ is defined as:

$$C_g(A) = D - D_g(A) \quad (2)$$

This definition corresponds merely to an extension of the (integer) codimension definition for vector sub-spaces, i.e., E_1 and E_2 being in direct sum (i.e., $E_1 \cap E_2 = \emptyset$):

$$E = E_1 \oplus E_2 \Rightarrow \text{codim}(E_1) = \dim(E_2) \quad (3)$$

This definition (Eq. (2)) bounds above the codimension by the dimension of the embedding space, since the fractal dimension (as the Hausdorff dimension) should be non-negative, i.e.:

$$D_g(A) \geq 0 \Leftrightarrow C_g(A) \leq D \quad (4)$$

In fact this constraint does not hold anymore as soon as we consider the codimension to be more fundamental than the notion of fractal dimension. This obviously requires to introduce directly the notion of codimension. One obtains such a definition considering the scaling behavior of the probability of events, rather than their number, therefore leaving from enumerations to probabilities.

2.2.2 Statistical definition of a fractal codimension:

Let us consider a sequence of events A_λ defined with higher and higher resolution λ , i.e. with smaller and smaller inner scale: $\ell = \frac{L}{\lambda}$. In the simplest case; it will correspond to a fractal

geometric set defined by a deterministic or random iterative procedure (e.g. the Cantor set, illustrated in Fig. 1). A more general framework is discussed in Appendix A. In a general manner, we expect that the measure of the fraction of the probability space Ω occupied by A_λ is thinner and thinner, as well as scaling. Therefore, us let define its fractal (statistical) codimension by the asymptotic scaling exponent c -when it exists- of their probability (denoted by Pr in the following):

$$\lambda \gg 1: \text{Pr}(A_\lambda) \sim \lambda^{-c} \quad (6)$$

Let us emphasize that c should not depend on the details of the sequence events A_λ , but rather their asymptotic behavior, as well as the one of their probabilities. When the A_λ 's have a well-defined limit A , it is rather convenient to use the short hand notation $c = C(A)$. In Appendix A, we discuss this as well as other generic cases, which for instance involve the upper limit of the A_λ 's (i.e. the set of points that belong to infinitely many A_λ). In any case, the A_λ 's, as well as their possible limit, no longer need to be compact, and their embedding (probability) space Ω can be an infinite dimensional space. There is no upper bound to the statistical codimension, since:

$$C(\Omega) = 0, \quad C(\emptyset) = \infty \Rightarrow C(A) \in [0, \infty] \quad (7)$$

Ω, \emptyset are particular cases of almost sure events, respectively null events.

A rather generic and useful example corresponds to the intersection by (fractal) random balls B_λ , of finer and finer resolution λ (smaller and smaller size $\ell = \frac{L}{\lambda}$), of a given (possibly random) set G :

$$A_\lambda = B_\lambda \cap G \quad (8)$$

In order to fully explore (in fact cover) the set G , the centers of the balls are independently and uniformly distributed (with respect to the Lebesgue measure of the embedding space E) and independently from the probability distribution of G (if any). If E is not bounded, one must consider the corresponding Poisson distribution. When G has some scaling property (e.g. is a fractal geometric set) we expect that the probability of A_λ 's defined by Eq. (8) will have a

scaling behavior (Eq. (7)). Furthermore, when G is a geometric set we expect that its statistical codimension, denoted by $C(G)$, correspond to its geometrical codimension ($C_g(G)$).

Example:

The rather academic Cantor set (Fig. 1) is an illuminating example. Here the balls correspond to sub-segments defined by the iteration of the division a segment into $\lambda_1 = 3$ sub-segments. Due to the fact that only 2 sub-segments over 3 are kept, when the ball resolution is increased by the factor $\lambda_1 = 3$, its probability of intersecting G decreases by a factor $2/3$:

$$\Pr(B_{3\lambda} \cap A) = \frac{2}{3} \Pr(B_\lambda \cap A) \quad (9)$$

therefore:

$$C(G) = \frac{\text{Log}(3/2)}{\text{Log}(3)} = 1 - \frac{\text{Log}(2)}{\text{Log}(3)} = C_g(G) \quad (10)$$

This is a result, which is not only easy to derive but also holds for random Cantor sets. Furthermore, the latter do not need to be restricted to a segment, but could be defined for the full real axis.

2.2.3 Intersection theorem:

It is not only straightforward to evaluate the codimension of the intersection of two events $E_1 \in F$ and $E_2 \in F$, but important for many applications. For instance it corresponds to the measurement by a fractal network (e.g. World Meteorological Organization network, [Lovejoy *et al.*, 1986], or a local monitoring network [Salvadori *et al.*, 1994]) of a fractal set (occurrences respectively of rain and pollution). If the series of two events $E_{1,\lambda}, E_{2,\lambda}$ are independent, then the (statistical) codimension of their intersection is:

$$C(E_1 \cap E_2) = C(E_1) + C(E_2) \quad (11)$$

i.e. codimensions just add for the intersection of independent fractal processes. This is an immediate consequence of the fact that the probability of the intersection (for any λ) factors into:

$$\Pr(E_{1,\lambda} \cap E_{2,\lambda}) = \Pr(E_{1,\lambda}) \Pr(E_{2,\lambda}) \quad (12)$$

therefore the corresponding exponents (Eq. (6)) just add. It is worth to note that the derivation and the validity of this "theorem" is far from being obvious when using the deterministic and geometric definition (see for discussion [*Falconer, 1990*]). Indeed, there are many cases that are rather troublesome, which can be perceived by considering simple examples with integer dimensions (e.g. the intersection of two planes in a three dimensional embedding space does not always yield a geometric codimension equals to 2). However, these annoying cases are irrelevant for statistics.

Furthermore, the theorem of intersection can be extended to the case of dependent events, with the help of conditional codimensions ([*Schertzer and Lovejoy, 1993*]; [*Salvadori, 1993*]; [*Salvadori et al., 2001*]). The latter corresponds to the exponent of the conditional probability in a rather straightforward extension of Eq. (5):

$$\Pr(E_{1,\lambda} | E_{2,\lambda}) \sim \lambda^{-C(E_1|E_2)} \quad (13)$$

which yields:

$$C(E_1 \cap E_2) = C(E_1|E_2) + C(E_2) \quad (14)$$

due to the fact that (for any λ):

$$\Pr(E_{1,\lambda} \cap E_{2,\lambda}) = \Pr(E_{1,\lambda} | E_{2,\lambda}) \Pr(E_{2,\lambda}) \quad (15)$$

2.2.4 Union theorem:

One obtains readily a similar theorem for the intersection of two events:

$$C(E_1 \cup E_2) \leq \inf(C(E_1), C(E_2)) \quad (16)$$

where the equality is obtained when the series of two events $E_{1,\lambda}, E_{2,\lambda}$ are independent. This results from the fact that for any λ :

$$\Pr(E_{1,\lambda} \cup E_{2,\lambda}) \leq \Pr(E_{1,\lambda}) + \Pr(E_{2,\lambda}) \quad (17)$$

where the equality is achieved for independence. With the help of Eq. (5), it yields Eq. (16). This theorem immediately demonstrates that enlarging an event E_1 with the help of a null event ($C(E_2) = \infty$) will not change its codimension.

2.2.5 Relating the two definitions of codimension:

In order to relate the two definitions of codimension in the case of a finite D -dimensional embedding space, it is convenient to use the fact that the probability of the event $(B_\lambda \cap G)$ is defined as the ratio of the number of balls intersecting G and of the total number of balls (indeed intersecting the embedding space), we have:

$$\Pr(B_\lambda \cap G) \sim \frac{N(B_\lambda \cap G)}{N(B_\lambda)} \quad (18)$$

since each of the numbers involved in the ratio defining the probability (Eq. (18)) admits a dimension as a scaling exponent (Eq. (1)):

$$\frac{N(B_\lambda \cap G)}{N(B_\lambda)} \sim \frac{\lambda^{-D_g(G)}}{\lambda^{-D}} \quad (19)$$

As far as this estimate is valid, it yields with the help of Eq. (8) that:

$$C_g(G) < D = \dim(E) < \infty \Rightarrow C_g(G) = C(G) \quad (20)$$

However, whereas there is no limitation on C (Eq. (7)), there is an upper bound on the geometrical codimension (Eq. (4)). Therefore, the equivalence between the two definitions does not hold any longer as soon as $C(G) > D$:

$$C(G) > D \Rightarrow C(G) > C_g(G) (= D) \quad (21)$$

A straightforward consequence is that the fractal dimension $D(G)$ computed with the help of the statistical codimension (i.e. by inverting Eq. (2) with $C(G)$ instead of $C_g(G)$) will be non positive:

$$\left. \begin{array}{l} C(G) > D \\ D(G) \equiv D - C(G) \end{array} \right\} \Rightarrow D(G) < 0 \quad (22)$$

The non-positiveness of this apparent dimension corresponds to the so-called “latent” dimension “paradox” (e.g.[Mandelbrot, 1991]) which is then immediately clarified since $D(G)$ cannot be understood as a deterministic geometric dimension¹. It is only a statistical exponent,

¹ In particular, there is no possible definition of a negative Hausdorff dimension.

which is furthermore defined with the help of the (statistical) codimension, only the latter statistical being intrinsic since directly defined (Eq. (5)).

This is not surprising: the statistical definition overcomes many limitations of the Hausdorff dimension which is defined for compact sets (hence bounded sets): the codimension measures the *relative scarcity* of a phenomenon (the frequency of its occurrence), whereas the dimension measures its *absolute scarcity* (the number of its occurrence). Obviously, we do not need to know the latter in order to be able to determine the former.

2.2.6 The sampling dimension:

We emphasized the fact that the (statistical) codimension can be defined in a rather more general manner than the (geometric) fractal dimension, since it needs not be restricted to a finite dimensional embedding space E nor to component sets A . However, empirically we never deal directly with infinities. Especially, this is true since when doing statistical analysis we always use finite size samples. It is thus quite important to understand what happens when we more and more explore the probability space, which can be understood as the set of all possible realizations (as illustrated by Fig. 3), by studying more and more samples. Obviously, the "effective" dimension of this subspace of probability space (the "effective" embedding space) should increase. Indeed, considering N_s (more or less²) independent samples each of dimension D and resolution λ (i.e. the ratio of the largest scale to the smallest resolved scale), the total number of pixels examined will be of the order:

$$N \cdot N_s = \lambda^{d+D_s} \tag{23}$$

where the "sampling dimension" D_s [Lavallée *et al.*, 1991; Schertzer and Lovejoy, 1989] is defined as:

$$D_s \sim \frac{\log N_s}{\log \lambda} \tag{24}$$

²A more precise condition will be discussed later.

This shows how the effective dimension can be increased above D (a unique sample) and this allows us (for large enough N_s , D_s) to render positive any negative (statistical) dimension! Indeed, consider an event sufficiently rare so that $C(A) > D$, we will nonetheless obtain a positive intersection dimension with our sample as soon as D_s is large enough, indeed the (statistical) dimension being:

$$D_s(A) = D + D_s - C(A) \quad (25)$$

it becomes positive for large samples:

$$D_s(A) > 0 \text{ for any } D_s > C(A) - D \quad (26)$$

The limits case, $D_s(A) = 0$ ($D_s = C(A) - D$), corresponds to the presence of isolated points in our sample: when $D_s < C(A) - D$ almost surely A is not present in our sample, it is almost surely present when $D_s > C(A) - D$.

2.3 Beyond fractal geometry

Fields having different levels of intensity rarely reduce to the oversimplified binary question of occurrence or non-occurrence. The latter is relevant only if the fractal dimension of occurrences does not depend in a sensitive manner with respect to the threshold defining a negligible intensity. Otherwise, we have to address the fundamental question: what is the field at different intensities and at different scales? In the case of rain, the dimension of the rain occurrence depends ([Hubert and Carbonnel, 1991; Hubert et al., 1993; Hubert et al., 1995]) indeed on the threshold defining a negligible rain rate. Generalizations of fractal/scale invariance ideas going well beyond geometry were desperately needed and appeared in 1983 when the dogma of a unique dimension was finally abandoned ([Hentschel and Procaccia, 1983], [Grassberger, 1983], [Schertzer and Lovejoy, 1984]).

3 Phenomenology of turbulent cascades

The phenomenology of (scalar) turbulent cascades had been first discussed in the context of hydrodynamic turbulence (since [Richardson, 1922]) and where the structures were considered as eddies. However, this phenomenology is much more general and not restricted to a hierarchy

of eddies, since we simply follow how the "activity" of turbulence becomes more and more inhomogeneous at smaller and smaller scales. The phenomenology of turbulent cascades thus corresponds to a general paradigm, for fields where the activity tends to be concentrated more and more at smaller and smaller scales. In the case of turbulence, this activity can be estimated in a rather precise manner by the rate at which energy is transferred to smaller scales, hence the fundamental importance of the density of the energy flux to smaller scales (ε)³

We will see that the most general property will be that a scaling field cannot be characterized by a unique (fractal) geometric set, but by an infinite hierarchy of them, hence the generic name "multifractal" (a term coined by Parisi [*Benzi et al.*, 1984; *Parisi and Frisch*, 1985]). However, we will show that under this innocent expression there exists a much richer diversity of multifractal processes and phenomena than is usually realized.

The key assumption in phenomenological models of turbulence (which became explicit with the pioneering work of Yaglom ([*Yaglom*, 1966]) is that successive steps define (independently) the fraction of the flux of energy distributed over smaller scales. Note that it is clear that the small scales cannot be regarded as adding energy; they only modulate the energy passed down from larger scales. The explicit hypothesis is that the fraction of the energy flux (or "activity") from a parent structure to an offspring will be determined in a scale invariant way.

In the (pedagogical) case of "discrete cascade models" (the much more realistic continuous scales model will be discussed in Sect.6), "eddies" are defined by the hierarchical and iterative division of a D -dimensional cube into smaller sub-cubes, with a constant ratio of scales λ_1 (greater than 1, very often equal to 2). More precisely, the initial D -dimensional cube Δ_0^0 of size L is divided step by step for each $n \in \mathbb{N}$ into smaller sub-cubes Δ_n^i ($i_j = 0, 1, \dots, \lambda_1^n - 1$; $j = 1, 2, \dots, D$), which form a disjoint cover of Δ_0^0 and are of size $\ell_n = \frac{L}{\lambda_1^n}$.

In other words, the D coordinates i_j of a sub-cube at step n are defined in base λ_1 with the

³ However as discussed by [*Schertzer and Lovejoy*, 1995], the scalar cascade framework is insufficient to deal with the vectorial nature of turbulence, but can be extended to 'Lie cascades' framework.

help of only n first digits. The density of the flux energy ε_n at the step n is supposed to be strictly homogeneous on each "sub-eddies" of scale ℓ_n , i.e. ε_n is a step function:

$$\varepsilon_n(\underline{x}) = \sum_{i=0}^{\lambda^n - 1} \varepsilon_n^i 1_{\Delta_n^i}(\underline{x}) \quad (27)$$

where $1_{\Delta_n^i}$ is the characteristic function of the sub-cube Δ_n^i . The energy density ε_{n-1} at step $n-1$ will be multiplicatively distributed to sub-eddies:

$$\varepsilon_n(\underline{x}) = \mu \varepsilon_n(\underline{x}) \varepsilon_{n-1}(\underline{x}) \quad (28)$$

with the help of a multiplicative increment⁴:

$$\mu \varepsilon_n(\underline{x}) = \sum_i \mu \varepsilon_n^i 1_{\Delta_n^i}(\underline{x}) \quad (29)$$

where the variables $\mu \varepsilon_n^i$ are usually assumed to be identically and independently distributed (i.i.d.), as well as independent of the variables ε_n^i .

In spite of their over-simplistic and somewhat awkward discrete discretization, these models are already able to give key understanding of some of the fundamentals of cascade processes, which will be confirmed for continuous scale cascades (see Sect. 6), which are indispensable to take into account other (statistical) symmetries (e.g. translation invariance).

3.1 Unifractal insights and the simplest cascade model (β -model)

The simplest cascade model, often called β -model, takes the intermittency of turbulence into account by assuming [Novikov and Stewart, 1964]; [Mandelbrot, 1974]; [Frisch et al., 1978] that eddies are either dead (inactive) or alive (active). This corresponds⁵ to the fact that the multiplicative increments $\mu \varepsilon$'s have two states (see Fig. 4 for an illustration) :

$$\begin{aligned} \Pr(\mu \varepsilon = \lambda_1^c) &= \lambda_1^{-c} && \text{(alive)} \\ \Pr(\mu \varepsilon = 0) &= 1 - \lambda_1^{-c} && \text{(dead)} \end{aligned} \quad (30)$$

The boost $\mu \varepsilon = \lambda_1^c > 1$ is chosen so that the ensemble averaged ε is conserved:

⁴The notation μ for multiplicative increments, is analogous to the symbol δ for additive increments.

⁵The β -model is often defined more vaguely than this. We follow the more precise stochastic presentation by [Schertzer and Lovejoy, 1984].

$$\langle \mu \varepsilon \rangle = 1 \Leftrightarrow \langle \varepsilon_n \rangle = \langle \varepsilon_0 \rangle \quad (31)$$

where $\langle . \rangle$ denotes the ensemble average. At each step in the cascade the fraction of the alive eddies decreases by the factor $\beta = \lambda_1^{-c}$ (hence the name " β -model") and conversely their energy flux density is increased by the factor $1/\beta$ to assure (average) conservation. After n steps, this drastic and simple dichotomy is merely amplified by the total scale ratio λ_1^n :

$$\begin{aligned} \Pr(\varepsilon_n = \lambda_1^{nc}) &= \lambda_1^{-nc} && \text{(alive)} \\ \Pr(\varepsilon_n = 0) &= 1 - \lambda_1^{-nc} && \text{(dead)} \end{aligned} \quad (32)$$

Hence either the density goes on to diverge with an (*algebraic*) order of singularity c , or is *at once* calmed down to zero! Following our discussion (and definitions) given in Sect. 2.2, c is the codimension of the alive eddies, hence their corresponding dimension D_s is (when $c < D$:

$$D_s = D - c \quad (33)$$

This is the dimension of the support of turbulence and corresponds to the fact that the average number of alive eddies (in the β -model is

$$\langle N_n \rangle = (\lambda_1^n)^{d-c} \quad (34)$$

3.2 The simplest multifractal variant (α -model)

We already pointed out that on the empirical level, occurrences of rain are not so much informative. For instance, a 1 mm daily rain rate is rather negligible compared to a 150 mm daily rain rate! Fig. 5 displays the rain rate at Nîmes (France) during a few years, and averaged over varying time scales T (from a day to a year). This figure illustrates the great intermittency of rain rates: most of the time it is negligible, while sometimes it reaches 200 mm (even 228 mm in a few hours—the famous October 1988 catastrophe!)— in comparison the daily average is ~ 2.1 mm. The variability is so significant in this time series that [Ladoy *et al.*, 1993] and [Bendjoudi *et al.*, 1997] find evidence of divergence of high order statistical moments (a subject we will discuss more in Sect. 7). Qualitatively this variability seems strikingly analogous to that of the energy flux cascade in turbulence (as displayed in Fig. 6), an analogy that turns out to be quite profound.

On the theoretical level the β -model turns to be a poor approximation to turbulence because it is unstable under perturbation: as soon as we consider a more realistic alternative to the caricatural dead/alive dichotomy, most of the peculiar properties of the β -model are lost. To show this, the " α -model" ([Schertzer and Lovejoy, 1984]) was introduced. It was named this way because of the divergence of moments exponent α it introduces. In the notation used below, this exponent is rather denoted q_D , where the "D" emphasizes that it depends on the dimension of space D over which the multifractal is averaged. In any case, this exponent should not be confused with the Lévy, nor with the strange attractor notation.

Rather than only allowing eddies to be either "dead" or "alive" we consider a more realistic α -instability allowing them to be either "more active" or "less active" according to the following binomial process:

$$\begin{aligned} \Pr(\mu\varepsilon = \lambda_1^{\gamma_+}) &= \lambda_1^{-c} && \text{(increase)} \\ \Pr(\mu\varepsilon = \lambda_1^{\gamma_-}) &= 1 - \lambda_1^{-c} && \text{(decrease)} \end{aligned} \tag{35}$$

with $\gamma_+ = \frac{c}{\alpha}$ (> 0) and $\gamma_- = -\frac{c}{\alpha'}$ (< 0). The β -model is recovered with $\alpha = 1$, $\alpha' = 0$.

The ensemble "canonical" conservation (Eq. (31)) implies that here are really only two free parameters out of c, γ_+, γ_- , since it corresponds to:

$$\lambda_1^{\gamma_+} \cdot \lambda_1^{-c} + \lambda_1^{\gamma_-} \cdot (1 - \lambda_1^{-c}) = 1 \tag{36}$$

The "p-model" [Meneveau and Sreenivasan, 1987] and the "binomial multifractal measure" correspond to microcanonical versions of the α -model, i.e. which means that the flux of energy is strictly conserved, not only on the average. This constraint fundamentally changes the properties of the processes, as we shall see below.

The *pure orders of singularity* γ_- and γ_+ lead to the appearance of *mixed orders of singularity*, as soon as $\gamma_- > -\infty$ (the " β -model"), mixed singularities of different orders γ ($\gamma_- \leq \gamma \leq \gamma_+$), are built up step by step through a complex succession of γ_- and γ_+ . For instance consider two steps of the process, the various probabilities and random factors are:

$$\begin{aligned}
\Pr(\mu\mathcal{E} = \lambda_1^{2\gamma_+}) &= \lambda_1^{-2c} && \text{(two boosts)} \\
\Pr(\mu\mathcal{E} = \lambda_1^{\gamma_+ + \gamma_-}) &= 2\lambda_1^{-c}(1 - \lambda_1^{-c}) && \text{(one boost and one decrease)} \\
\Pr(\mu\mathcal{E} = \lambda_1^{2\gamma_-}) &= (1 - \lambda_1^{-c})^2 && \text{(two decreases)}
\end{aligned} \tag{37}$$

This process has the same probability and amplification factors as the three states α -model with a new scale ratio of λ^2 *i.e.*,

$$\begin{aligned}
\Pr(\mu\mathcal{E} = (\lambda_1^2)^{\gamma_+}) &= (\lambda_1^2)^{-c} \\
\Pr(\mu\mathcal{E} = (\lambda_1^2)^{(\gamma_+ + \gamma_-)/2}) &= 2(\lambda_1^2)^{-c/2} - 2(\lambda_1^2)^{-c} \\
\Pr(\mu\mathcal{E} = (\lambda_1^2)^{\gamma_-}) &= 1 - 2(\lambda_1^2)^{-c/2} + (\lambda_1^2)^{-c}
\end{aligned} \tag{38}$$

Iterating this procedure, after $n = n^+ + n^-$ steps we find:

$$\begin{aligned}
\gamma_{n^+, n^-} &= \frac{n^+ \gamma_+ + n^- \gamma_-}{n^+ + n^-}, \quad n^+ = 1, \dots, n; \quad n^- = n - n^+ \\
\Pr(\mathcal{E}_n = (\lambda_1^n)^{\gamma_{n^+, n^-}}) &= \binom{n}{n^+} \lambda_1^{-cn^+} (1 - \lambda_1^{-c})^{n^-}
\end{aligned} \tag{39}$$

where $\binom{n}{n^+}$ is the number of combinations of n objects taken k at a time. This implies that we may write:

$$\Pr(\mathcal{E}_n \geq (\lambda_1^n)^{\gamma_i}) = \sum_j p_{ij} (\lambda_1^n)^{-c_{ij}} \tag{40}$$

The p_{ij} 's are the ‘‘submultiplicities’’ (the prefactors in the above), c_{ij} are the corresponding exponents (‘‘subcodimensions’’) and λ_1^n is the total ratio of scales from the outer scale to the smallest scale. Notice that the requirement that $\langle \mu\mathcal{E} \rangle = 1$ implies that some of the $\lambda_1^{\gamma_i}$ are greater than one (boosts) and some are less than one (decreases), that is, some $\gamma_i > 0$ and some $\gamma_i < 0$.

In other words, leaving the simplistic alternative dead or alive (‘‘ β -model’’ for the alternative weak or strong (‘‘ α -model’’) leads to the appearance of a full hierarchy of levels of survival, hence the possibility of a hierarchy of dimensions of the set of survivors for these different levels. Therefore the field can be understood as ‘multifractal’, *i.e.* defined by an (infinite) hierarchy of fractal sets.

4 The general multifractal framework

4.1 The codimension function $c(\gamma)$

The pedagogical example of the α -model is helpful to get insights for a general formalism adequate for more general cascade processes. For instance, as the number n of cascade steps becomes large in the α -model, one obtains asymptotic expressions (Eq. (40)) which are independent of the steps, but depend only on the total ratio of scale, denoted now by $\lambda = \frac{\ell}{L}$ instead of λ_1^n :

$$\Pr(\varepsilon_\lambda \geq \lambda^\gamma \varepsilon_0) \sim \lambda^{-c(\gamma)} \quad (41)$$

This is a basic multifractal relation for multifractal processes, which merely states, in the light of our earlier discussion on the notion of codimension (see Sect. 2.2, in particular Eq. (5)), that the measure of the fraction of the probability space corresponding to the events

$$A_\lambda(\gamma) = \{(\underline{x}, \omega) \in Ex\Omega \mid \varepsilon_\lambda(\underline{x}, \omega) \geq \lambda^\gamma \varepsilon_0\} \quad (42)$$

has a (statistical) codimension $c(\gamma)$. As already emphasized, in general there is no upper bound on $c(\gamma)$. On the other hand, due to the nested hierarchy of these events ($\forall \lambda, \gamma \leq \gamma' : A_\lambda(\gamma) \subset A_\lambda(\gamma')$) $c(\gamma)$ is an increasing function of γ .

Other fundamental properties, which will be readily derived with the help of statistical moments (next section, Sect. 4.2), are that $c(\gamma)$ must be convex and that if the process is conservative (i.e. any $\lambda : \langle \varepsilon_\lambda \rangle = \varepsilon_0$), then $c(\gamma)$ has the fixed point: $c(C_1) = C_1$, where C_1 is at the same time a singularity corresponding to the mean of the process and its codimension: $c(\gamma)$ is at this point tangent the first bisectrix. Fig. 7 illustrates these properties of the codimension function $c(\gamma)$. This graphical representation helps also to estimate the limitations due to the finite size of a sample. Indeed, corresponding to our discussion on the sampling dimension (Sect. 2.2.6), there is a "sampling singularity" γ_s ; i.e. the maximum almost sure maximum singularity presents in a sample of sampling dimension D_s . This singularity has a codimension equal to the effective dimension of sampling (see Fig. 8), therefore:

$$\gamma_s(D_s) = c^{-1}(D + D_s) \quad (43)$$

With no surprise, this heuristic estimate can be secured, at least for $D_s = 0$, by rigorous mathematical derivation .

4.2 The Multiscaling of Moments $K(q)$ and the Legendre Transformation

Under fairly general conditions its probability distribution or (all) its statistical moments may equivalently specify the properties of a random variable. More precisely, for a non-negative random variable x , these two representations are linked by a Mellin transformation M , which is:

$$\langle x^{q-1} \rangle = M(p) = \int_0^\infty x^{q-1} p(x) dx \quad (44)$$

$$p(x) = M^{-1}(\langle x^{q-1} \rangle) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \langle x^{q-1} \rangle x^{-q} dq \quad (45)$$

(essentially these are simply the Laplace and inverse Laplace transforms for the logs). In fact, if the moments are not increasing too quickly with the order q (more precisely, when they satisfy the ‘‘Carleman criterion’’—see [Feller, 1971]), only the knowledge of the moments of integer orders is required. The relevance of this condition for turbulence have been discussed ([Orszag, 1970]), but it is important to note that the Mellin duality is nevertheless ([Schertzer and Lovejoy, 1993]) relevant for cascades and somewhat more general than the Legendre duality pointed out by [Parisi and Frisch, 1985] in a restrictive multifractal framework (see Sect. 4.3) than the stochastic one we are presently discussing.

However, it is useful to check that the latter is an asymptotic ($\lambda \rightarrow \infty$) result from the former for the corresponding exponents. Since $c(\gamma)$ is the exponent that characterizes the scaling of the probabilities, we introduce the corresponding function $K(q)$ to characterize the moments, anticipating that the two will be related:

$$\langle \varepsilon_\lambda^q \rangle \approx \lambda^{K(q)} \quad (46)$$

For large $\text{Log } \lambda$, we can use the saddle point approximation (Laplace's method, see for example [Bender and Orszag, 1978]) which yields asymptotic approximations to integrals of exponential form. One obtains that $K(q)$ is related to $c(\gamma)$ by:

$$\langle \varepsilon_\lambda^q \rangle = \int d\text{Pr}(\varepsilon_\lambda) \varepsilon_\lambda^q \sim \int d\text{Pr}(\varepsilon_\lambda) \lambda^{q\gamma} \sim \int_{-\infty}^{\infty} \text{Log}(\lambda) dc(\gamma) \lambda^{q\gamma} \lambda^{-c(\gamma)} \quad (47)$$

which yields the asymptotic behavior ($\lambda \rightarrow \infty$):

$$\int_{-\infty}^{\infty} d\gamma e^{\ln(\gamma)(q\gamma - c(\gamma))} \sim \exp\left[\ln(\lambda) \cdot \max_{\gamma}\{q\gamma - c(\gamma)\}\right], \quad \lambda \gg 1, \quad (48)$$

as well as the prefactor, which we do not consider here. A similar expansion can be done for the inverse Mellin transform Eq. (45), and we have therefore the (involutive) Legendre duality for the exponents:

$$K(q) = \max_{\gamma}\{q\gamma - c(\gamma)\} \Leftrightarrow c(\gamma) = \max_q\{q\gamma - K(q)\} \quad (49)$$

This demonstrates that both curves are convex (due to the fact that iterating twice the Legendre transform on a non-convex curve yields only the "convex hull" of this curve). One may note that it is rather straightforward to directly demonstrate it for $K(q)$. This also means that the curve $c(\gamma)$ is the envelop of the tangencies of $K(q)$ and reciprocally (see Fig. 9). Hence there is a simple one-to-one correspondence between moments and orders of singularities.

4.3 Comparison with other multifractal formalisms

Until now, we discussed multifractal notions within a codimension framework [*Schertzer and Lovejoy, 1987b; Schertzer and Lovejoy, 1992*] it is therefore timely to compare it with dimension frameworks. In relation to the nonlinear scaling of the velocity structure functions [*Anselmet et al., 1984*], i.e. statistical moments of the velocity increments, [*Parisi and Frisch, 1985*] introduced a notion of multifractals by considering the geometric distribution of the singularities of the velocity increments. With the help of the so-called refined self-similar hypothesis [*Kolmogorov, 1962; Obukhov, 1962*], the latter can be related linearly to the singularities of the energy flux. However, they consider neither a probability space nor a cascade process, but rather a geometric distribution of the singularities. The popular $f(\alpha)$ formalism was introduced by [*Halsey et al., 1986*] who dealt with multifractal "geometric attractors" and in many respect emphasized the implicit non-random framework developed by [*Parisi and Frisch, 1985*]. Rather than considering the density of the multifractal measure p_λ (the non-random analog of the turbulent ε_λ), they considered the measure itself integrated over a ball (box) size L / λ .

In both cases, it was assumed that the support of the singularities, which in our notations could be defined as:

$$S_\lambda(\gamma) = \{x | \varepsilon_\lambda(x) \approx \lambda^\gamma \varepsilon_0\} \quad (50)$$

has a well-defined limit (fractal) dimension, as well as its (lower) limit:

$$S(\gamma) = \varliminf_{\Lambda \rightarrow \infty} S_\Lambda(\gamma) \equiv \bigcup_{\lambda} \bigcap_{\Lambda > \lambda} S_\Lambda(\gamma) \quad (51)$$

which corresponds to the set of points, where there exists a given resolution λ , after which they have a singularity γ . In fact, these definitions and assumptions are too much demanding. First, due to the (approximate) equality sign in Eq. (50), instead of the inequality sign involved in Eq. (42), the supports $S_\lambda(\gamma)$, contrary to the events $A_\lambda(\gamma)$, are not in general hierarchically nested. Therefore [Parisi and Frisch, 1985] were compelled to add an ad-hoc hypothesis to assure this feature as well as the convexity of the analogue of $c(\gamma)$. If we change this equality sign to the inequality of Eq. (42), $S_\lambda(\gamma)$ corresponds to a D -dimensional cut of the event $A_\lambda(\gamma)$, i.e. the restriction of the latter for a given $\omega \in \Omega$. As a consequence of Appendix A, whereas the (upper) sequence $\bar{A}_\lambda(\gamma)$ and its (upper) limit $\bar{A}(\gamma)$ have always a well-defined statistical codimension, they do not have always a well-defined dimension. [Frisch, 1995; Parisi and Frisch, 1985] acknowledged that within their formalism they could get *only* a bounded range of singularities (in fact $c(\gamma) \leq D$) for the so-called lognormal model. This is a generic limitation of their formalism. However, the consequences of this limitation were not discussed, whereas we will see that they are of prime importance (Sect. 7). There is another limitation, which is rather related to the type of limit that is considered for the supports or the events of a given singularity γ . In the stochastic framework, it is more than likely that when as we add in more and more cascade steps, γ will undergo random walks as λ is increased. Therefore, the relevant notion limit is the upper limit (Eq. (6)) rather than the most stringent lower limit (Eq. (51)). For applications it means that the multifractal field is *nonlocal*, and one cannot track a given singularity value by locally refining the analysis of the field, e.g. with the help of wavelet analysis. The latter could yield spurious results.

Let us mention the relation between the codimension notations and $f(\alpha)$ dimension notations.

Due to the fact that in the latter case, the measure rather than its density is considered:

$$\int_{B_\lambda} p_\lambda d^D \mathbf{x} = p_\lambda \lambda^{-D} \sim \lambda^{-\alpha_D} \quad (52)$$

we have:

$$\alpha_D = D - \gamma; \quad f(\alpha_D) = D - c(\gamma) \quad (53)$$

Let us emphasize that this correspondence is valid only for deterministic singularities, i.e. satisfying $f(\alpha_D) \geq 0$ or $c(\gamma) \leq D$ (Sect. 2.2). We introduced the subscript “ D ”, which was not used in the original, to α , in order to underscore its dependence on the dimension D of the system. On the other hand, [Halsey *et al.*, 1986] used a partition function introduced by [Hentschel and Procaccia, 1983], whose scaling exponent $\tau(q)$ can be related to the scaling moments function $K(q)$ (Sect. 4.2), with the help of the Trace Moment (which is discussed in Sect.7.3.2) in the following way:

$$\tau_D(q) = (q-1)D - K(q) = (q-1)(D - C(q)) \quad (54)$$

5 Universality

5.1 The concept of universality

This issue of universality for multifractal processes had been the subject of a hot debate, whose main steps and conclusions are discussed at length by [Schertzer and Lovejoy, 1997], who emphasized that "due to the growing number of attempts at modeling and analyzing multifractals in rain (and elsewhere) - it is becoming central for applications". . In the following, we summarize this discussion and highlight its conclusion.

Let us first emphasize that there is only a convexity constraint on the nonlinear functions $K(q)$ and $c(\gamma)$, therefore *a priori*, an infinity of parameters is required to determine a multifractal process. For obvious theoretical and empirical reasons physics abhors infinity! This is the reason why in many different fields of physics the theme of *universality* appears: among the infinity of parameters it may be possible that only very few of them are relevant. This is especially true as soon as we consider not only ideal systems, but more realistic systems

subjected to perturbations or interactions with itself. Indeed, such perturbations or interactions may wash out many of the peculiarities of the theoretical model, retaining only some essential features. The system can be expected to converge to some *universal attractor*⁶, in the sense that a whole class of models/processes, belonging to the same domain of attraction, will converge to the same process defined by (far) fewer (relevant) parameters (see Fig. 3.6).

Although the term is not always explicitly used, the notion of universality is fairly widespread in physics. It corresponds to the fact that among the many parameters of a theoretical model, very few will in fact be relevant. For instance, in critical phenomena most of the many exponents describing phase transitions will depend only on the dimensionality of the system. Loosely speaking, a theoretician may imagine a model depending on a very large number of parameters for an isolated system, but most natural systems are open and it is the existence of these interactions which leads most of the details introduced by the fantasy of the theoretician to be washed out, just leaving the (few) essentials. The general idea, exploited for instance in the Renormalizing Group approach is that repeated iterations of a given process with itself, converges towards a limit, and this limit will be reached starting with quite different processes. More precisely, all the processes belonging to the “same basin of attraction” will converge toward the same limit or “attractor”, although they could be originally quite different, henceforth the notion of universality: the larger the basin, the more universal the attractor.

5.2 Universality in multiplicative processes?

The study of multiplicative random processes has a long history (see [*Aitchison and Brown*, 1957]), going back to at least [*McAlister*, 1879], who argued that multiplicative combinations of elementary errors would lead to lognormal distributions. [*Kapteyn*, 1903] generalized this somewhat and stated what came to be known as the “law of proportional effect”, which has been frequently invoked since, particularly in biology and economics (see also [*Lopez*, 1979])

⁶ Indeed, it was the realization that low dimensional systems (such as nonlinear mappings or coupled nonlinear ordinary differential equations) had universal behavior (such as the famous Feigenbaum constant) that led to an explosion of interest in deterministic chaos. Universal multifractals may be considered as analogies with large numbers of degrees of freedom.

for this law in the context of rain). This law was almost invariably used to justify the use of lognormal distributions i.e. it was tacitly assumed that the lognormal was a universal attractor for multiplicative processes. Although [Kolmogorov, 1962] and [Obukhov, 1962] did not explicitly give the law of proportional effect as motivation, it was almost certainly the reason why they suggested a lognormal distribution for the energy dissipation in turbulence. Since then, culminating in the multifractal processes, we have seen that there have been many proposals for explicit multiplicative cascade models that would reproduce the strong intermittency in turbulence. Unfortunately, in the course of development of these models the basic issues of universality were obscured by various technical questions.

If we simply iterate the model step by step with a fixed ratio of scale λ , we indefinitely increase the overall range of scales $\Lambda \rightarrow \infty$ posing already a non trivial mathematical problem (weak limit of random measures, see [Kahane, 1985]). In his pioneering work, [Yaglom, 1966] claimed that iterating the process to smaller scales may lead to the (universal) lognormal model. The claim of universality of the lognormal model was first criticized by [Orszag, 1970] and then by [Mandelbrot, 1974]. Whereas the former was on the grounds that the (infinite) hierarchy of integer order moments would not determine a lognormal process, the latter pointed out that even if the cascade process was lognormal at each finite step, that in the small scale limit, the spatial averages of the cascade process would *not* be lognormal⁷. Furthermore, since the particularities of the discrete models (e.g. the α -model) remain as a discrete cascade proceeds to its small-scale limit, the opposite extreme claim has since been made: that multiplicative cascades could not admit *any* universal behavior. For instance, Mandelbrot stated ([Mandelbrot, 1989]): “in the strict sense, there is no universality whatsoever... this fact about multifractals is very significant in their theory and must be recognized...” (see also [Mandelbrot, 1991] for more antiuniversality statements). More recently, [Gupta and Waymire, 1993] repeated the same kind of claim. In both cases, their rejection of universality was based on a misunderstanding of

⁷ Indeed, we already noted that the particularities of the discrete models (e.g., the α -model) remain as the cascade proceeds to its small scale limit ($\lambda \rightarrow \infty$) and this *non universal* limit already poses a *non trivial* mathematical problem (that of weak limits of random measures).

the alternatives discussed by [Schertzer and Lovejoy, 1987a] and [Schertzer and Lovejoy, 1991].

5.3 Universal Multifractals

On the contrary, *keeping the total range of scale fixed and finite, mixing* (by multiplying them) *independent processes of the same type*, (preserving certain characteristics, e.g. variance of the generator), and *then seeking the limit $\Lambda \rightarrow \infty$: a totally different limiting problem is obtained!* For instance, this may correspond to *densifying* the excited scales by introducing more and more intermediate scales (see Fig.11), and seeking thus the limit of continuous scales of the cascade model. Alternatively, we may also consider the limit of multiplications of identically independently distributed (i.i.d.) discrete cascades models leading also to universal multifractal processes. [Schertzer and Lovejoy, 1997] established rigorous demonstrations of the fact that the renormalized nonlinear mixing over a finite range of scales of i.i.d. cascade processes, as well as renormalized scale densification of a given multifractal processes, converge to a universal multifractal.

6 Continuous scale cascade

6.1 Limitations of discrete scale cascades

One important consequence of universality is the possibility to obtain a continuous scale process from a discrete cascade model with the help of a scale densification, i.e. introducing more and more intermediate scales between the discrete cascade. Continuous scale processes are rather indispensable, then discrete cascades have many limitations. Indeed, it is already questionable to have a scale ratio of the elementary cascade step λ strictly larger than 1, in fact larger or equal to 2, without any physical reason, e.g. a quantification rule. Furthermore the hierarchical splitting rule of structures into sub-structures introduces a notion of distance which is no longer a metric, but an ultra-metric. More precisely it corresponds to the λ -adic ultrametric: the distance between two structures at a given level of a discrete cascade process is defined by the level of the cascade where there is their first (and smaller) common ancestor, not the usual metric. This means for instance that the distance between the centers of two contiguous eddies

is not uniform. This fact has many drastic consequences, since all the statistical interrelations between different structures will depend on this ultra-metric, not on the usual metric. In particular, there is no hope to obtain a (statistically) translation invariant cascade, since a translation is related to the metric, not the ultra-metric. In other words, discrete cascades have been useful to grasp some fundamentals, but one has to take care of not being blocked by some of their artifacts. As final note on discrete scale cascade, let us emphasize that almost all rigorous mathematical results on cascade processes have been derived in this restricted framework; this is presumably due not only because it is rather convenient, but also for some complex historical reasons, including the question of the biased debate on universality (see previous section). As a consequence, the question of continuous scale cascade has been not discussed enough.

6.2 Continuous scale cascades and their generators

The general idea of continuous scale cascade ([Schertzer and Lovejoy, 1987a]) corresponds to considering a stochastic one-parameter multiplicative group property for the densities ε_λ defined for arbitrary scale ratios λ instead of being defined only to (discrete) powers ($\lambda_1^n, n = 1, 2, \dots$) of the elementary step scale ratio (λ_1):

$$\forall \Lambda, \lambda \geq 1: \varepsilon_\Lambda = \varepsilon_\lambda \cdot T_\lambda(\varepsilon'_{\Lambda/\lambda}) \quad (55)$$

where ε'_{λ} and ε_λ , are independently and identically distributed for any λ . This means that not only a multiplicative cascade from scales L to L/Λ factors into the same given cascade from L to L/λ and from L/λ to L/Λ , but the latter corresponds to a cascade of the same type from L to $L\lambda/\Lambda$ rescaled with the help of the contraction operator T_λ . The simplest case, which will be considered until Sect. 8, corresponds to an isotropic self-similar cascade, where T_λ is the isotropic contraction: $T_\lambda(\underline{x}) = \underline{x}/\lambda$.

As for any one-parameter group, we are interested by its infinitesimal generator, which will be stochastic in the present case, and therefore, loosely speaking, to come back to an additive group. Let us consider the generator of the cascade over a (non-infinitesimal) scale ratio λ defined by:

$$\varepsilon_\lambda = \exp(\Gamma_\lambda) \quad (56)$$

And which should satisfy the corresponding additive group property:

$$\forall \Lambda, \lambda \geq 1: \Gamma_\Lambda = \Gamma_\lambda + T_\lambda(\Gamma_{\Lambda/\lambda}) \quad (57)$$

This gives a simple and very convenient meaning to the moment scaling function $K(q)$ (Eq. (46)): it is nothing else than the (Laplace) second characteristic function-or cumulant generating function- of the generator and the latter should be logarithmically divergent (with the scale ratio) in order to satisfy Eq. (46). The latter property can be satisfied by considering 'colored' generators obtained by fraction integration of a white noise γ_0 , called for rather obvious reason the sub-generator. The logarithmic divergence is obtained by selecting the appropriate order of integration to be performed.

For a concrete and generic example, let us consider the case of universal multifractals (sect. 5.3). As a consequence of their universality, their generators should be (colored) stable Lévy noises ([Schertzer and Lovejoy, 1987a]). The appropriate order of fractional integration to obtain the logarithmic divergence is D/α' for a stable white noise of Lévy's stability index $0 < \alpha \leq 2$, where α' is the conjugate of α : ($\frac{1}{\alpha} + \frac{1}{\alpha'} = 1$). In order to get some convergent moments, this stable white noise should be furthermore extremely asymmetric ([Schertzer and Lovejoy, 1989]) for $\alpha < 2$, i.e. with a skewness $\beta = -1$, whereas it is obviously symmetric ($\beta = 0$) for the gaussian case (i.e. $\alpha = 2$).

7 The extremes

7.1 The singular limit of a cascade process

The small-scale limit $\lambda \rightarrow \infty$ of a cascade process is very singular since for any positive singularity γ , the density $\varepsilon_\lambda \approx \lambda^\gamma$ diverges. These divergences are statistically significant for $\gamma > C_1$, since we have $\langle \varepsilon_\lambda \rangle = \lambda^{K(q)} \rightarrow \infty$ for all $q > 1$, due to the fact that $K(q) > 0$ for $q > 1$. This singular behavior means that if a limit exists, it is not in the sense of functions. We really have something similar to the Dirac δ -measure, which can be defined as a "generalized function" as a limit of functions, without being itself a function and is indeed only meaningful if

we integrate over it. It is rather obvious that the β -model does correspond to a (random) generalization of the Dirac δ -measure for non isolated points belonging to a fractal set of codimension $c = D - D_s > 0$. Conversely the Dirac δ -measure can be understood as the particular (deterministic) case corresponding to a codimension $c = 0$, i.e. A is a set of isolated points.

As a consequence, one has to consider the limit of the corresponding measures $\Pi_\lambda(A) \rightarrow \Pi_\infty(A)$ over compact sets A of dimension D , i.e. the D -dimensional integration of the density ε_λ over A :

$$\Pi_\infty(A) = \lim_{\lambda \rightarrow \infty} \Pi_\lambda(A) = \lim_{\lambda \rightarrow \infty} \int_A \varepsilon_\lambda d^D \underline{x} \quad (58)$$

In agreement with turbulent denominations, the integrals Π_λ can be called fluxes (of energy through the scale $\ell = L / \lambda$), whereas ε_λ can be called flux density (of energy at the scale $\ell = L / \lambda$). Therefore, we expect a convergence in fluxes, but not in densities. Due to the singularity of the limit, we may furthermore expect that there will be convergence for only a limited range ($0 < q < q_D$) of moment orders of the flux, since higher moments are related to higher singularities (see below for a detailed discussion) i.e.:

$$\exists q_D > 1, \forall q \geq q_D: \langle \Pi_\infty(A)^q \rangle = \infty \quad (59)$$

whereas:

$$\forall \lambda < \infty: \langle \Pi_\lambda(A)^q \rangle < \infty \quad (60)$$

The sub-index D of the critical order q_D underscores its dependence on the dimension of the integration which is performed. This dependence can be used ([Schertzer and Lovejoy, 1984]) in order to demonstrate that cascades processes are generically multifractals: increasing order q_D of convergence defines a hierarchy of fractal sets having larger and larger fractal dimension D . It is very important to note that the critical order q_D of divergence of statistical moments is also the exponent of the power-law fall-off of the probability distribution:

$$\exists q_D > 1, \pi \gg 1: \Pr(\Pi_\infty(A) > \pi) \approx \pi^{-q_D} \quad (61)$$

and that the two equations Eq. (59) and Eq.(61) are equivalent. The latter has many practical implications that we will review below.

7.2 Bare and dressed cascades

The singular limit of the cascade process underscores the necessity to distinguish the properties of a cascade stopped at a finite resolution λ , from those corresponding to the limit. [Schertzer and Lovejoy, 1987a] argued that in a very general manner this difference is related to the importance of the interaction with finer scale activity, which 'dresses' the former to yield the latter, in similarity with what happens in renormalization when higher and higher order of interactions are taken into account. Therefore, it is rather appropriate to distinguish between the “bare” cascade quantities obtained after the cascade has proceeded down to a finite resolution λ , and the corresponding “dressed” quantity obtained after integrating a completed cascade over the same scale ($\ell = L/\lambda$). See Fig.12 for an illustration for a finite resolution Λ , although we are primarily interested by $\Lambda \rightarrow \infty$. Due to the group property of a multiplicative cascade (see Sect. 6.2), a dressed cascade factors into its bare part and an hidden part, which corresponds to a flux of a cascade from L to $L\lambda/\Lambda$ rescaled with the help of the contraction operator T_λ . Bare and dressed properties are similar, as far as the latter flux remains a finite prefactor with $\Lambda \rightarrow \infty$. A drastic change occurs as soon as this prefactor scale with Λ/λ since it will diverge with $\Lambda \rightarrow \infty$.

7.3 Scale dependence and divergence of the flux:

7.3.1 Heuristics

Let us first consider some simple heuristics ([Schertzer *et al.*, 1993]), whose main interest is that they are model independent. They are based on the fact that a D -dimensional integration of a singularity γ just corresponds to shift the latter by $-D$, which corresponds to the scaling exponent of the elementary volume of integration. As a consequence, all singularities of order $\gamma < D$ will be smoothed out. This already explains why this question of statistical divergences is beyond the scope of deterministic-like multifractal formalisms (see Sect. 6.2). On the contrary those corresponding to $\gamma \geq D$ will not be smoothed out and therefore the scale of observation is irrelevant: the flux will scale with the inner scale of activity of the cascade and therefore will diverge with $\Lambda \rightarrow \infty$. However, this divergence may remain statistically

insignificant, due to its low statistical weight. Nevertheless, one may reach a critical $\gamma_D \geq D$ where the divergence becomes statistically significant. Above this critical singularity, the observed dressed codimension function $c_d(\gamma)$ does not correspond any longer to $c(\gamma)$: dressed quantities will have much larger fluctuations than the bare quantities. $c_d(\gamma)$ can be therefore estimated ([Schertzer, 2001]) by considering that $c_d(\gamma)$ should maximize the occurrences of high singularities, respecting nevertheless the convexity constraint. This means that $c_d(\gamma)$ should be the tangency of $c(\gamma)$ in γ_D :

$$\gamma < \gamma_D: c_d(\gamma) = c(\gamma_D); \quad \gamma \geq \gamma_D: c_d(\gamma) = c(\gamma_D) + q_D(\gamma - \gamma_D) \quad (62)$$

The divergence of the statistical moments for $q \geq q_D$ - q_D being the critical order corresponding to γ_D in the framework of the Legendre duality- results from the fact that a straight line is singular for the Legendre transform, therefore:

$$q < q_D: K_d(q) = K(q); \quad q \geq q_D: K_d(q) = \infty \quad (63)$$

7.3.2 Trace moments

The previous heuristics are secured by introducing ([Schertzer and Lovejoy, 1987a]) Trace Moments of the flux which are simpler to handle than the statistical moments of the flux. Indeed, the latter are rather complex since already for integer order $q > 1$, they correspond to a q -multiple D -dimensional integration:

$$\left\langle \left[\int_A \varepsilon_\lambda d^D \underline{x} \right]^q \right\rangle = \left\langle \int_A \varepsilon_\lambda(\underline{x}_1) d^D \underline{x}_1 \cdots \int_A \varepsilon_\lambda(\underline{x}_q) d^D \underline{x}_q \right\rangle \quad (64)$$

The "trace moments" are obtained by performing the same integration, but only over the "diagonal" $\Delta(A^q) = \{\underline{x}_1 = \underline{x}_2 \dots = \underline{x}_q\}$ of A^q , the domain of integration of Eq. (64), i.e.:

$$\mathbf{Tr}_A [(\varepsilon_\lambda)^q] = \int_A \langle \varepsilon_\lambda^q \rangle d^{qD} \underline{x} \quad (65)$$

This quantity, which is defined also for non-integer orders q (including negative orders), is rather easy to handle since it corresponds to a simple D -dimensional integration, and indeed, its scaling behavior is readily obtained:

$$\mathbf{Tr}_A [(\varepsilon_\lambda)^q] \sim \sum_{A_\lambda} \langle \varepsilon_\lambda^q \rangle \lambda^{-qD} = \sum_{A_\lambda} \lambda^{K(q)} \lambda^{-qD} \sim \lambda^{K(q)-(q-1)D} \quad (66)$$

and this yield a twin divergence rule for the trace moments (illustrated in Fig. 13):

$$\mathbf{Tr}_A \mathcal{E}_\infty^q = \begin{cases} 0 & \text{for } 1 < q < q_D \\ \infty & \text{for } q > q_D \text{ or } q < 1 \end{cases} \quad (67)$$

which results from the fact that due to the convexity of $K(q)$, the exponent $K(q) - D(q-1)$ (Eq. (66)) has only two zeroes corresponding respectively to $q=1$ (due to $K(1)=0$, which corresponds to the conservation of the density: $\langle \varepsilon_\lambda \rangle = 1$) and $q = q_D \geq 1$, where q_D will be shown below to correspond to the critical exponent of divergence discussed in Sect. 7.3.1. Indeed, we have the following inequalities between moments and trace-moments ([Schertzer and Lovejoy, 1987a]):

$$\langle \Pi_\lambda^q(A) \rangle \geq \mathbf{Tr}_A[\varepsilon_\lambda^q] \quad (q \geq 1) \quad (68)$$

$$\langle \Pi_\lambda^q(A) \rangle \leq \mathbf{Tr}_A[\varepsilon_\lambda^q] \quad (q \leq 1) \quad (69)$$

due to the convexity of the function $f(x) = |x|^q$ for $q \geq 1$ and its concavity for $q \leq 1$. We therefore obtain with the help of Eq. (69):

$$\left\{ \mathbf{Tr}_A \mathcal{E}_\infty^q = \infty \right\} \Rightarrow \langle \Pi_\infty(A) \rangle = \infty \quad (q \geq q_D) \quad (70)$$

which confirms that q_D is the critical order of divergence of moments as well as of the trace moments, since it is rather straightforward to check that when a divergence of moments occurs, its leading term corresponds to the trace-moment. On the other hand, Eq. (68) implies that:

$$\left\{ \langle \Pi_\infty(A)^q \rangle > 0 \right\} \Rightarrow \left\{ \mathbf{Tr}_A \mathcal{E}_\infty^q = \infty \right\} \quad (q \leq 1) \quad (71)$$

which means that the low-order divergence ($q=1$) of the trace-moments is indispensable in order to ensure that the multifractal process is non-degenerate, i.e. the bare process is too sparse to be observed in the space D and converges almost surely to 0.

7.4 Sample finite size effects

In practice we are able only to examine finite size samples, hence, instead of computing the theoretical moments,

$$\langle X^q \rangle = \int x^q dP_X \quad (5)$$

one only deals with estimates, the most usual ones being an average over the N_s independent samples

$$\{X^q\}_s = \frac{1}{N_s} \sum_{i=1}^{N_s} X_i^q \quad (6)$$

As long as the law of large numbers applies, these estimates usually converge ($N_s \rightarrow \infty$) towards the theoretical moments:

$$\langle X^q \rangle = \lim_{N_s \rightarrow \infty} \{X^q\}_s \quad (7)$$

One may also consider space/time averages and ergodicity assumptions. In our case, we will have to consider a combination of statistical and space/time averaging, in particular when estimating the trace moments (Sect. 7.3.2). A first consequence of finite N_s is that only a limited range of moment orders q 's can in fact be safely explored: as we will now show, estimates of moments (or of trace moments) of higher order give no real information about the process and may even lead to an erroneous understanding of the real statistics if this limitation is not taken into account ⁸.

The finite sampling limitation can be best understood with the help of the sampling dimension D_s (Sect. 2.2.6). Indeed, consider a sample consisting of N_s independent realizations, each of dimension D , each covering a range of scales λ . As we increase N_s , we gradually explore the entire probability space encountering extreme but rare events that would almost surely be missed on any finite sample (Fig 3). This corresponds to the fact that we are increasing the dimension of observation D to an (overall) effective dimension Δ_s , which may be quantified, with the help of the sampling dimension D_s (Eq.(24), $D_s = 0$ in case of a unique sample). The latter help us to determine the highest order singularity (γ_s) we are likely to observe on N_s realizations:

$$c(\gamma_s) = D + D_s = \Delta_s; \quad (72)$$

⁸Indeed, various authors have speculated on the significance of the $q \rightarrow \infty$ limit on the basis of finite empirical samples of turbulence data!

The Legendre transform of $c(\gamma) = c(\gamma_s)$ with $\gamma \leq \gamma_s$ leads to a spurious linear estimate K_s instead of the nonlinear K for $q > q_s$ where $q_s = c'(\gamma_s)$ is the maximum moment that can accurately be estimated:

$$q \geq q_s: K_s(q) = \gamma_s(q - q_s) + K(q_s), q < q_s: K_s(q) = K(q) \quad (73)$$

In Sect.7.5.2, we will show that this linear behavior corresponds to the analogue of a phase transition and therefore is rather model -independent.

7.5 Multifractal phase transitions

7.5.1 Fluxdynamics and thermodynamics

As discussed by different authors ([*Tel*, 1988], [*Schuster*, 1988], [*Schertzer*, 2001; *Schertzer and Lovejoy*, 1992]), there are strong analogies between multifractal exponents and standard thermodynamic variables. However, there are notable differences in viewpoints, depending on the chosen multifractal framework. Table 1 displays the analogies, within the codimensional multifractal formalism, between what can be called (statistical) "fluxdynamics", due to the fact that the quantity of main interest is a flux of energy, and the classical thermodynamics. We believe that these are easier to be obtained in a codimensional framework, since it originates from the analogies between the exponents of probability density and of number density, which define respectively the codimension $c(\gamma)$ of a singularity γ and the $D - S(E)$ entropy of a state energy E . The conjugate variable of the singularity and the energy for the Legendre transform corresponds respectively to the moment of order q and the (reciprocal) temperature $\beta = T^{-1}$, and the scaling moment function $K(q)$ is the analogue of a (Massieu) potential.

Discontinuities of the analogues of the free energy (the dual codimension function $C(q)$) and the thermodynamic potential ($K(q)$) can be understood as corresponding to multifractal phase transitions. However, there is a large difference between fluxdynamics and thermodynamics, the latter is related to systems in equilibrium and without dissipation, while the former corresponds to a system out of equilibrium and strongly dissipative. A practical consequence related to this distinction is that a multifractal process is fundamentally a system requiring an infinite hierarchy of temperatures, not a unique one, in order to define its statistics. Therefore *observing* a multifractal process at a given temperature yields only a very partial information,

and a multifractal phase transition corresponds rather to a qualitative change of *observation* of the *same* system when one changes the observation temperature, whereas a thermodynamic phase transition rather corresponds to a qualitative change of the system *behavior* under observation.

7.5.2 Second order phase transition

Sample finite size effects (Sect. 7.4) can be now understood as corresponding to a phase transition of second order and in fact a "frozen free energy" transition which have been discussed in various contexts [Derrida and Gardner, 1986], [Mesard et al., 1987], [Brax and Pechanski, 1991]. Indeed, we saw that the almost sure highest order singularity (γ_s) which can be observed on N_s realizations, yields with the help of the Legendre transform a linear behavior of the observed K_s (Eq. (73)) for $q > q_s$, whereas it is nonlinear as $K(q)$ for $q < q_s$. Therefore, K_s has a discontinuity of second order at q_s . On the other hand, this linear behavior implies that the observed analogue of the free energy $C_s(q)$ seems to be "frozen" for low temperature ($q \rightarrow \infty$), since we have:

$$C_s(q) \equiv \frac{K_s(q)}{q-1} \approx \gamma_s (1 + q^{-1} (1 - q_s / \gamma_s)) \quad (74)$$

Further to the heuristics derivation we have presented here, some exact mathematical results have been obtained, which are however restricted to discrete cascades and furthermore to $D_s = 0$. On the contrary, the notion of second order phase transition is interesting, because it is rather model-independent since based on the analogies of the statistical exponents of the cascade. Indeed, it should occur as soon as there are no bounds on the singularities or their range exceeds the critical γ_s .

7.5.3 First order phase transition

We can now revisit the question of the divergence of moments (Sect. 7.3) taking care now of the sample size finite effects, in the heuristic and very general framework we discussed in Sect. 7.3.1. We pointed out that above a critical singularity γ_D , the dressed codimension $c_d(\gamma)$ becomes linear (Eq. (62)). Due to the definition of the codimension (Eq. (41)), this

corresponds to a power-law for the probability distribution, and by consequence to a divergence of statistical moments. However, due to the finite size of the samples, one obviously cannot observe directly this divergence, but in fact a first order transition, instead of the second order transition discussed above (Sect.7.5.2). Indeed, following the argument for Eq. (72), the maximum observable dressed singularity $\gamma_{d,s}$ is the solution of:

$$c_d(\gamma_{d,s}) = \Delta_s. \quad (75)$$

By taking the Legendre transform of c_d with the restriction $\gamma_d \leq \gamma_{d,s}$, we no longer obtain the theoretical $K_d(q) = \infty$ for $q > q_D$, (Eq. (63)), but then obtain the finite sample dressed $K_{d,s}(q)$:

$$q \leq q_D: K_{d,s}(q) = K(q); \quad q \geq q_D: K_{d,s}(q) = \gamma_{d,s}(q - q_D) + K(q_D) \quad (76)$$

As expected, Eq. (63) is recovered for $N_s \rightarrow \infty$, due to the fact that $\gamma_{d,s} \rightarrow \infty$. For N_s large but finite, there will be a high q (low temperature) first order phase transition, whereas the scale breaking mechanism proposed for phase transitions in strange attractors ([Szépfalussy *et al.*, 1987]; [Csordas and Szépfalussy, 1989]; [Barkley and Cumming, 1990]) is fundamentally limited to high and negative temperatures (small or negative q). This transition corresponds to a jump in the first derivative $K'(q)$ of the potential analogue ([Schertzer *et al.*, 1993]):

$$\Delta K'(q_D) \equiv K'_{d,s}(q_D) - K'(q_D) = \gamma_{d,s} - \gamma_D = \frac{\Delta_s - c(\gamma_D)}{q_D} \quad (77)$$

On small samples ($\Delta_s \approx c(\gamma_D)$), this transition will be missed, the free energy simply becomes frozen and we obtain: $K_{d,s}(q) \approx (q-1)D$, which was already discussed with help of some experiments ([Schertzer and Lovejoy, 1984]), whereas Eq. (76) corresponds to an improvement of earlier works on "pseudo scaling" ([Schertzer and Lovejoy, 1984; 1987a]). Note that the above relations, especially Eq. (77) were tested numerically with the help of lognormal universal multifractals ([Schertzer *et al.*, 1993; 1994]).

7.5.4 The big image: hard and soft multifractal phases

Now, we can display the different multifractal phases in the (q^{-1}, D) plane where q is the order of the statistical moment and D the dimension of space, which is also the integration dimension

yielding the dressed quantities⁹. The latter is rather the analogue of an external field h , since it has a smoothing role as for instance a magnetic field applied to an antiferromagnet. In the latter case, by increasing the magnetic field one may succeed in preventing this inflation of the microscopic world, maintaining a finite border line down to a transition temperature (the Néel temperature) lower than the (zero-field) critical T_c . Therefore, the transition lines delineating the phases in the (q^{-1}, D) plane (Fig. 14) are quite similar to the (T, h) transition lines of an antiferromagnet [*Coniglio and Stanley*, 1986], [*Nagamiya et al.*, 1955].

The transition line (q_D^{-1}, D) corresponds to the first order transition (Sect. 7.5.3) which separates the "soft" and "hard" phases. These phases are rather the respective analogues of the disordered and ordered phases. The soft phase corresponds to the common sense presupposition that the flux will converge without any sensitivity to the small scale activity, i.e. that cutting off hidden fluctuations/interactions involving scale ratio larger than λ does not induce major changes, i.e. there is no significant difference between bare and dressed properties. This soft phase is the analogue of a classical disordered phase, since each sub-domain of integration A of same scale ratio λ gives rather similar contributions.

But there is the possibility of a hard phase in which on the contrary small scale activity cannot be ignored: it becomes fundamental to distinguish between the bare (theoretical) and dressed (observed) fields. The contribution to the flux by the sub-domains can be quite uneven, rather in analogy to a classical ordered phase, some of them can yield overwhelming contributions thereby creating dominant large-scale structures. As we discussed it (Sect. 7.3.1), this corresponds to the fact that the space/time integration is not able to impose its own scale ratio λ and that the effective scale ratio is the (divergent) scale ratio of the process itself $\Lambda \rightarrow \infty$.

The critical transition line (q_D^{-1}, D) ends at the critical point $(1, 0)$ after a sharp vertical bend at the point $(1, C_1)$. This bend arises because when D is smaller than the codimension C_1 of the mean of the process, the mean of the D -dimensional intersection ($D_1 = D - C_1$) has an apparent negative dimension. Any D -dimensional observation will therefore almost surely have huge fluctuations before collapsing to a null process. The very singular statistics corresponding to this "degeneracy" are the following: while the mean of the process is kept constant and finite,

⁹Despite a slight complication in notations, it is rather straightforward to consider two distinct dimensions.

simultaneously, all moments of order $q > 1$ diverge to infinity while those < 1 converge to zero. The analytical continuation of the transition line $(1, D)$ for $D > C_1$ corresponds to the divergence not of the moments of the flux (implied by the divergence of the trace moments), but only to the divergence of the trace moments (see Sect. 7.3.2). Therefore, the second continuation \bar{q}_D^{-1}, D indicated in Fig. 14 for $q < 1$ remains the separation of the finite and infinite trace moments however the latter no longer imply divergence or convergence of the fluxes. The empirical evidence of these distinct phases is reviewed by [Schertzer, 2001].

8 Generalized Scale Invariance (GSI)

We now show that all the previous results can be extended in a rather straightforward manner to strongly anisotropic processes, whereas the usual approach to scaling is first to posit (statistical) isotropy and only then scaling, the two together yielding self-similarity. Indeed this approach is so prevalent that the terms scaling and self-similarity are often used interchangeably! Perhaps the best known example is Kolmogorov's hypothesis of "local isotropy" from which he derived the $k^{-5/3}$ spectrum for the wind fluctuations. The GSI approach is rather the converse: it first posits scale invariance (scaling), and then studies the remaining non-trivial symmetries. For instance, Fig. 15 gives a (scaling) anisotropic version of the isotropic cascade scheme (Fig. 4). One may easily check that this type of anisotropy—which reproduces itself from scale to scale—does not introduce any characteristic scale. The straightforward generalization of scaling shown in fig. 4 involving scaling anisotropy in fixed direction is called “self-affinity”. As far as we know this anisotropic scheme ([Schertzer and Lovejoy, 1983; Schertzer and Lovejoy, 1985a]) seems to be the first explicit model of a physical system involving a fundamental self-affine fractal mechanism.

GSI corresponds to the fact that the contraction operator T_λ , which was introduced in Sect.6.2 (Eq. (55)), is no longer an isotropic contraction: $T_\lambda(\underline{x}) = \underline{x} / \lambda$. Linear GSI corresponds to the fact that T_λ is a linear one-parameter group; i.e. it admits a linear generator G distinct from the identity, which generates to isotropic contractions:

$$T_\lambda(\underline{x}) = \lambda^G \equiv \exp[\text{Log}(\lambda)G] \quad (78)$$

One can define a generalized notion of scale ([Schertzer and Lovejoy, 1984; 1985b; 1987b; Schertzer et al., 1997; 1999]), associated to the one-parameter (linear) contraction T_λ , which satisfies the following:

- *nondegeneracy*, i.e. :

$$\|\underline{x}\| = 0 \Leftrightarrow \underline{x} = \underline{0} \quad (79)$$

linearity with the contraction parameter $1/\lambda$, i.e.:

$$\|T_\lambda \underline{x}\| = \|\underline{x}\| / \lambda \quad (80)$$

- *Balls defined by this scale are strictly decreasing*, i.e.

$$\forall \lambda, \lambda' \in \mathbb{R}^+ : \lambda \geq \lambda' \Rightarrow B_\lambda \subset B_{\lambda'} \quad (81)$$

where the balls B_λ defined by the contraction T_λ satisfy:

$$B_\lambda = \{\underline{x} \mid \|\underline{x}\| \leq L / \lambda\} \quad (82)$$

It is important to note that the scaling of the volume of the balls B_λ , defines an effective dimension D_{el} , which has been called 'elliptical dimension' in reference to the shape of the balls under GSI contraction. It is indeed an invariant merely defined with the help of the Jacobian of the contraction:

$$\det[T_\lambda] = \lambda^{-D_{el}} \quad (83)$$

and due a well-known matrix identity, it corresponds to the trace of its generator:

$$D_{el} = Tr(G) \quad (84)$$

It is straightforward to check that the usual Euclidean norm $|\underline{x}| = (\sum_i x_i^2)^{1/2}$ of a metric space is the scale associated to the isotropic dilation ($G = 1$) and $D_{el} = D$. On the other hand, whereas the two first properties are rather identical to those of a norm, the last one is weaker than the triangular inequality, which is required for a norm.

The conditions of existence of a generalized scale should depend on the generator G of the one-parameter group of (linear) contractions. Indeed we have the following property ([Schertzer and Lovejoy, 1985b, Schertzer et al., 1999]):

- let the unit ball defined as an ellipsoid generated by a positive symmetric matrix A :

$$B_1 = \{\underline{x} \mid (\underline{x}, A\underline{x})^{1/2} \leq L\} \quad (85)$$

- The contraction group T_λ defines a generalized scale, if and only if: its generator G satisfies:

$$\text{Spec}(\text{sym}(AG)) > 0 \quad (86)$$

where $\text{Spec}(\cdot)$ and $\text{sym}(\cdot)$ denote respectively the spectrum and the symmetric part of a linear application. One can show furthermore that when the unit ball is an ellipsoid defined by a positive symmetric matrix A , which belongs to a given neighborhood of a scalar linear application, i.e. $A = \mu 1$, $\mu \in R^+$, the dilation group T_λ defines a generalized scale, if and only if its generator G satisfies:

$$\text{Spec}(\text{sym}(G)) > 0 \Leftrightarrow \text{Re}(\text{Spec}(G)) > 0 \quad (87)$$

9 Conclusion

The codimension multifractal formalism that we introduced in this course, was initially developed, and discussed with respect to a scalar stochastic measure, e.g. the turbulent flux of energy, which has a fundamental property of conservation (on ensemble average). We limited our introduction to this question, which already requires many theoretical developments.

However, in turbulence directly observable quantities are rather the (vector) velocity field or the temperature field. Both of them are non-conservative, i.e. their ensemble averages have scaling laws. Let us point out that a rather straightforward extension to (scalar) non-conservative fields corresponds to Fractionally Integrated Flux models (FIF, [Schertzer et al., 1997]). On the other hand, we briefly mentioned that the scalar cascade framework is insufficient to deal with the vector nature of turbulence, but can be extended to 'Lie cascades' framework [Schertzer and Lovejoy, 1995].

Let us finally mention the important question of multifractal space-time processes, which can be approached ([Marsan *et al.*, 1996]) by combining the General Scale Invariance notions (Sect. 8, in order to take into account the strong scaling anisotropy between time and space) within the Fractionally Integrated Flux models, by taking care of the causality with the help of causal Green functions.

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Appendix A

A.1) A General Framework

One considers a sequence of events A_λ ($A_\lambda \in F$, where F is a σ -field of events of a probability space (Ω, F, P) with probability P), where the parameter $\lambda \rightarrow \infty$, will in fact correspond to the (finer and finer) resolution of a stochastic process. The resolution is related a notion of scale defined on the embedding (topological) set E where the stochastic process $X_\lambda(\omega)$ is valued, and to its Borelian σ -field $B(E)$. In the simple cases E is a bounded subset of R^n and $X_\lambda(\omega)$ corresponds to a geometric subset of points (e. g. a (random) fractal set at resolution λ), but in the most interesting cases E is a functional space and $X_\lambda(\omega)$ is a random field or random measure (e. g. the energy flux at resolution λ). The resolution λ is in general related to a scale $\ell = \frac{L}{\lambda}$ (L being the outer scale) of homogeneity, either of observation (e.g. fluctuations are not estimated below this scale) or in simulations (the fluctuations are not computed below this scale).

It is rather convenient to use the notation \Pr for probability on any space, either on the original σ -field F (i.e. P) or its image on the Borelian σ -field $B(E)$. (i.e. P_X).

In the simplest case, the sequence of events A_λ is defined by a discrete iterative process (e.g. a discrete cascade process), and the λ 's follow a power law:

$$\lambda = \lambda_n = \lambda_1^n, n \in N, \lambda_1 = \text{Const.} > 1. \quad (\text{A.1})$$

In general, F is in fact defined by a filtration, i.e. it contains the σ -field generated by an increasing family of σ -fields F_λ :

$$\sigma\left(\bigcup_{\lambda} F_\lambda\right) \subseteq F \quad (\text{A2})$$

and $A_\lambda \in F_\lambda$ is defined by a process $X_\lambda(\omega)$ adapted to this filtration.

The asymptotic behavior indicated by Eq. 6 is more precisely defined by:

$$\lim_{\lambda \rightarrow \infty} \frac{\log \Pr(A_\lambda)}{\log \lambda} = -c \quad (\text{A.3})$$

if this limit exists. For heuristic reasons, c corresponds to a notion of (statistical) codimension.

Indeed, loosely speaking a probability corresponds to the frequency of events:

$$\Pr(A_\lambda) \approx \frac{\#(A_\lambda)}{\#(A_\lambda \text{ and } A_\lambda^c)} \quad (\text{A.4})$$

and each number (denoted by #) of events scales like a power of a dimension, therefore the probability scales like a power of a difference of dimension, i.e. a codimension (this will be more discussed in Sect. ??)

A.2) Existence of a codimension

The limit c in Eq.A3 exists when the upper and lower limits, which are always defined on $[0, \infty]$:

$$\limsup_{\lambda \rightarrow \infty} \frac{\log \Pr(A_\lambda)}{\log \lambda} = \downarrow \limsup_{\lambda \rightarrow \infty} \frac{\log \Pr(A_\Lambda)}{\log \Lambda} \equiv -\bar{c} \quad (\text{A5})$$

$$\liminf_{\lambda \rightarrow \infty} \frac{\log \Pr(A_\lambda)}{\log \lambda} = \uparrow \liminf_{\lambda \rightarrow \infty} \frac{\log \Pr(A_\Lambda)}{\log \Lambda} \equiv -\underline{c} \quad (\text{A6})$$

are equal, whereas generally $\underline{c} \geq \bar{c}$. \underline{c} , \bar{c} can be called respectively the upper and lower codimensions of the sequence of events $\{A_\lambda\}$.

However, more general results can be obtained by considering the following monotonous sequence of events $\{\bar{A}_\lambda\}, \{\underline{A}_\lambda\}$ (respectively the upper and lower sequences of $\{A_\lambda\}$) and their corresponding limits \bar{A}, \underline{A} :

$$\bar{A}_\lambda \equiv \bigcup_{\Lambda \geq \lambda} A_\Lambda : \bar{A} \equiv \downarrow \lim_{\lambda \rightarrow \infty} \bar{A}_\lambda \quad (\text{A7})$$

$$\underline{A}_\lambda \equiv \bigcap_{\Lambda \geq \lambda} A_\Lambda : \underline{A} \equiv \uparrow \lim_{\lambda \rightarrow \infty} \underline{A}_\lambda \quad (\text{A8})$$

\bar{A}, \underline{A} correspond respectively to the upper limit of the A_λ 's, i.e. the infinitely often A_λ , and their lower limit, the set of eventually all A_λ (i.e. for large enough λ).

Proposition 1:

We have the following sufficient conditions of existence of a codimension for a (discrete) sequence of events $\{A_\lambda\}$, λ 's being a discrete increasing sequence on R^+ (generally defined by Eq. (A.1):

- (a) if $\{A_\lambda\}$ is a non increasing sequence of events, then it has a well defined codimension c ,
- (b) the upper sequence $\{\bar{A}_\lambda\}$; generated by $\{A_\lambda\}$ with the help of Eq.(A7), has a well defined codimension $c(\{\bar{A}_\lambda\})$, which is a lower bound of the lower codimension \bar{c} (as well of the codimension c , if the latter exists),
- (c) the lower sequence $\{\underline{A}_\lambda\}$; generated by $\{A_\lambda\}$ with the help of Eq.(A8), has a well defined upper codimension $\underline{c}(\{\underline{A}_\lambda\})$, which is an upper bound of the upper codimension \underline{c} (as well of the codimension c , if the latter exists),
- (d) if $\{A_\lambda\}$ has a well defined set limit A , then it has a well defined codimension :

$$c = c(\{\bar{A}_\lambda\}) = \underline{c}(\{\underline{A}_\lambda\}) \quad (\text{A9})$$

- (e) if c exists and is positive and the λ 's follow a power law (Eq.A1), $c(\{\bar{A}_\lambda\})$ is also (strictly) positive.

Proof:

Proposition (1,a) results from the monotone convergence property of the probability measure. Indeed, the sequences $\Pr(A_\lambda)$ and $\text{Log} \Pr(A_\lambda) / \text{Log} \lambda$ are both monotonously non increasing and therefore the following two limits are always defined (respectively on $[0,1]$ and on $[0, \infty]$):

$$\Pr(A) = \lim_{\lambda \rightarrow \infty} \downarrow \Pr(A_\lambda) \quad (\text{A10})$$

$$c \equiv - \downarrow \lim_{\lambda \rightarrow \infty} \frac{\log \Pr(A_\lambda)}{\log \lambda} \quad (\text{A11})$$

Due to the fact that the sequence $\{\bar{A}_\lambda\}$, as defined by Eq. (A7), is non increasing, Proposition (1.a) implies that the following codimension is always defined on $[0, \infty]$:

$$c(\{\bar{A}_\lambda\}) \equiv - \downarrow \lim_{\lambda \rightarrow \infty} \frac{\log \Pr(\bar{A}_\lambda)}{\log \lambda} \quad (\text{A12})$$

We will demonstrate that:

$$\bar{c} \geq c(\{\bar{A}_\lambda\}) \quad (\text{A13})$$

Indeed, we have also:

$$\Pr(\bar{A}_\lambda) \geq \sup_{\Lambda \geq \lambda} \Pr(A_\Lambda) \quad (\text{A14})$$

as well as:

$$\frac{\text{Log} \Pr(\bar{A}_\lambda)}{\text{Log} \lambda} \geq \sup_{\Lambda \geq \lambda} \frac{\text{Log} \Pr(A_\Lambda)}{\text{Log} \Lambda} \quad (\text{A15})$$

Therefore:

$$\downarrow \lim_{\lambda \rightarrow \infty} \frac{\text{Log} \Pr(\bar{A}_\lambda)}{\text{Log} \lambda} \geq \downarrow \lim_{\lambda \rightarrow \infty} \sup_{\Lambda \geq \lambda} \frac{\Pr(A_\Lambda)}{\text{Log} \Lambda} = \lim_{\Lambda \rightarrow \infty} \sup \frac{\Pr(A_\Lambda)}{\text{Log} \Lambda} \quad (\text{A16})$$

which corresponds to the announced inequality (Eq.A13) and therefore demonstrates Proposition (1.b).

The monotone convergence property of the probability measure still imply that the sequences $\Pr(\underline{A}_\lambda)$ and sequence $\text{Log} \Pr(\underline{A}_\lambda)$ are monotone non decreasing and:

$$\Pr(\underline{A}) = \uparrow \lim_{\lambda \rightarrow \infty} \Pr(\underline{A}_\lambda) \quad (\text{A17})$$

However, it does not imply that the sequence $\text{Log} \Pr(\underline{A}_\lambda) / \text{Log} \lambda$ is monotone non-decreasing. Nevertheless, the following codimension is always defined:

$$\underline{c}(\{\underline{A}_\lambda\}) \equiv - \liminf_{\lambda \rightarrow \infty} \frac{\text{Log} \Pr(\underline{A}_\lambda)}{\text{Log} \lambda} \quad (\text{A18})$$

and we have (note the difference with Eq. A14):

$$\Pr(\underline{A}_\Lambda) \leq \Pr(A_\Lambda) \quad (\text{A19})$$

as well as:

$$\inf_{\Lambda > \lambda} \frac{\Pr(\overline{A}_\Lambda)}{\text{Log} \Lambda} \leq \inf_{\Lambda \geq \lambda} \frac{\Pr(A_\Lambda)}{\text{Log} \Lambda} \quad (\text{A20})$$

Therefore:

$$\uparrow \lim_{\lambda \rightarrow \infty} \inf_{\lambda' \geq \lambda} \frac{\text{Log} \Pr(\overline{A}_\Lambda)}{\text{Log} \Lambda} \leq \uparrow \lim_{\lambda \rightarrow \infty} \inf_{\Lambda \geq \lambda} \frac{\Pr(A_\Lambda)}{\text{Log} \Lambda} = \liminf_{\Lambda \rightarrow \infty} \frac{\Pr(A_\Lambda)}{\text{Log} \Lambda} \quad (\text{A21})$$

and by consequence:

$$\underline{c} \leq \underline{c}(\{A_\lambda\}) \quad (\text{A22})$$

This demonstrates Proposition (1.c). Finally, Proposition (1.d) results from (1.b) and (1.c), as well as the fact that if the limit set limit A is defined, then:

$$A = \overline{A} = \underline{A} \quad (\text{A23})$$

One may note that (1a) is in fact a particular case of (1.d).

Proposition (1.e) is a straightforward consequence of the first Borel Cantelli lemma, since:

$$\sum_n \Pr(\overline{A}_{\lambda_n}) \approx \sum_n \lambda_n^{-c} < \infty \Rightarrow \Pr(\overline{A}) = 0 \Rightarrow c(\{\overline{A}_{\lambda_n}\}) > 0 \quad (\text{A24})$$

A.3) Consequences

A.3.1 Some rather general consequences:

Propositions (1.a) or (1.d) are useful for the simplest cases, for instance for geometrical set defined recursively, in particular when it corresponds to a repeated truncation, since Proposition 1.a is then relevant. This is in particular the case for the Cantor set discussed in the main text.

On the contrary, the slightly more involved Proposition (1.b) is always relevant. It implies that, given a sequence $\{A_\lambda\}$ (in general, this sequence is chosen for some physical reasons), we have in general to consider its upper sequence $\{\overline{A}_\lambda\}$. Indeed, the latter has always a well defined codimension, whereas it is not always the case for the original sequence, and at the same time it is in general physically relevant, since its limits corresponds to the event of infinitely often the original events.

A.3.2 Intersections and covers with balls

In general, the notion of scale defined on the embedding space E with the help of balls B_λ of size $\ell = \frac{L}{\lambda}$ (e.g. the non-classical notion of scale will be defined in Sect.8), which generate its topology and are in general translation invariant. It is therefore rather important to evaluate the statistics of intersection of these balls (of finer and finer resolution) with a given (measurable) subset G of the embedding set E . Indeed in of Sect. 2.2.2, we propose to consider a rather generic way (Eq. 8) to defining a sequence of events $\{A_\lambda\}$ (belonging to the σ -field $(S(E))$) as the intersection of a sequence of $\{B_\lambda\}$ of balls of decreasing size $\ell = \frac{L}{\lambda}$ (e. g. λ 's constitute a power law sequence, as in Eq. A1):

$$A_\lambda = B_\lambda \cap G \quad (\text{A25})$$

and whose centers $x_{c,\lambda}$ are identically and independently distributed according to the uniform distribution (or to the Poisson distribution, if E is not bounded) with respect to the Lebesgue measure of the embedding space E . Furthermore, if G is random, its distribution is independent of the center distribution. As a rather straightforward example, let us take $G = \{x\}$, i.e. a given single point, therefore of zero dimension and $d = \dim E$ (geometrical) codimension:

$$\Pr(\{x\} \cap B_\lambda(x_{c,\lambda})) = \Pr(\{x_{c,\lambda}\} \cap B_\lambda(x)) = \lambda^{-d} \Pr(\{x\} \cap B_1(x_{c,1})) \quad (\text{A26})$$

where $B_\lambda(x)$ is deterministic, contrary to $B_\lambda(x_{c,\lambda})$. Therefore the statistical codimension is equal to the geometrical codimension. One obtains a similar agreement, when on the contrary G is d -dimensional and has a zero codimension. Indeed;

$$\Pr(G \cap B_\lambda) \approx \Pr(\{x_{c,\lambda}\}) \propto \int_G d^d x \quad (\text{A27})$$

Note that similar results hold for a Poisson distribution of ball centers when considering elementary volumes, i.e. the asymptotic limit $\lambda \rightarrow \infty$. In respect to the heuristic arguments discussed in Sect. 2.2.5, let us point out that more rigor can be obtained by refining the above arguments (Eq. A27) and by considering the small-scale power-law of the d -dimensional Hausdorff measure of G , when G is no longer d -dimensional.

A.3.3 Covers with balls

The upper sequence $\{\bar{B}_\lambda\}$ generated by the sequence $\{B_\lambda\}$ corresponds to a (partial) random cover of the embedding space. In turn, by intersection with a given subset G , they generate a sequence of (partial) random cover $\{\bar{A}_\lambda\}$ of the set G :

$$\bar{A}_\lambda = \bar{B}_\lambda \cap G; \quad \bar{A} = \bar{B} \cap G \quad (\text{A28})$$

we already discussed in Sect. A.3.3 on the physical and mathematical interest to consider the upper sequence $\{\bar{A}_\lambda\}$ instead of the original one $\{B_\lambda \cap G\}$. In the present case, we are shifting our interest from random intersection to random cover. It is important to estimate how dense or sparse is this cover. This obviously depends on the rate of decrease of the size of balls B_λ . Indeed, due to Proposition 1, we have in general:

$$c(\{\bar{B}_\lambda\}) \leq c(\{B_\lambda\}) = d \quad (\text{A29})$$

where the right hand side equality corresponds to Eq. A26. On the other hand, due to the independence of these balls, both Borel Cantelli lemma apply:

$$\sum_n \lambda_n^{-d} < \infty \Rightarrow \Pr(\bar{B}) = 0 \Rightarrow c(\{\bar{B}_{\lambda_n}\}) > 0 \quad (\text{A30})$$

$$\sum_n \lambda_n^{-d} = \infty \Rightarrow \Pr(\bar{B}) = 1 \Rightarrow c(\{\bar{B}_{\lambda_n}\}) = 0 \quad (\text{A31})$$

In case of a power law discretization of the λ 's (Eq.A1), $\{\bar{B}_\lambda\}$ corresponds to a sparse cover (Eq.A30). Nevertheless, interesting problems arise when one densifies the process, i.e. considers the limit $\lambda_1 \rightarrow 1_+$ (see Sect. 5.3).

Appendix B

B.1) Numerical implementation of Universal Multifractal

This numerical implementation has been discussed with some details by [Wilson et al. 1991, Pecknold et al. 1995, Tchiguirinskaia et al. 2000]. However, it turn out that is rather indispensable to have an adequate large wave- number cut-off in the Fourier space . In this appendix, we will focus on this question. Indeed, one needs to perform, in the physical space, a convolution of a Lévy white noise $\gamma(\underline{x})$ (the subgenerator) of Lévy stability index α by a power-law Green function:

$$G(x) \propto \|x\|^{-d/\alpha} \quad (\text{B1})$$

both being taken with a given (finite) resolution λ . This corresponds in the Fourier space to a multiplication of a Lévy noise $\hat{\gamma}(k)$ (the hat denotes a Fourier transform, and k is the wave vector) by the dual power law:

$$\hat{G}(k) \propto \|k\|^{-d/\alpha} \quad (\text{B2})$$

with $\frac{1}{\alpha} + \frac{1}{\alpha'} = 1$. The main question is how to define the resolution, since a sharp cut-off in the Fourier space -i.e. multiplication by the characteristic function of the sphere $S_\lambda = \{k \mid 1 \leq \|k\| \leq \lambda\}$ - will generate convolutions with Bessel functions in the physical space. A rule of thumb is to introduce an exponential cut-off. It turns out that there is a rather convenient way to do it in the physical space with the help of "Cauchy wavelet", which is a misnomer since the physical space function is not localized as a wavelet should be. Let us consider for simplicity the 1D case, due to the fact that a translation in the physical space corresponds to phase shift in the Fourier space:

$$G_a(x) = G(x+a) \Rightarrow \hat{G}_a(k) = e^{-ika} \hat{G}(k) \quad (\text{B3})$$

it suffices to take a pure complex number, e.g. $a = i\xi$, and consider the real part of the inverse Fourier transform to obtain the desired exponential cut-off in the Fourier space.

B.2) A Mathematica package for 2D Universal Multifractals simulation:

In order to illustrate how easy it is to implement an algorithm for generating Universal Multifractals, we display the few lines contained in a corresponding Mathematica package:

```
<< Statistics`ContinuousDistributions`
```

```
<< Statistics`DescriptiveStatistics`
```

```
<< Graphics`Graphics`
```

```
Needs["Graphics`Animation`"]
```

```
Needs["Graphics`ImplicitPlot`"]
```

Below are the statistical distribution definitions.

```
nuni = UniformDistribution[-Pi/2, Pi/2]
```

`\!(UniformDistribution[(-\([Pi]/2)\), \([Pi]/2])`

this gives a uniform between $-\pi/2$ and $+\pi/2$

`ndist[lam_] := ExponentialDistribution[lam]`

`RanE[lam_] := Random[ndist[lam]]`

RanE gives an exponential random deviate lam is mean

`Levy[\[Alpha]_] :=`

`Module[{\[Phi], \[Phi]0}, {\[Phi] =`

`Random[nuni], \[Phi]0 = -(\pi/2) (1 - Abs[1 - \[Alpha]])^\[Alpha];`

`Sign[\[Alpha] -`

`1] Sin[\[Alpha] (\[Phi] - \`

`\[Phi]0)] Cos[\[Phi]^(1/\[Alpha])(Cos[\[Phi] - \[Alpha] (\[Phi] - \`

`\[Phi]0)]/RanE[1])^\[Alpha]]`

This gives a unit Levy r.v.

`epsmodule[lam_, \[Alpha]_, C1_, H_] :=`

`Module[{sin, kalpha, kH, gen, sgen,`

`ep}, {sin =`

`Table[(1 - I ({x, y}.{x, y})), {x, (-lam/2 + 1),`

`lam/2}, {y, (-lam/2 + 1), lam/2}],`

`kalpha = Re[sin^(1/\[Alpha])], kH = Re[sin^-((2 - H)/2)],`

`sgen = ((C1 Log[`

`lam]/((Abs[\[Alpha] - 1) lam^2 Mean[`

`Flatten[kalpha^\[Alpha]]))^\[Alpha]) Table[`

`Levy[\[Alpha]], {lam}, {lam}],`

`gen = ListConvolve[kalpha, sgen, 1], ep = E^gen};`

`ListConvolve[kH, ep, 1]/Mean[Flatten[ep]]]`

epsmodule calculates the fractionally integrated multifractal, of order H. It uses microcanonical normalization.

`topotest2 = epsmodule[2^6, 1.9, 0.12, 0.5];`

ListPlot3D[topotest2, PlotRange -> All]

\[SkeletonIndicator]SurfaceGraphics

11 Tables

<u>Flux Dynamics</u>	<u>Thermodynamics</u>
probability space	phase space
moment order: q	(reciprocal) temperature: $\beta = T^{-1}$
singularity order: γ	(negative) energy: $-E$
generator	(negative) Hamiltonian
singularity codimension: $c(\gamma)$	codimension of entropy: $D - S(E)$
scaling moment function: $K(q) = \max_{\gamma} (q \cdot \gamma - c(\gamma))$	(negative) Massieu potential: $-\Sigma(\beta) = -\min_E (\beta E - S(E))$
dual codimension function: $C(q) = \frac{K(q)}{q-1}$	(negative) free energy $-F(\beta) = -\Sigma(\beta) / \beta$
dimension of integration: D	external field: h
ratio of scales: λ	correlation length: ξ

Table 1 Correspondence between fluxdynamics and thermodynamics (setting for notation simplicity $k_B = 1$ for the Boltzman's constant k_B): $\Sigma(\beta)$ being the Massieu potential (e.g. [Balian, 1987]) $F(\beta)$ the Helmholtz free energy.(from [Schertzer and Lovejoy, 2001])

12 Figures

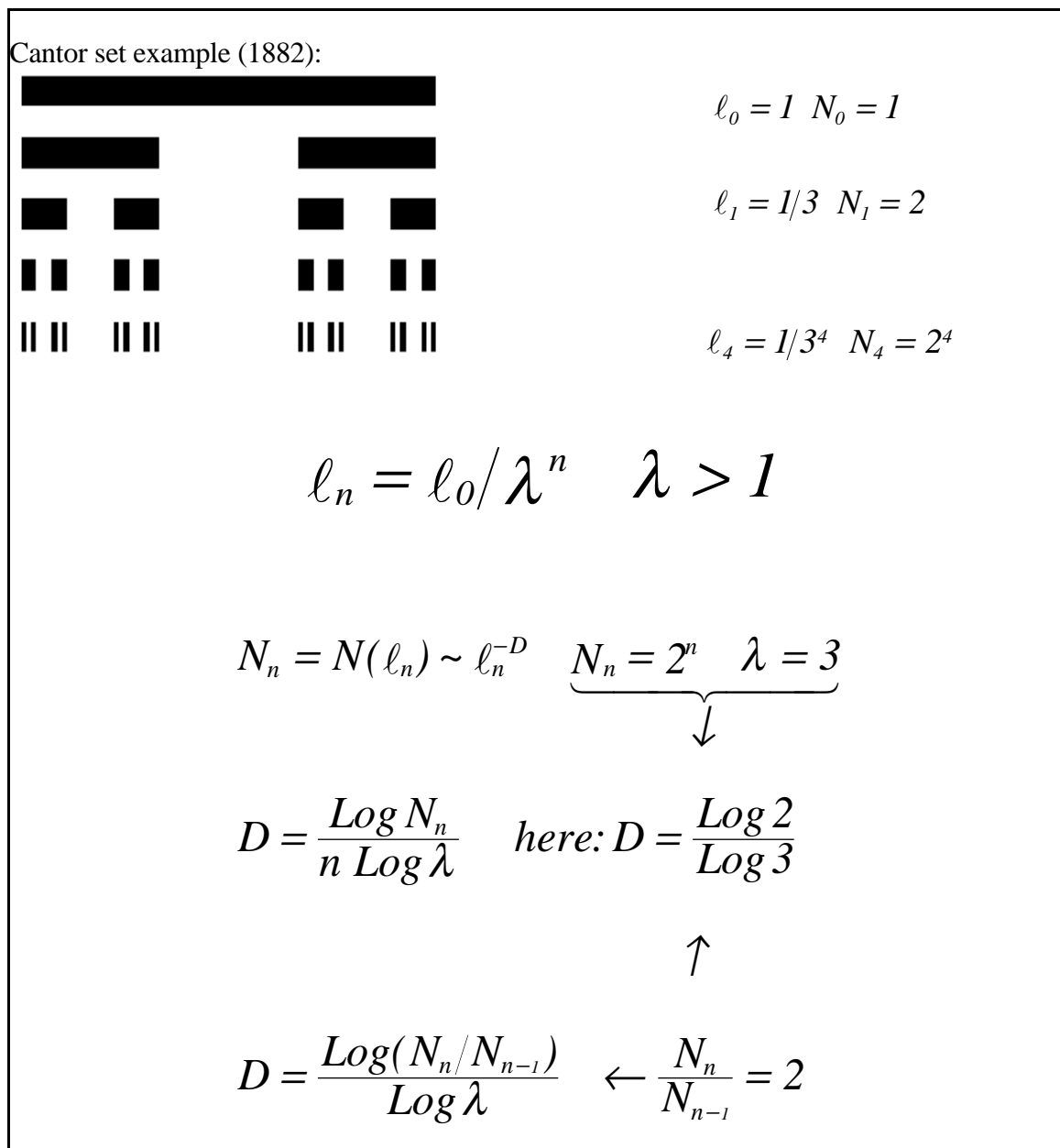


Fig. 1 Summary of Cantor set. (From [Schertzer and Lovejoy, 1993]).

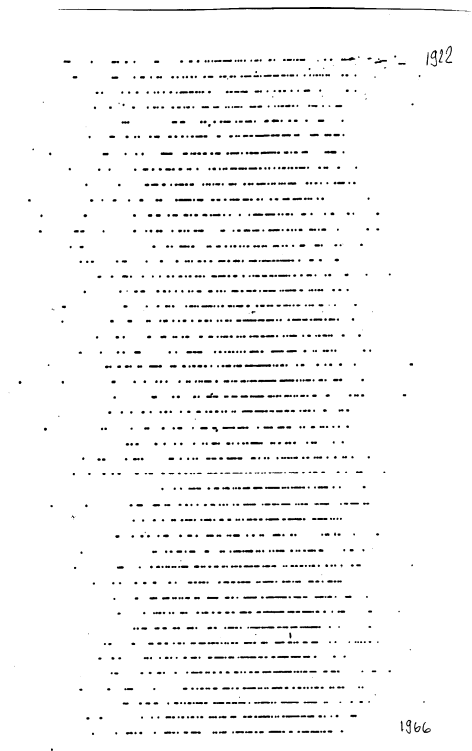


Fig. 2: Rainfall data from Dedougou for a period of 45 years. Each line is a different year, each point a rainy day (From [*Hubert and Carbonnel, 1989*]).

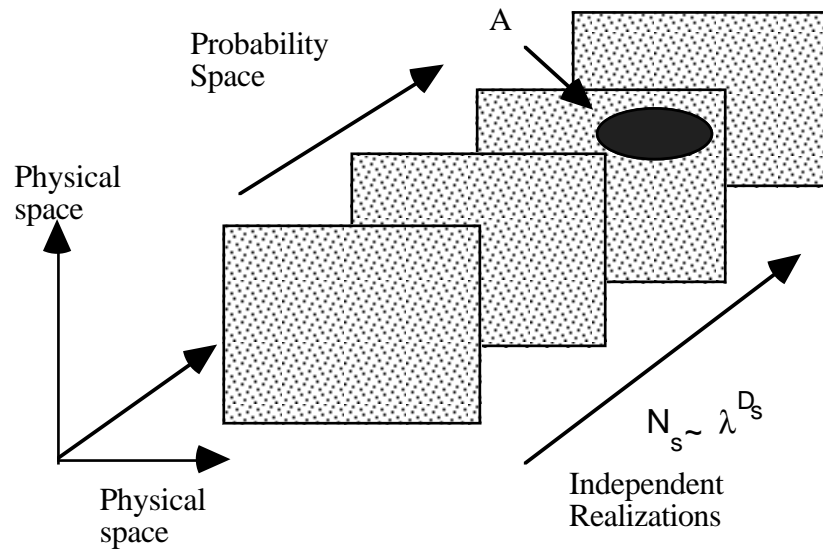
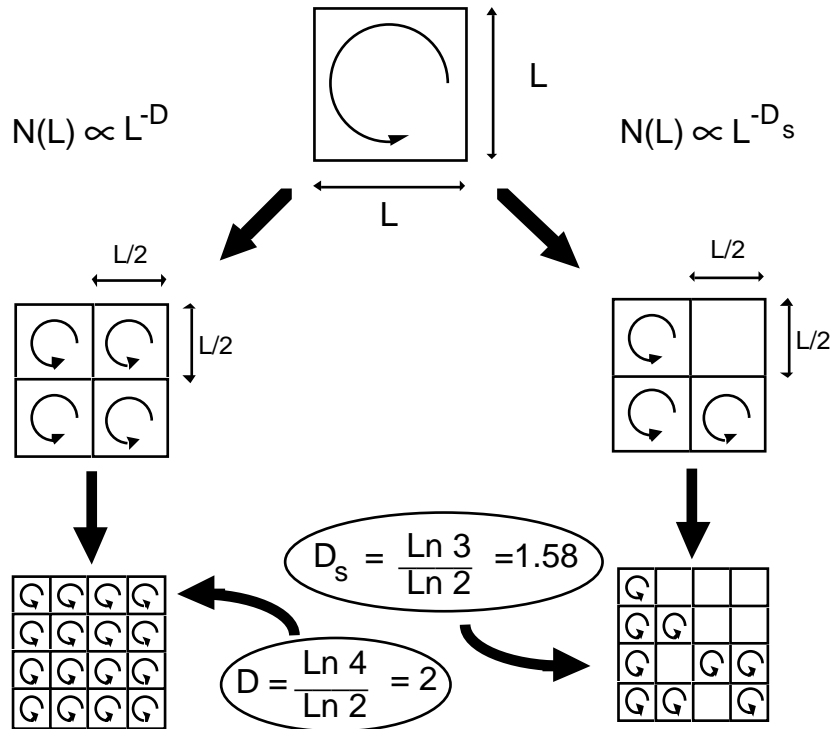


Fig. 3 Illustration showing how in random processes the effective dimension of space (D) can be augmented by considering many independent realizations (N_s). As $N_s \rightarrow \infty$, the entire (infinite dimensional) probability space is explored. (From [Schertzer and Lovejoy, 1993]).

ISOTROPIC
= self similarity



4 Isotropic Cascade. The left hand side shows an non-intermittent (“homogeneous”) cascade, the right hand side shows how intermittency can be modeled by assuming that not all sub-eddies are “alive”. This is an implementation of the “ β -model”. (From [Schertzer and Lovejoy, 1993]).

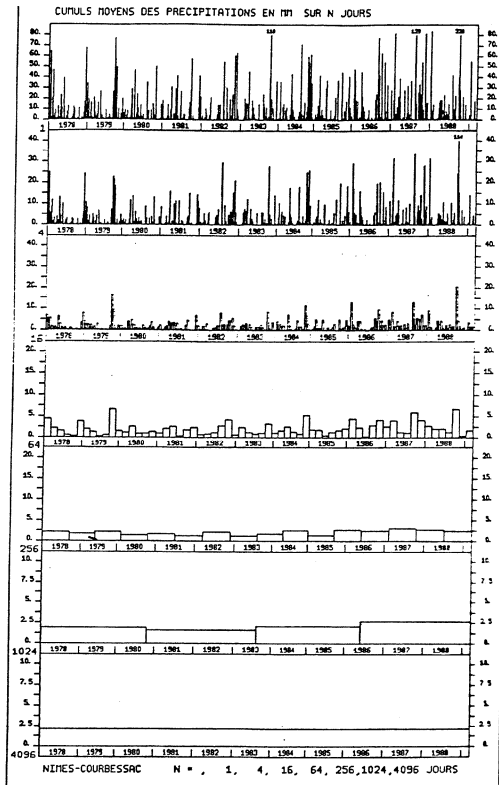


Fig. 5 Rain rate at Nîmes (France) for years 1978 to 1988 and averaged over time scale varying from (top to bottom): 1, 4, 16, 64, 256, 1024 and 4096 days. (From [Ladoy et al., 1993])

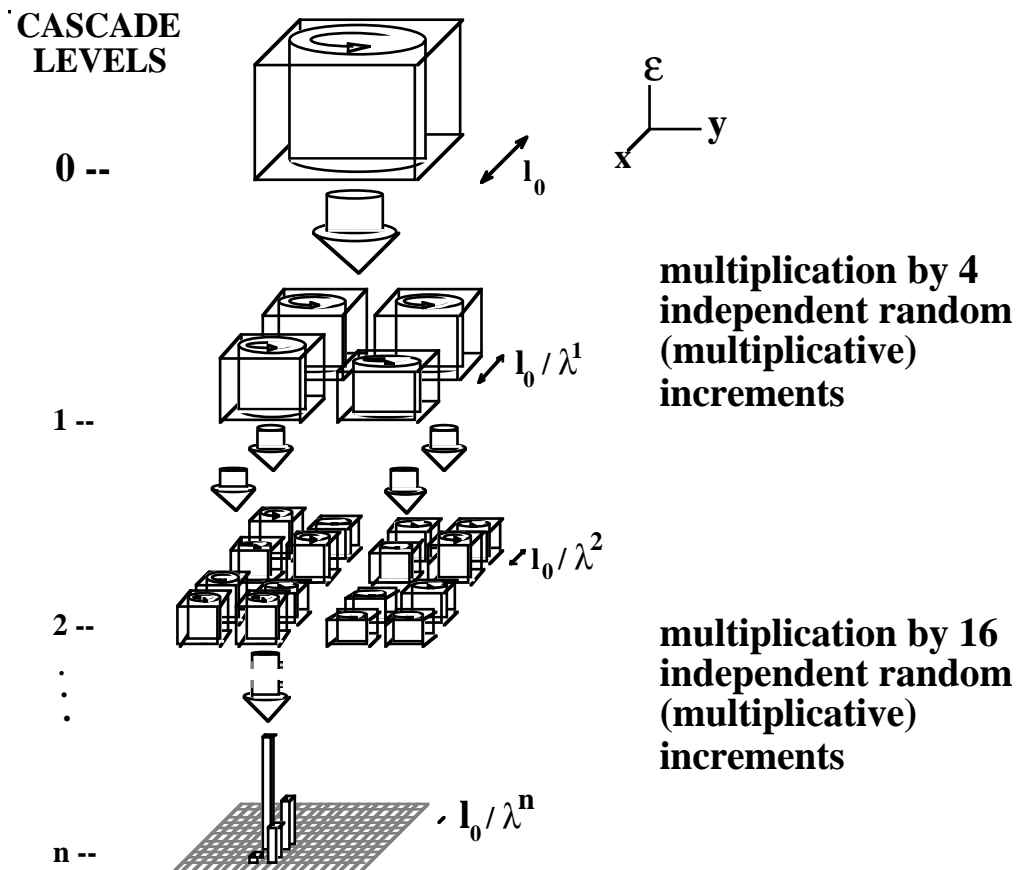


Fig. 6: A schematic diagram showing few steps of a discrete multiplicative cascade process, here the " α -model" with two pure orders of singularity $\gamma_- (> -\infty)$ and γ_+ (corresponding to the two values taken by the independent random increments, $\lambda^{\gamma_-} < 1$ and $\lambda^{\gamma_+} > 1$) leading to the appearance of mixed orders of singularity $\gamma (\gamma_- \leq \gamma \leq \gamma_+)$. (From [Schertzer and Lovejoy, 1989]).

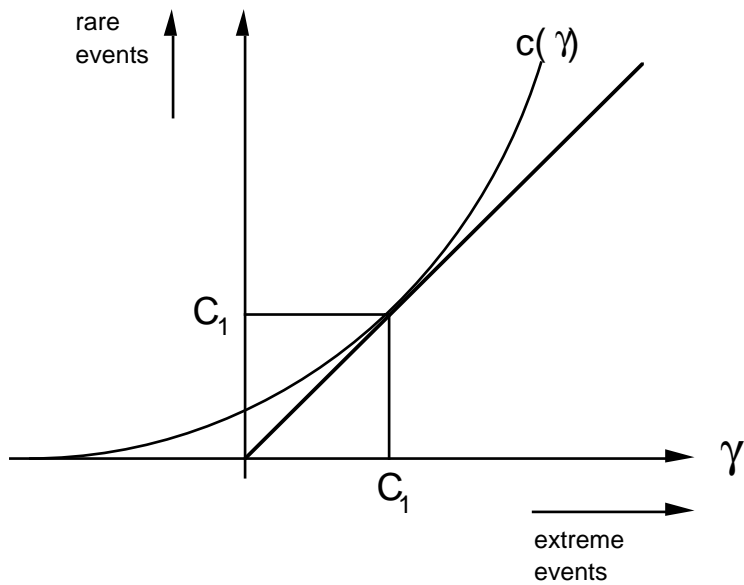


Figure 7: A schematic illustration of a conserved multifractal $c(\gamma)$, showing the relations $c(C_1) = C_1$ and $c'(C_1) = 1$, where C_1 is the singularity of the mean. (From [Schertzer and Lovejoy, 1993]).

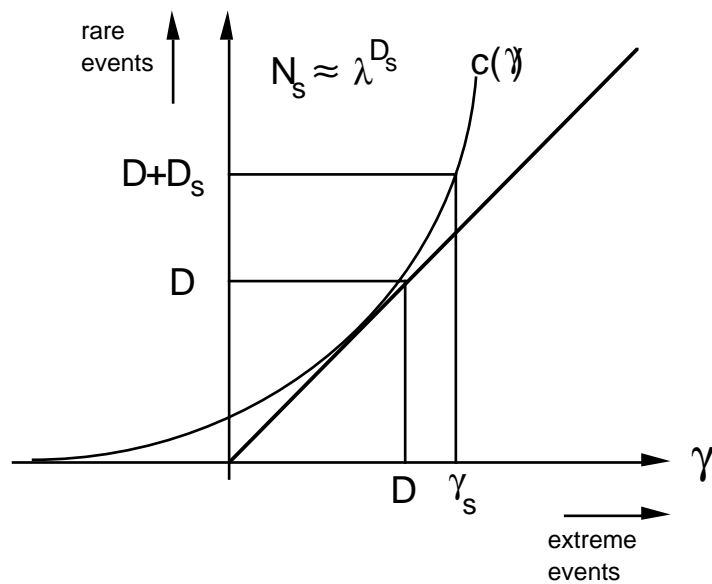


Figure 8: Schematic illustration of sampling dimension and how it imposes a maximum order of singularities γ_s . (From [Schertzer and Lovejoy, 1993]).

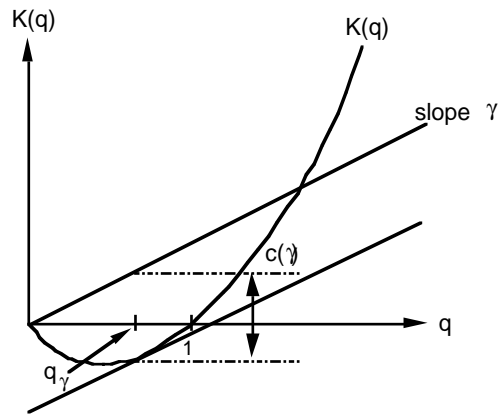


Figure 9— $K(q)$ versus q showing the tangent line $K'(q_\gamma) = \gamma$ with the corresponding chord γ_q (From [Schertzer and Lovejoy, 1993]).

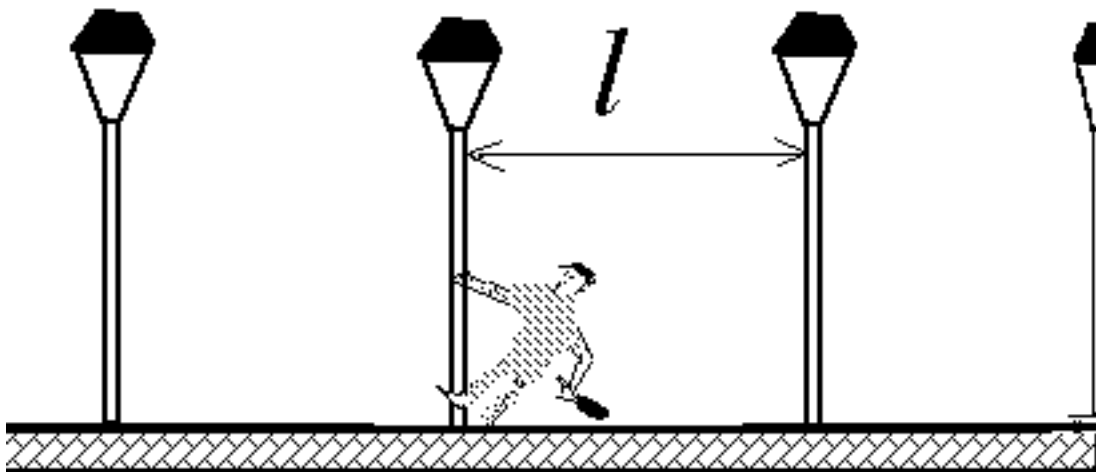


Figure 10 The universal limits of drunkards walks: as soon as the distance between lamp posts ℓ tends to zero, the details of the precise rule of choice left/right at each lamppost becomes less and less relevant to the walk which converges either to the usual Brownian motion ($\alpha = 2$) or to a Lévy flight” ($0 < \alpha < 2$, $\langle |\Delta x|^q \rangle = \infty$, $q \geq \alpha$). (From [Schertzer and Lovejoy, 1993]).

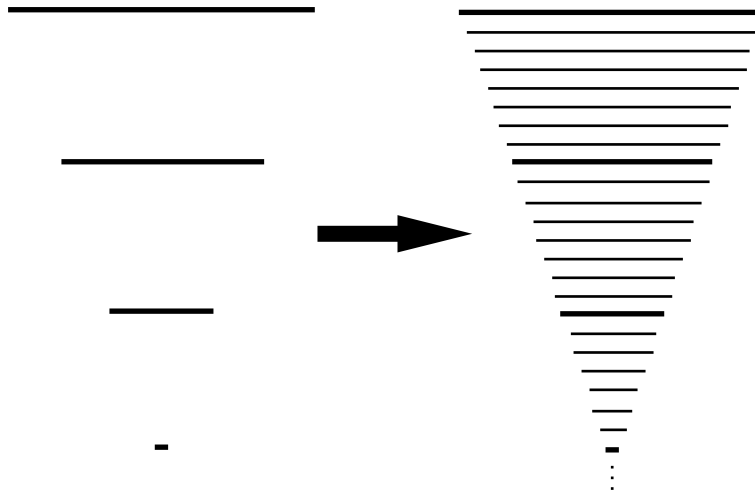


Fig. 11— Scheme of densification of scales.(From [Schertzer and Lovejoy, 1997]).

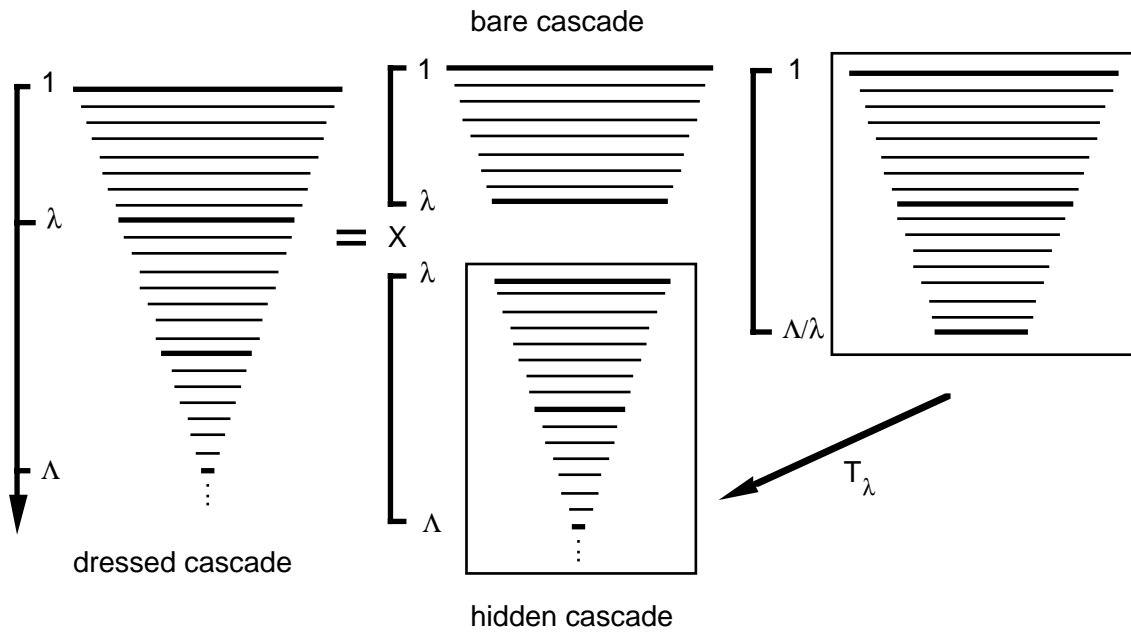


Figure 12—A schematic diagram showing a cascade constructed down to scale ratio Λ , dressed (averaged) up to ratio λ . This is equivalent to a bare cascade constructed over ratio λ , multiplied by a hidden factor obtained by reducing by factor λ , with the help of the operator T_λ , a cascade constructed from 1 to Λ/λ . (From [Schertzer, 2001]).

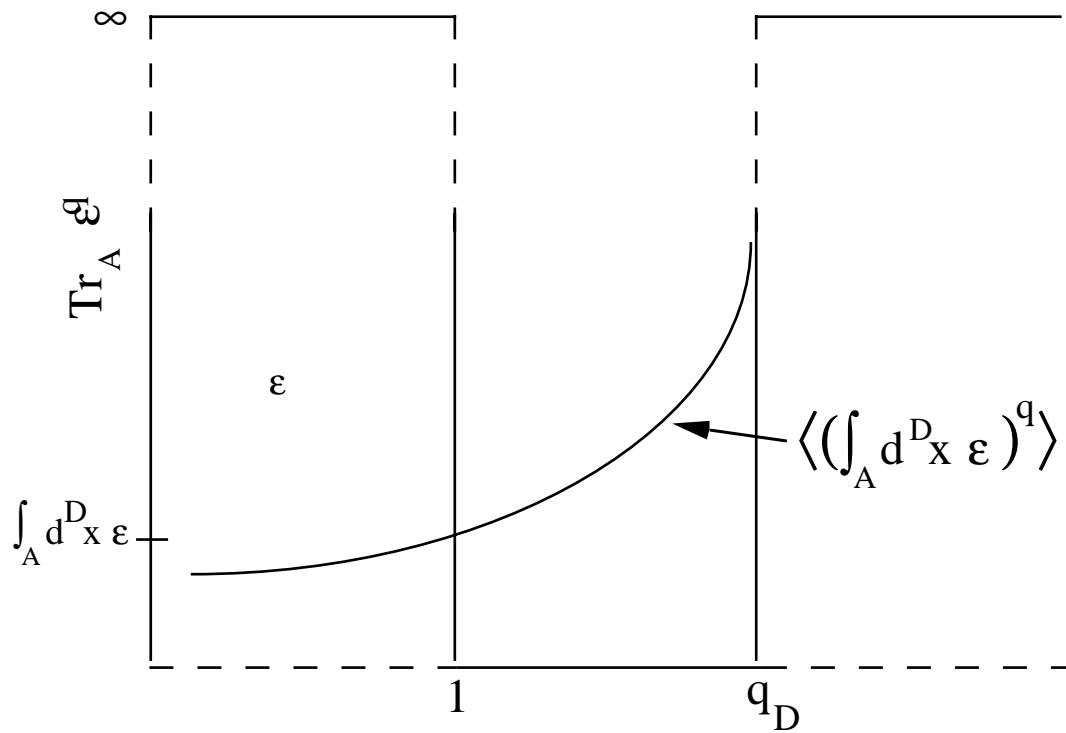


Figure 13—Twin divergences of trace moments. (From [Schertzer and Lovejoy, 1987a]).

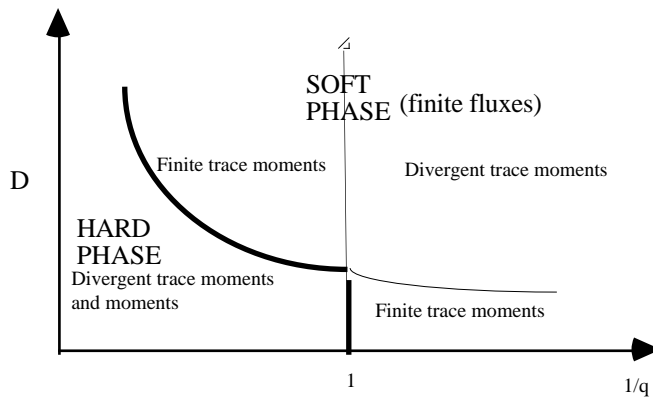


Fig. 14: Hard and soft multifractal phases. The bold line represents the transition line (q_D^{-1}, D) for fluxes and trace-moments, whereas its analytical continuation (\bar{q}_D^{-1}, D) (light line) concerns only the trace-moments. One may note that for any given D , we have the twin divergence of the trace moments rule (see Fig. 13 and Sect. 7.3.2): trace moments are convergent only for intermediate values of q . (From [Schertzer, 2001]).

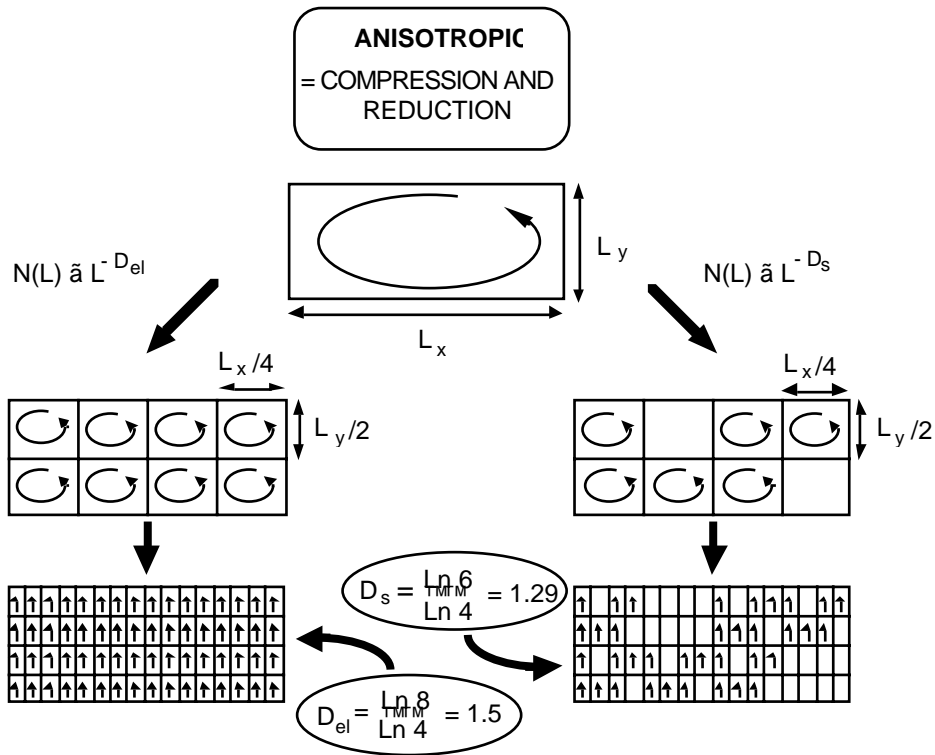


Figure 15—Anisotropic cascade scheme: compare with Fig. 4 (From [Schertzer, 2001]).

13 References

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