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GENERALISED SCALE INVARIANCE IN TURBULENT PHENOMENA

D. Schertzer and S. Lovejoy[1]
EERM/CRMD
2 avenue Rapp 75007 Paris
FRANCE

ABSTRACT :

Many geophysical fluid dynamical systems are highly anisotropic and intermittent over a wide range of scales. In the following, we develope a formalism called generalised scale invariance (GSI) which is necessary when the statistical properties are no longer symmetric with respect to rotation, but remain symmetric under general scale changing operators which can no longer correspond toself-similar dilations. The physically significant invariants are densities of scaling measures which are symmetric under these generalised scale changes. By relating GSI to existing cascade models, we show that scaling measures are characterised in general by multiple fractal dimensions, and are associated with the interesting phenomena of the divergence of high order statistical moments. Finally, we analyse radar rain fields showing not only scaling but also dimensional dependence of statistical averages.

1. Introduction :

Scale invariance is a notion widely used in isotropic systems with many scales such as turbulence. However, many natural flows, exhibit strong anisotropy which results from the existence of prefered directions (e.g. in the atmosphere due to gravity or rotation). In meteorology this common and unfortunate association of scaling with isotropy has raised the question of whether a single scaling regime exists at all : the classical scheme of atmospheric dynamics postulates a quasi-two dimensional regime at large scales and a quasi-three dimensional regime at small scales.

Recently, we have proposed an alternative scaling theory((1, 2, 3) see also (4, 5) for non- mathematical reviews) in which the anisotropy introduced by gravity via the buoyancy force results in a differential stratification and a consequent modification of the metric. This leads to a reduction of the effective dimension of space (from the isotropic value D=3 to 23/9=2.5555...). The metric is modified because in a cascade process, the most natural metric to use is the one in which the "balls" it defines coincide with the average eddies. In the isotropic case, the balls are self-similar spheres, but when there is a priviledged direction, we expect these to be replaced by self-affine ellipsoids (see fig. 1 and 2).

In order to take into account this and other effects such as the differential rotation introduced by the Coriolis force, a general formalism of scaling is required. In fact, as pointed out in Section 3 and 5, only measurable, not metric properties are necessary. This is because the scale notion may be extended so as to depend only on measurable properties of the balls.

As indicated and illustrated in Section 2, the fundamental problem is that of finding a family of "balls" representing the statistical properties of eddies at different scales. These balls define physically important, scale invariant (mathematical) measures such as the flux (or dissipation) of energy through structures of a given scale. In Section 3, a general formalism is deduced from

[1] current address : Physics dept., McGill University, 3600 University st., Montreal Que. H3A 2T8, CANADA

these phenomenological considerations to take into account both anisotropy and intermittency. This formalism is given a sound foundation by the linear metric case which is explored in section 4. Section 5 is devoted to several of the numerous implications for scale invariant measures, in particular we discuss "multidimensional" intermittency which is described not by a single dimension but rather by a sequence of dimensions. Section 6 gives direct experimental support for multidimensionality in the rainfield (obtained from remotely sensed radar data).

2. Phenomenology of turbulent cascades :

2.1. Isotropic energy cascades :

Since Richardson (6) the phenomenology of turbulence has been closely associated with self-similar cascades. In this section we review simple cascade schemes, outlining several variations in order to capture both anisotropy and intermittency. In an isotropic, homogeneous cascade, non-linear interactions break-up large eddies into smaller sub-eddies, transfering their energy (without dissipation) in the process (energy is thus an invariant of the process). Fig. A, B schematically shows a single step of such a cascade. The initial eddy (A), represented for convenience as a square, is transformed into B. Each of the sub-eddies are copies of the original reduced by the linear ratio λ (here taken $\lambda = 2$) and each containing a fraction λ^{-2} of the original energy. If the process is continued indefinately, it is clear that the energy distribution remains homogeneous and isotropic.

In order to account for the "spottiness" (7) of turbulence (the fact that the active regions only occupy a small frnction of the total volume available), this cascade scheme has been elaborated, through the work of Novikov and Stewart (8), Yaglom (9) to the more general scheme of Mandelbrot (10). The simplest case (known as the "β model" (11) is illustrated in fig. 1C. As before, the large eddy is broken up isotropically. Now however, the sub-eddies are randomly chosen to be either "dead" or "alive" (active), with the energy at each step being divided equally only between the N active sub-eddies with $\langle N \rangle < \lambda^2$. When the the number of steps tends infinity, the energy is eventually distributed over a set of points (called the "support of the turbulence"), with (Hausdorff) fractal dimension $D_s = \log\langle N \rangle / \log(\lambda)$. In fully developed three dimension turbulence, it is found empirically that, $D_s \sim 2.5$ (12).

This simple scheme can readily be extended to account for the more realistic case involving turbulence with a continuum of intensities. This leads to a number of interesting implications (10, 1, 2, 3, 13), including the hyperbolic nature of extreme fluctuations (divergence of the high moments of the density of energy flux), and the multidimensional nature of the intermittency, both of which we discuss in Section 5. For the moment, we rather concentrate on showing how these simple schemes can deal not only with intermittency, but also the strong anisotropy.

2.2. Anisotropic cascades :

The strong anisotropy in the atmosphere is primarily due to gravity which induces a differential stratification and the Coriolis force which induces a differential rotation. The simplest way to deal with this (3, 13) is to consider anisotropic cascades in order to account for the vertical stratification. This natural idea leads to the surprising conclusion that the effective dimension (called an elliptical dimension D_{el} - see Section 4) of the atmosphere is 23/9 = 2.5555 rather than 2 or 3 as in the usual models. To see how this intermediate dimension can arise, consider the schematic illustration of a simple anisotropic cascade shown in fig. 1D, E. Rather than producing sub-eddies by dividing both axes in fig. 1 by the same factor, we divide one by λ and the other by λ^{Hz}. Fig. 1D, E shows this with $\lambda = 4$, $H_z = 1/2$. The resulting elliptical dimension is $1+H_z = 1.5$ rather than 2 as in the isotropic case. In the intermittent case, (E), the support has an effective dimension (also of the elliptical type) $D_s < D_{el}$. Note that at each step of the process, the initial rectangular eddy is reduced in size and elongated. The transformation from one scale to another now involves a compression as well as a reduction. Note that as in the atmosphere, the structures at the largest scales are the most horizontally stratified. In the atmosphere, theoretical and empirical results show $H_z = 5/9$, hence $D_{el} = 2+H_z = 23/9$ (3).

One of the motivations for the formalism described below, is to go beyond these square and rectangular eddy shapes, which are instructive, but hardly realistic. Using GSI, the squares and rectangles can be replaced by nearly any shape, the simplest of which are circles. Fig 2 illustrates this with a family of average eddies in a simple example of anisotropic scaling involving both differential rotation and stratification. For comparison recall that under isotropy, we transform from one member of the family to another by simple multiplication by the ratio λ, hence the balls are concentric circles. In Fig. 2, we rather multiply by λ^G where G is a matrix.

3. Generalised notion of scale :

As noted above, in geophysics, the notion of scale has to be generalised in order to take into account anisotropy. However, Geophysical quantities are also often extremely variable, hence at the very least, we require measures which are both anisotropic and intermittent.

The previous examples outlined the basic properties associated with the notion of scale which can be restated in the following abstact way : there exists a family $\mathcal{B}$ of "balls" B generating the topology of a set M and an application $\emptyset$ from M to R^+ which is increasing (i.e. $B \subset B' \Rightarrow \emptyset(B) \leq \emptyset(B')$). $\emptyset(B)$ defines the scale of B.

The balls of $\mathcal{B}$ can be generated by a scale transformation of ratio λ from those of a sub-family $\mathcal{B}_1$ (covering M) bounded for $\emptyset$ (i.e. there exists a positive and finite real number A such that : $\forall B_1 \in \mathcal{B}_1, \emptyset(B_1) < A$). This (abstract) scale transformation of ratio λ corresponds to an operator T_λ (from M to M) such that : $\emptyset(T_\lambda B) = \lambda^D \emptyset(B)$.

We are thus lead to the following abstract definition in terms of a group (the "scaling group") of operators T_λ for a topological space M :

Generalized scale transformation (global definition) :

(i) T_λ is a multiplicative group ($\lambda \in R_+^*$) of transformation from M to M, i.e:

(3.1) $T_{\lambda\lambda'} = T_\lambda \circ T_{\lambda'}$

in particular : $T_1 = 1$ = the identity and $T_\lambda^{-1} = T_{\lambda^{-1}}$,

(ii) there exists a family $\mathcal{B}_1$ of "balls" (open sub-sets of M) such that $\mathcal{B} = T_\lambda \mathcal{B}_1$ is a basis for the topology of M,

(iii) there exists an increasing function $\emptyset$ from $\mathcal{B}$ to R_+, bounded on $\mathcal{B}_1$ and which factorizes in ($D \in R_+$) :

(3.2.) $T_\lambda \emptyset = \lambda^D \emptyset$

(in (3.2.) $T_\lambda \emptyset$ is naturally defined by : $T_\lambda \emptyset B = \emptyset(T_\lambda B)$, $\forall \lambda$, B.

Note in (3.2.) the expression λ^D results from the group property of T_λ since it is implied by the assumption of the existence of a continous function $g(\lambda)$ in (3.2.).

As is easily shown, in case of a metric space, D plays the role of a dimension and $\emptyset(B)$ can be taken as the radius of the balls defined by the distance $d(x,y)$ (i.e. if $B_{x,L}$ is the ball centred at x, radius L, then : $B_{x,L}= \{ y/d(x,y) \leqslant L$, $\emptyset(B_{x,L})=L$ and D = 1). d (x,y) can be called the scaling metric. More generally we can use the measurability property of the balls. For instance on R^d, we can take $\emptyset$ as the Lebesgue (d volume) measure, by supposing that the B's are Lebesgue measurable, and D equals d if the balls are the usual spheres or cubes. As we will see, this is no longer true with strongly anisotropic balls (such as self-affine, but not self-similar, ellipsoïds). Even more anisotropic (and/or irregular) balls can be addressed in this formalism (see Figs. 3a, b : the balls need not be convex). Interesting examples are isotropic or anisotropic Cantor sets or more generally fractal sets which are not Lebesgue measureable but measurable by a D-dimensional Hausdorff measure. In all these cases $\emptyset$ can be taken as the correct Hausdorff measure (that which is finite and positive on the balls) and the scale is given by $\emptyset^{1/D}$. $\emptyset$ can be called the scaling measure.

The last example already shows that although it is usually based on metrics the measurability notion of scale is more general than the metric notion. It also has the advantage of being immediately transposable to the space K(M) of the "test functions" of compact support on M, since the equality in (3.2.) is also true in terms of test functions :

(3.3) $f \in K(M),\ T_\lambda\ f(x) = f(T_\lambda^{-1}\ x)$

not only in terms of balls i.e. :

(3.4) $\emptyset \in K'(M),\ f \in K(M) \Rightarrow T_\lambda\ \emptyset\ (f) = \lambda^D\ \emptyset\ (f) = \emptyset\ (T_\lambda\ f)$.

We now establish several precise results in the linear case.

4. Linear GSI case :

4.1. Introduction :

In this section we will explore the necessary and sufficient conditions for obtaining a scaling group on a vector space M, where the T_λ will be a linear application from M to M.

In this case, it is well known that any multiplicative group T_λ is generated by a (bounded) linear application G according to :

$$(4.1.)\quad T_\lambda = \exp\ (G\log\lambda\ ) = \sum_{n=0}^{\infty} (\log\lambda)^n \frac{G^n}{n!}$$

To generate a generalized scaling measure we proceed as follows : start with a given scaling measure (defined by $G_0, \emptyset_0, D_0$), which may correspond to the usual scaling such as an isotropic metric. Next deduce whether a given generator G defines a new scaling (with $\emptyset$, D). This requires the use of unit balls $\mathcal{B}_1$ (i.e. $\emptyset(B_1)=\emptyset_0\ (B_1) = 1$) which should generate, through T_λ defined by (4.1.), the whole family $\mathcal{B}$ of balls of the new scaling, i.e. :

$$(4.2.)\qquad \emptyset(B)=\lambda \Leftrightarrow \emptyset_0(T_\lambda^{-1}B)=1$$

4.2. The measurable case :

This case is rather easy to handle. If we start with a measure, ϕ_o then ϕ will be the image measure of ϕ_o though T_λ (which is continuous), and thus will satisfy the desired properties if (3.2.) is satisfied with a positive D. If we take ϕ_o as the Lebesgue measure on R^d, then we obtain :

(4.3.) $D_{el} = \text{Trace}\,(G)$

since the Jacobian of the transformation T_λ is :

(4.4.) $\det\,(T_\lambda) = \exp\,(\text{Trace}(G)\text{Log}\,\lambda)$

D_{el} can be considered as the effective dimension of the space, and when the balls are ellipsoïdswe may continue to call it the elliptical dimension of the space (cf. 1, 2, 3). See Fig. 3a, b for examples obtained by various non-linear generators G.

4.3. The metric case :

This case is more demanding since the image of a metric is not usually itself a metric. Nevertheless it is possible to establish (13) the following proposition (starting with an initial metric d(x,y)) :

(4.5.) $\inf \text{Re}\, \sigma(G) \geqslant 1$

where $\sigma(G)$ is the spectrum of G :

(4.6.) $\sigma(G) = \{\mu \in C \mid G - \mu 1\}$ non invertible on $C \otimes M$

and $C \otimes M$ is the complexified space of M.

If we started with a unit ball defined by the ellipsoïd generated by a symmetric operator A and euclidean product $(\cdot\,,\cdot)$:

(4.7) $x \in B_1 \Leftrightarrow (Ax,x)^{1/2} \leqslant 1$

Then we obtain the following condition :

(4.8) $\inf \sigma(\text{sym}\,(AG)) \geqslant 1$

where Sym (AG) denotes the symmetric part of AG.

4.4. Some simple examples on the plane :

A particularly simple example of linear GSI may be obtained by the use of quaternions. Of the many possible representations of quaternions (such as the Pauli matrices) we choose the following four 2x2 matrices : the identity (1) and :

(4.9) $I = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad K = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

these satisfy the following anticomutation relations :

(4.10)
$$IJ = -JI = -K$$
$$JK = KJ = I$$
$$KI = -IK = -J$$

and :

(4.11) $1 = -I^2 = J^2 = K^2$

If we decompose G on this basis, then : G=d1+cK+eI+fJ also, taking $a^2=c^2+f^2-e^2$, we obtain, with u = Logλ:

(4.12) $\lambda^G = \lambda^d(1 \cosh(au) + (G - d\,1) \sinh(au)/(a))$

Of course, I and K corespond to the elementary linear operation of multiplication by i and complex conjugation. J has the same effect as K coupled with a rotation, namely we have :

(4.13) $cK + fJ = c'R^*KR$; $c'^2 = (c^2+f^2)$, $R=e^{I\theta}$, $\theta = \tan^{-1}(f/c)$

If a is imaginary, the rotation effect (due to I) is dominant, otherwise, the stratification effect (due to K and J) dominates. Fig. 2 shows an example of families of B_λ for the limiting case where the ellipsoids touch along a log-spiral, f=1, e=2, $a^2=-3$. When rotation dominates, the axes of the ellipsoids rotate indefinately, otherwise the total rotation is only $\tan^{-1}(f/c)$. The existence of such metrics are assured by :

(4.14) $d^2 \geq c^2 + f^2$

Lovejoy and Schertzer (14) exploit the stochastic fractal model discussed in Lovejoy and Mandelbrot (15) to give examples of (mono-dimensional) fields respecting linear metric GSI.

5. GSI is a natural framework for multiplicative chaos and multidimensional intermittency :

5.1. Introduction :

Usual stochastic process (such as Brownian motion) are obtained by the (weighted) addition of independent identically distributed (i.i.d.) random variables (e.g. integrals of white noise). Conversely, the multiplicative group T_λ suggests that in GSI the most natural type of process to use are those obtained by multiplication.

The difference in nature of additive and multiplicative processes is profound since the former is mono-dimensional, while the latter generally leads to multiple dimensions. This difference needs underlining since many efforts have been made to relate the most obvious aspect of intermittency-its "spottiness" (7) to a turbulent support with a single fractal dimension (10, 11) . If we define active (turbulent) regions as those exceeding an arbitrary threshold, then the active regions may indeed be characterised in this way: the turbulence occupies a much smaller space than that available. However, as pointed out in (1, 2), phenomenological models of intermittency (8, 9, 10) lead, more generally to supports characterised by multiple dimensions, corresponding to the different (tensorial) powers of the measure of the flux of energy. Indeed, a sequence of dimensions is easily obtained (2, 3) by considering the divergence of high moments of the density ϵ_A of this flux with respect to different D_A-dimensional Hausdorff measures:

(5.1) $\epsilon_A = \infty$ for $D_A < C(h) = D_{el} - D(h)$
$C(h) = \log \langle W^h \rangle / (h-1)$

where W is the random variable which distributes the density during a step of the cascade (i.e. from an eddy to a sub-eddy in fig. 1). Increasing h corresponds to studying the more intense regions. Since D(h) is a decreasing function of h, the most intense regions are the most sparsely distributed.

5.2 GSI and additive processes :

Combining the action of T_λ with addition creates random structures of different sizes and intensities. Indeed, take an i.i.d. random test functions $V^{(\lambda)}$ of a given spatial resolution (e.g. $V^{(\lambda)}$= constant over the unit ball: the process is therefore the sum of i.i.d. indicator functions of unit balls). Thus:

$$(5.2) \quad V_\lambda = T_\lambda^{-1}(V^{(\lambda)}) \qquad (\lambda>1)$$

will be a random test function of lower spatial resolution λ^{-1}. Summing these different V_λ (renormalising their intensities by $\lambda^{-\gamma}$, $\gamma>0$ if necessary), we obtain a random density with respect to the (fundamental) scaling measure m ($T_\lambda m = \lambda^{D_{el}} m$), thus define m_Λ as:

$$(5.3) \quad m_\Lambda = \left(\int_1^\Lambda \lambda^{-\gamma} V_\lambda d\lambda/\lambda\right) m \qquad (\gamma>0)$$

is a random measure corresponding to a hierarchy of structures of scale ratio λ, and the action of T_λ (for any λ) will obviously leave this property unchanged. More precisely, the density of $T_\lambda m$ will also be the sum of i.i.d. densities of the same type as for m except for a magnification λ^{-C} where C depends on γ and the probability distribution of the V_λ). Thus:

$$(5.4) \quad T_\mu m_\Lambda \overset{d}{=} \mu^D m_\Lambda$$

($\overset{d}{=}$ meaning equality in distributions, $C=D_{el}-D$) and D is the unique dimension characterising the supports of all the any moments of m . Note that we have implicitely supposed that both $\langle V\rangle=0$ and $\langle m\rangle=0$. This is not restrictive since we can normalise m_Λ by adding to m_Λ, $fm=\langle m_\Lambda\rangle$where f is the density of the average of m_Λ).

5.3 GSI and multiplicative processes-multiplicative chaos :

Instead of adding random increments of finer and finer resolution along the cascade, one may multiply by random increments of finer and finer resolution. This multiplicative procedure corresponds to the non-linear break-up of eddies into sub-eddies. The resulting random density will be of the form:

$$(5.5) \quad m'_\Lambda = f_\Lambda m = \exp\left(\int_1^\Lambda V_\lambda d\lambda/\lambda\right) m$$

where the V_λ will still result from the action of T_λ^{-1} on i.i.d. $V^{(\lambda)}$ (of resolution 1). Mandelbrot's cascade model of intermittency on a rigid grid corresponds to a discrete summation ($\lambda_n = \delta^n$, $d\lambda/\lambda=\delta-1$, the i.i.d. random variable $W_n^{(i)}$ being the intensity of $\exp(V_n(\delta-1))$ on the i^{th} cube of resolution λ^{-n}).

In such processes, one is interested in $m_\infty = \lim_{\Lambda\to\infty} m_\Lambda$ (even when f has no limit in the sense of functions), which represents a difficult mathematical problem where few results have been obtained (16). Nevertheless, due to the multiplicative property of both T_λ and the way the process is constructed, we may introduce the co-dimension function C(h):

$$(5.6) \quad \langle T_\lambda^{-1} f_\Lambda^h\rangle \sim \lambda^{(h-1)C(h)} \langle f_\Lambda^h\rangle$$

In this paper, we will not develope the formalism further, but only note that it clearly indicates that multidimensionality is quite general. Other work on multidimensionality may be found in Hentschel and Procaccia (17), Grassberger (18), Mandelbrot (19), Parisi and Frisch (20).

6. Intermittent multidimensional measures in the radar determined rain field :

6.1. The integral structure function :

From the preceeding it is clear that the most obvious way of empirically studying scale invariance is by measuring various powers of cascade quantities over different scales and dimensions. We therefore introduce the integral structure function $S(h, L, D_A)$ defined for a quantity $X(r)$ as :

$$S(h, L, D_A) \simeq \langle ((\int X(r) d^{D_A} r)/L^{D_A})^h \rangle$$

where L is the size of the D_A-dimensional hypercube over which the averages are take. Scale invariance implies :

$$S(h, \lambda L, D_A) = \lambda^{-p(h,D_A)} S(h,L,D_A)$$

where λ is our usual enlargement ratio and $p(h,D_A)$ is a function not only of h, but also of D_A. Note that $d^{D_A}r$ denotes a Hausdorff measure, dimension D_A : the averaging can clearly be performed over any fractal set.

The function $p(h,D_A)$ contains all the information about the scale and dimension dependence of both intense (large h) and weak (small h) phenomena.

For the simple case where the phenomenon is mono-dimensional with dimension D_S (co-dimension = $D_{el}-D_S=C_S$) (e.g. the "β model") it can be shown (21) that $p(h, D_A)$ takes the following simple form (for $h>0$) :

$$p(h, D_A) = \inf(C_S, D_A).(h - 1)$$

i.e. for $D_A>C_S$, p is independent of the averaging dimension and is linear in h. When $D_A>C_S$, the averaging set and the phenomena intersect : when the dimension of the averaging set is large enough to intersect the phenomenon then averages are independent of D_A.

This has immediate consequences for $p(h,D_A)$ of multidimensional phenomena : as D_A is decreased from its maximum possible value more and more of the intense regions (with lowest dimension) will fail to intersect the averaging set. Hence, $p(h,D_A)$ will be sensitively dependent on D_A. Averages of multidimensional phenomena are therefore not only scale dependent, they are also dimension dependent. A related difference is that $p(h, D_A)$ is no longer linear in h.

6.2. The rain field :

Of all the geophysical fields, none are known with as high a resolutin in the four dimensions of space and time as the radar-determined rain field. For example, the data used in the study described below were from 5 series of Constant Altitude Z LOg Range maps (CAZLORs).

The radar measures the total backscatter from all the drops within a scattering volume, with an amplitude proportional to the drop volume squared, and with a random phase due to the random positions of the drops. The total integral Z is indirectly related to the rain rate (R), by an approximate formula (22) : $R \propto Z^{0.6}$.

Fig. 4 shows the functions $p(h,D_A)$ for D_A= 1, 1.5, 2, 3, 4 corresponding to a) azimuthal averaging, b) averaging over 1.5 dimensional fractals c) azimuth-range averaging, d) azimuth-range-elevation averaging, e) azimuth-range-elevation-time averaging. The curve for D_A = 1.5 is of interest because Lovejoy and Schertzer (21, 23) show that geophysical measuring networks are clustered at all scales (e.g. on continents, near cities) with $D_A< 2$ (e.g. in Canada the meteorological surface network has $D_A \sim 1.5$. In France the climate network has $D_A \sim 1.8$, the global network, $D_A \sim 1.75$).

7. CONCLUSION :

Motivated by the strong anisotropy and intermittency of the atmosphere, we have developed a formalism called generalised scale invariance. The formalism is based on two sets of elements and may be regarded as an extension of earlier work on cascade processes (especially 10, 3).

The first is a group of general scale changing operators, whereas the second are the intermittent measures invariant under the operators. In a turbulent cascade, the scale changing operator transforms eddies into sub-eddies, while leaving the physically significant energy flux invariant (here represented by a scaling measure). The scaling operators can be classified according to whether the balls associated with the eddy topology define a metric or are only measurable. It can further the c ssified according to whether the underlying space is homogeneous (i.e. translation invariant, linear GSI), or inhomogeneous (non-linear GSI). Examples of each are given. The scaling measures can be classified according to whether they involve a single fractal dimension (mono-dimensional measures), or whether as in the more general case) the measures are characterised by an infinite sequence of fractal dimensions with the most intense regions having the lowest dimension. The latter case is also associated with interesting phenomena of divergence of high order statistical moments.

Finally, we test some of these ideas directly on the radar determined rain field. The scale and dimension dependence of the averages of various powers of this field are clear support of its multidimensional nature.

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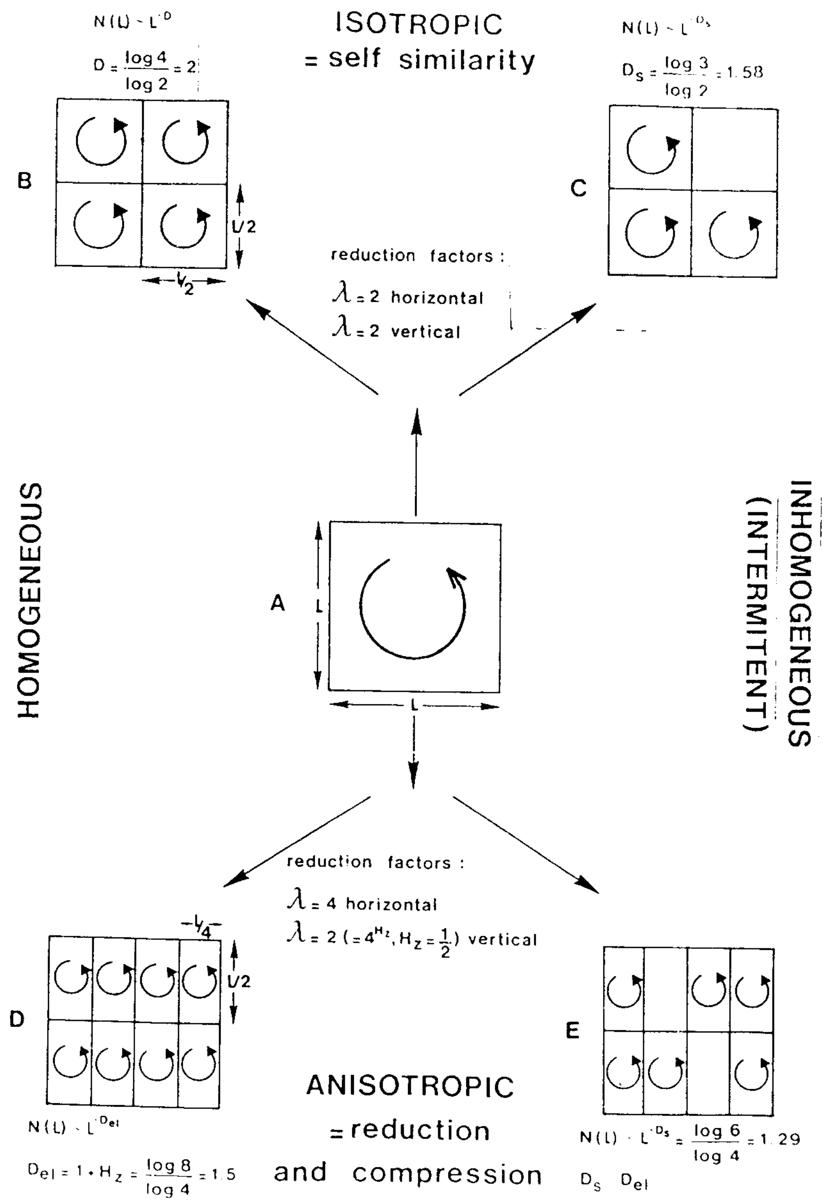

Fig. 1 : A schematic representation of how various turbulence models treat the break-up of an eddy (represented by the square in A) via non-linear interactions during a single step in the cascade process. The various schemes are divided from left to right into homogeneous and inhomogeneous (intermittent), and from top to bottom into isotropic and anisotropic cades. For each scheme, the formula giving the number of active eddies at size (L) (=N(L)) is shown.

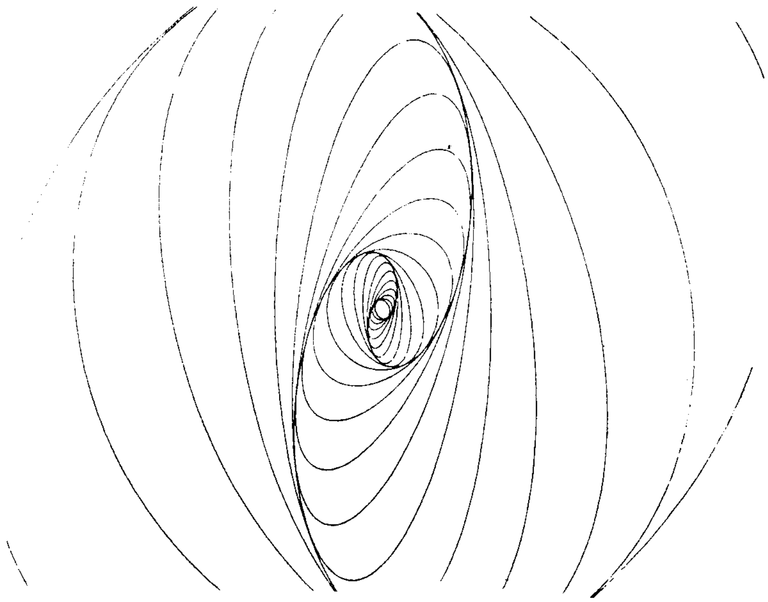

Fig. 2 : The shapes of the average eddies (the balls B_λ) for an example with both differential stratification and rotation (modelling the effect of the Coriolis force). Here, $D_{el}=2$.

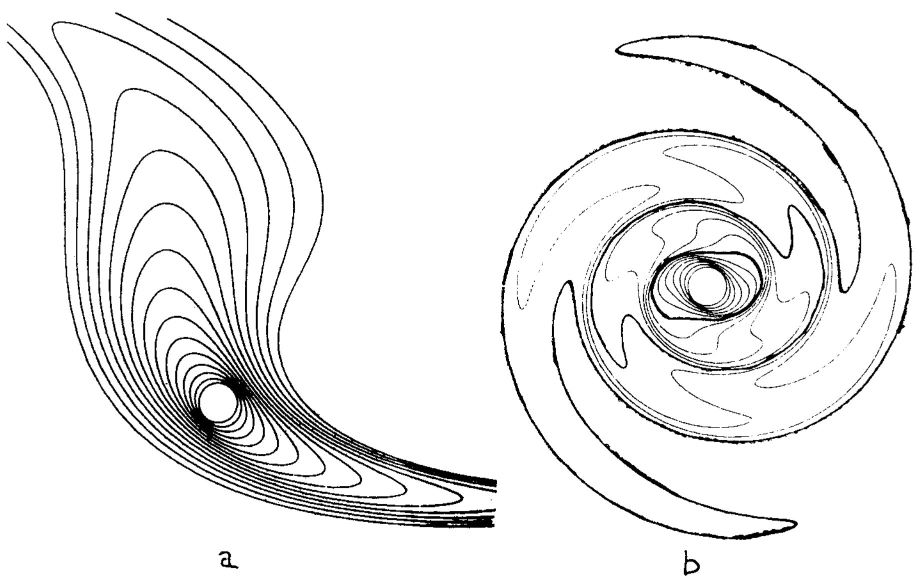

Fig. 3 : Example of the balls B_λ for non-linear, non-metric GSI, obtained with various (non-linear) generators **G**. 3b models a scale invariant "cyclone".

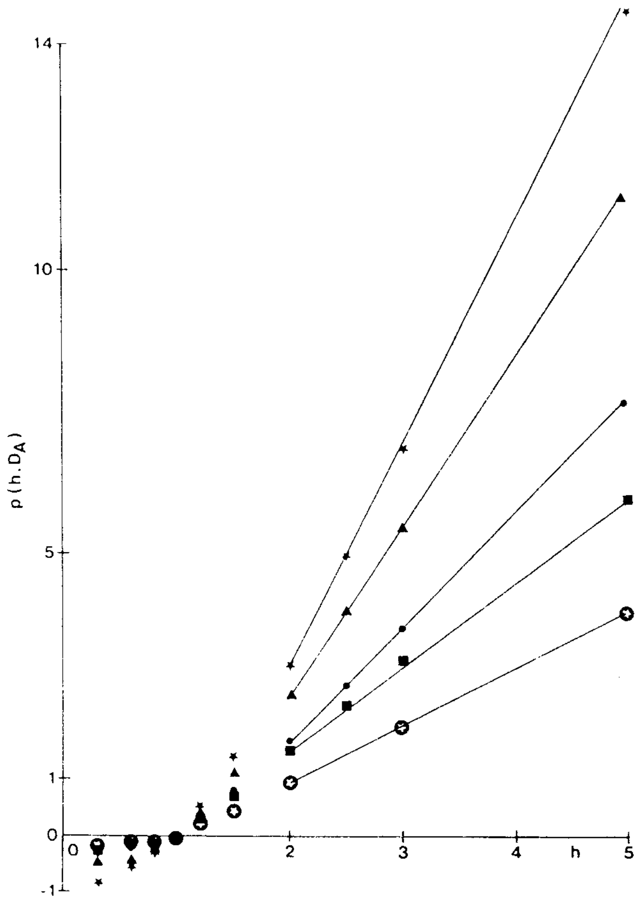

Fig. 4 : The structure function exponents $p(h,D_A)$ for (symbols bottom to top respectively), $D_A=1$, $D_A=1.5$ (using simulated fractal rain gage networks). $D_A=2$, $D_A=3$ (space), $D_A=4$ (space-time). The straight lines for large h have slopes D_A which indicate the most intense regions have dimension zero.