Our Multifractal Atmosphere: A Unique Laboratory for Non-linear Dynamics

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Errata

A number of subscripts and superscripts were not properly typed. Equations where this may cause ambiguity are corrected as follows:

$$\Pr(\mu\varepsilon=\lambda^{\gamma_i}) = \sum_j p_{ij}\lambda^{-C_{ij}} \tag{5}$$

$$\Pr(\varepsilon_\lambda=\lambda^{\gamma}) \approx \lambda^{-c(\gamma)} \tag{6}$$

$$\langle\varepsilon_\lambda{}^h\rangle = \lambda^{K(h)} = \int\lambda^{h\gamma}d\Pr \approx \int\lambda^{h\gamma-c(\gamma)}d\gamma = \int e^{(h\gamma-c(\gamma))\ln\lambda}d\gamma \tag{9}$$

Also, just below eq.5: "... ($c_i = \min_j c_{ij}$)...".

In eq. 2 the sign of α' is wrong, it should read: $\Pr(\mu\varepsilon=\lambda^{-C/\alpha'}) = 1-\lambda^{-C}$
The "dead' or "alive" "β model" is recovered with $\alpha=1$, $\alpha'=0$ (not $\alpha'=\infty$).

Our Multifractal Atmosphere: A Unique Laboratory for Non-linear Dynamics

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Abstract

We argue that the atmosphere in particular and geophysical systems in general, provide unique laboratories for studying non-linear dynamics and multifractals. Indeed it is possible that over a range of scales spanning as many as 9 orders of magnitude, the atmosphere is symmetric with respect to a scale changing operator involving only the scale ratio. Due to anisotropy introduced by gravity and the Coriolis force, this Generalized Scale Invariance is more complex than the usual "self-similar" symmetry associated with isotropic fractals. This wide range scaling is possible because the governing equations have no characteristic length from the outer planetary scale down to an inner viscous scale of the order of millimeters. Furthermore, the variability (intermittency) is due to cascade proceses which transfer the energy (and other conserved fluxes) to smaller scales. In this paper, which is largely a review, we show how these cascades generically lead to (universal) multifractals, when the cascades are taken to their continuous limit. We illustrate these ideas with simple models as well as with multifractal analyses of various geophysical fields.

1. INTRODUCTION

The atmosphere is probably our most familiar strongly non-linear dynamical system: the dimensionless parameter characterizing the strength of the non-linearity (the Reynold's number) is of order 10^{12}. It displays striking (multi) fractal structures (clouds, eddies, fronts etc.) spanning as many as nine orders of magnitude in scale (from planetary down to millimetric scales). It is also very accessible: for example over a dozen meteorological satellites operated by four different countries collect data at a total rate of 10^8-10^9 bits/s at over 50 different frequency channels and with spatial resolutions varying from 1 to 200km (other satellites not specifically meteorological such as SPOT or LANDSAT can also provide useful but infrequent measurements at resolutions of 10-30m). Nearly 10,000 ground stations perform routine daily (and many of these, hourly) measurements; every twelve hours six hundred of these loft radiosonde balloons yielding detailed vertical profiles of temperature, wind, pressure and humidity. In addition roughly 300 weather radars measure precipitation at a total rate of 10^6-10^7 bits/s. These rates of data acquisition are so large that systematic archiving — not to mention analysis — even of data from a single satellite is very costly; with the result that much of the data is only used "on the fly", usually quite superficially. In addition to these "operational" systems, many other large research oriented data sets also exist: for example if measured in terms of the precision of measurements and range of time and space scales covered, research weather radars (such as the one operated by McGill) probably produce the highest quality data set of turbulent fields anywhere (10^4 in time and $(2x10^2)x(2x10^2)x(10^1)$ in the two horizontal and one vertical directions respectively for measurements of the radar reflectivity of rain).

In this paper, we argue that physicists should seize the opportunity to exploit this unique non-linear dynamical laboratory. The advantages of the atmosphere (and of many other geophysical systems), go well beyond the possibility of exploiting readily available data sets: no other terrestrial non-linear system spans anywhere near the same range of scales: the comparable range for wind tunnels is $\approx 10^4$, and for three dimensional computer models, $\approx 10^2$. The range of scales is particularly important since it is the basic large parameter in the relevant cascade theories (discussed below). The rapprochement of physics and meteorology has already begun; in the last few years atmospheric applications have played a new role in developing ideas in scale invariance ("Generalized Scale Invariance" — see below), as well as multifractals (including universal multifractals). It has also lead to new data analysis techniques specifically designed for systems displaying extreme variability over large ranges in scale.

2. THE EQUATIONS OF ATMOSPHERIC DYNAMICS:

2.1 Scale invariance as an atmospheric symmetry principle:

A striking and immediate feature of the atmosphere and many other geophysical systems is the ubiquity of complex fractal structures. This complexity prompted Richardson (who is best known as the father of numerical weather prediction) to ask in 1926 "Does the wind have a velocity?" (are the trajectories of air particles smooth enough for derivatives to exist?), counterposing the example of the newly discovered Wierstrass function (whose graph is a fractal that is everywhere continuous but nowhere differentiable). Pursuing this idea, Richardson proposed that the variability in the atmosphere arises through a series of scale invariant cascade steps in which the energy flux from solar heating at large scales is redistributed over smaller and smaller scales by the non-linear dynamics (see fig. 1 for a modern model). This cascade idea is the basis of Kolmogorov's famous "scaling" (power law) $k^{-5/3}$ spectrum (for the energy in the wind at wavenumber k), and for a series of cascade models (see Monin and Yaglom 1975 for an early review) culminating in the multifractal models described below. During roughly the same period as Richardson, and in apparent contrast to the cascades which require a whole series of eddies of decreasing size, Bjerknes and the Norwegian school of meteorology emphasised the importance of a few large structures for forecasting, particularly the "fronts", while Leray and von Neumann in fluid mechanics in the 30's and 40's called for a better characterization of the singularities in fluid mechanics.

For some time (Schertzer and Lovejoy 1983a, 1985a, 1987a, 1988, 1989a, 1990a,b Lovejoy and Schertzer 1986, 1988, 1990b,c), we have argued that these seemingly disparate aspects can be united into a coherent framework if atmospheric dynamics respects a scale invariant symmetry principle in which the statistical properties of the large and small scale are related by a scale changing operation involving only the scale ratio. This idea is plausible since the basic equations governing the atmosphere (the Navier-Stokes equations) are

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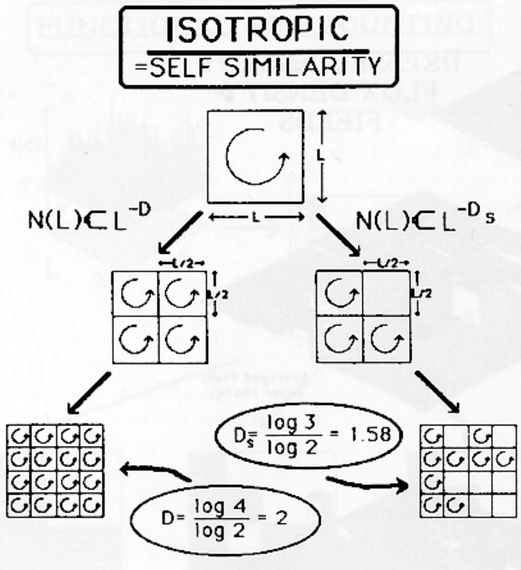


Fig. 1a: Schematic diagram showing two steps of the break-up of an eddy into sub-eddies.

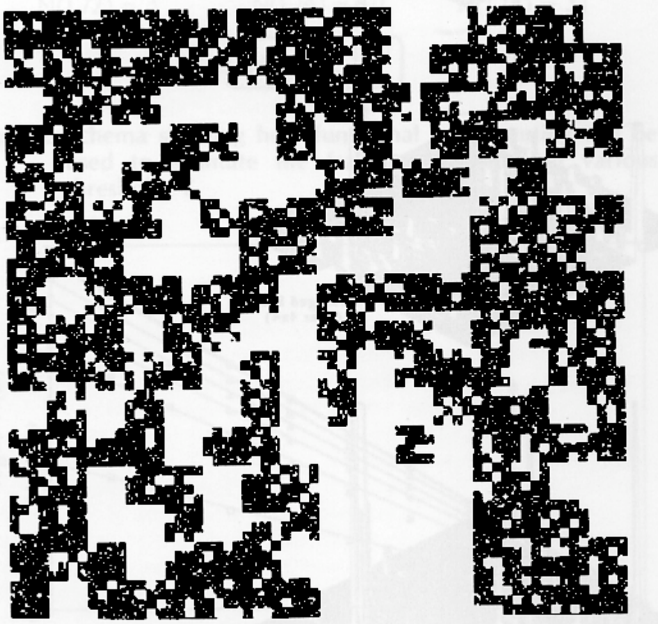


Fig. 1b: The result when λ=4, C=0.27, the back areas are "alive", the white areas "dead".

invariant under isotropic spatial dilations ("zooms"); $\underline{x} \rightarrow \lambda \underline{x}$, if the velocity ($\underline{v}$) is rescaled as $\underline{v} \rightarrow \lambda^{H} \underline{v}$, where λ is a scale ratio and H is an arbitrary scaling exponent (this allows for the possibility of multiscaling/multifractal solutions).

In real flows, viscosity will break this scaling at a small "viscous" scale (denoted η - typically $\approx$ 1mm in the atmosphere). Furthermore, energy injection is primarily at planetary scales since it comes from solar heating and the resulting equator/pole temperature difference defines the largest scale over which scaling can occur. Even though the energy input is modulated by cloud clover, the latter is also scaling and does not seem to break the overall scaling symmetry at any intermediate scale (Lovejoy 1981, 1982). As long as the surface boundary conditions do not break the scaling symmetry, we may therefore expect to find a range of scales exhibiting scale invariant statistics. Furthermore, the topography which is an important lower boundary condition is also (multiple) scaling over much of the same range (see fig. 2a,b for a "box-counting" analysis of the topography of France), and so again, the scaling symmetry is respected.

The atmosphere is not a simple fluid system: aside from the forcing and boundary conditions mentioned above, we must also take into account the anisotropy, notably due to gravity (which leads to differential stratification) and due to the Coriolis force (which leads to differential rotation). Other dynamical processes must also be considered, especially thermodynamic, radiative, and various processes involving water in its different forms. Taking all these factors into account, we will obtain a coupled system of non-linear partial differential equations describing the dynamics. However we may still expect the overall system to respect a scaling symmetry (even though the anisotropy requires such summetry to be more complex than the isotropic dilations discussed above). In fact, following a standard approach in physics, in the absence of specific symmetry breaking mechanisms, *the scaling symmetry is the only tenable assumption about the dynamics*. This argument (which applies also to many other geophysical systems, even when their governing partial differential equations are not known at all), is all the more plausible since below, we show how scale invariance can be generalized well beyond the restrictive self-similar systems associated with isotropic dilations ("zooms"). Furthermore, even self-similar scaling can be quite complex: since H is arbitrary, we may anticipate that its value will be different for weak and strong regions of our fields (the latter are multifractals).

2.2 *Phenomenology of fluid dynamics: flux conservation and cascades:*

Before proceeding to describe cascades in more detail along with the relevant multifractal formalism, we will need to appeal to two other aspects of the phenomenology of fluid dynamics. The first is the conservation property of the critical non-linear terms. The most important conserved quantity is the energy flux per unit mass, $\epsilon \left(= -\frac{\partial v^2}{\partial t}\right)$. Because of this conservation the energy flux sink for the energy flux injected at large scales is therefore provided by viscosity which is only important for very small scale structures. Furthermore, the mechanism for transfering flux from one scale to a smaller scales is only efficient if the scales are not very different from each other, hence the energy is "cascaded" from large to small scales ultimately being dissipated by viscosity. When we consider the other dynamical processes mentioned above, each new process will add a new conserved flux, and we may expect to obtain a set of nonlinearly coupled cascade processes.

The elements of scaling, conservation of flux, and cascades, lead directly to Kolmogorov's (1941) famous $k^{-5/3}$ energy spectrum for velocity fluctuations at wavenumber k. Expressed in terms of the size l of an "eddy" (fluid structure) this means the velocity $v \approx \epsilon^{1/3} l^{1/3}$ is an estimate of the velocity gradient across the eddy. Although Kolmogorov originally assumed that ϵ was not too variable in space, (that the cascade was "homogeneous"), it was soon realized that turbulence was in fact highly intermittent. In laboratory flows this is the "spottiness" analysed by Batchelor and Townshend 1949. In the atmosphere, the intermittency is expressed by the fact that most of the energy, water and other fluxes are concentrated in small (violent) regions: "storms", "cells" etc. Specific stochastic models of the variability of ϵ were subsequently developed. The elements of scale invariance, and flux conservation provided the basic ingredients of these multiplicative cascade processes (see fig. 1a,b) in which the large scale "parent" eddies multiplicatively modulate the energy flux to the smaller sub-eddies, while conserving the average energy

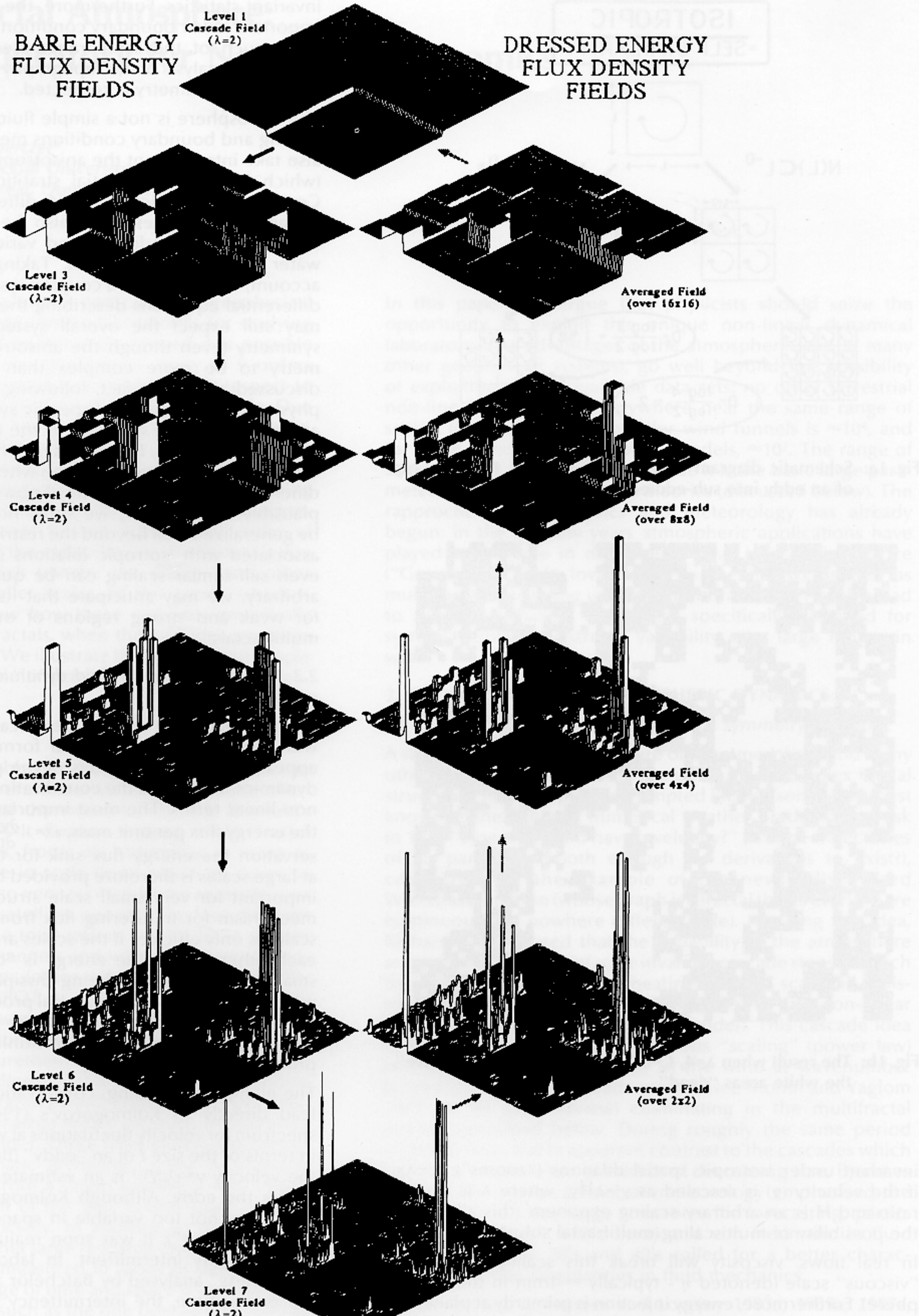


Fig. 1c: The left hand side shows the step by step construction of a ("bare") multifractal cascade (called an "α model") starting with an initially uniform unit flux density. The vertical axis represents the density of energy (ε) flux to smaller scales which is conserved by the non-linear terms in the dynamical equations governing fluid turbulence. At each step the horizontal scale is divided by two, and independent random factors are chosen either >1 or <1, normalized to ensure that <ε>=1. The developing spikes are incipient singularities of various orders. The right hand side shows the effect of smoothing (eq. 2) over larger and larger scales, it yields a "dressed" cascade.

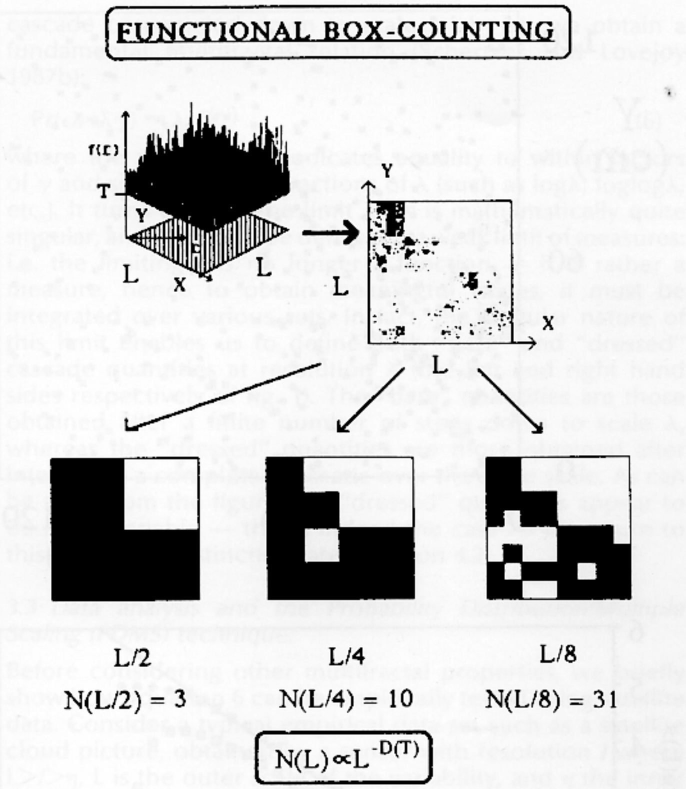


Fig. 2a: Schema showing how functional box counting can be used to estimate the fractal dimensions at various thresholds T.

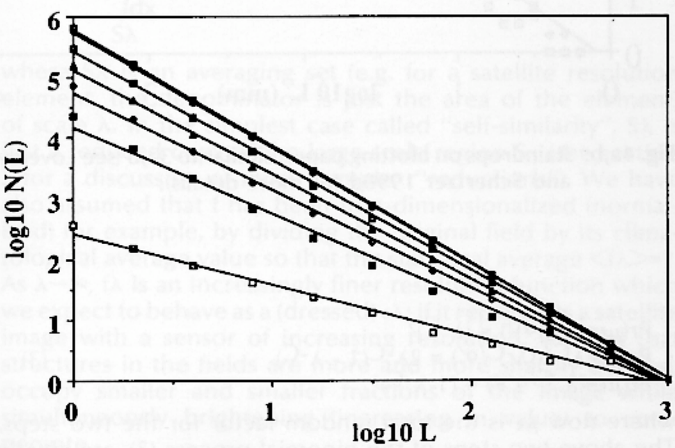


Fig. 2b: The results of functional box counting when applied to 1024x1024km topographic map of France at a 1km resolution (in collaboration with P. Ladoy, D. Lavallée). The lines (bottom to top) are the box counting results for altitude thresholds decreasing by factors of two from 3600m above sea level. The corresponding dimensions decrease from 0.84 (3600m) to 1,92 (28m).

flux. Interestingly, two seemingly quite different models were initially proposed to account for the intermittency: the first, the log-normal model in involving both log-normal probability distribution of ϵ and multiple scaling (Kolmogorov 1962, Yaglom 1966, Mandelbrot 1972 . . .), and the second, a birth/death type process involving only "dead" (calm) or "alive" (active) regions, now known as the "β model" (Novikov and Stewart 1964, Mandelbrot 1974, Frisch et al 1978 . . .). In the next section we will discuss cascade processes in more detail, showing how they generically yield multifractals. Furthermore, we will indicate how, by rendering them continuous in scale we obtain universal behaviour. The apparently disparate lognormal and β models are none other than the extremes of a continuous family of universal multifractals.

3. A REVIEW OF FRACTALS AND MULTIFRACTALS:

3.1 Sets, fractals, geometry:

Before returning to the atmosphere, we quickly survey some of the relevant fractal and multifractal notions. Consider the geometric idea of the dimension of a set of points. The notion that interest us here is that which relates the number of points in the set to its size. The intuitive (and essentially correct) definition of measure dimension that we will use below, is that the number of points $n(l)$ in a (fractal) set S at scale l (e.g. in a sphere of radius l) varies as:

$$n(l) \propto l^{d(S)} \quad (1)$$

where d(S) is the dimension of the set. Defining the "codimension" $c(S)=d-d(S)$ where d is the dimension of space in which the set is embedded, the set is a "fractal" if $c(S)>0$. A simple atmospherically relevant example is the number of in situ meteorological measuring stations on the earth in a circle radius l (fig. 3). If the measuring stations were uniformly distributed, over the surface ($d(S)=2$), we would obtain $n(l) \propto l^2$, however the actual distribution (fig. 3a) is highly non-uniform, empirically yielding $n(l) \propto l^{1.75}$. Alternatively, the density of points is proportional to $l^{d(S)}/l^{d}=l^{-c(S)}$ which for fractal sets decreases with l; the rate of decrease is characterized by $d-d(S)=c(S)$. The fractal dimension of a set therefore is a measure of its sparseness. Another example of fractal sets commonly occuring in the atmosphere is the set of rain drops. Fig. 4a,b shows the analogous analysis.

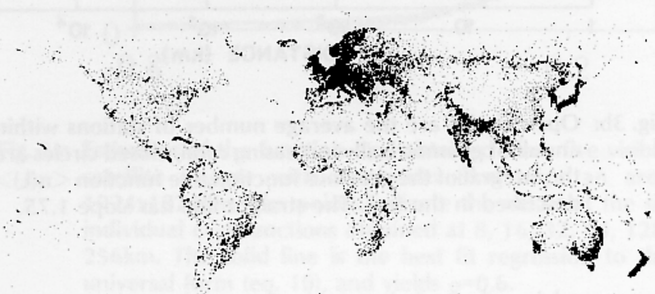


Fig. 3a: The locations of the 9,563 station in the global meteorological measuring network showing their high degree of sparseness (from Lovejoy et al 1986).

3.2 Measures, multifractals, dynamics:

Clouds are not sets, dynamics is not geometry; geophysical systems are usually fields (more generally measures) and their treatment takes us well beyond the geometry of sets, providing us with dynamical "multifractal" generators. Consider in more detail the cascade model shown in fig. 1b which was produced by dividing the unit square (the initial eddy, with $\epsilon=1$ everywhere), into four sub-squares (sub-eddies) each of scale $\lambda^{-1}x\lambda^{-1}$ where λ (=2 here) is the scale ratio. In each sub-eddy, we pick a random factor ($\mu\epsilon$) from the following binomial process (this just corresponds to a biased coin):

$$Pr(\mu\epsilon=\lambda^{C/\alpha}) = \lambda^{-C}$$
$$Pr(\mu\epsilon=\lambda^{C/\alpha'}) = 1 - \lambda^{-C} \quad (2)$$

where "Pr" indicates "probability" and the parameters C, α, α' are constrained so that on average the entry is flux

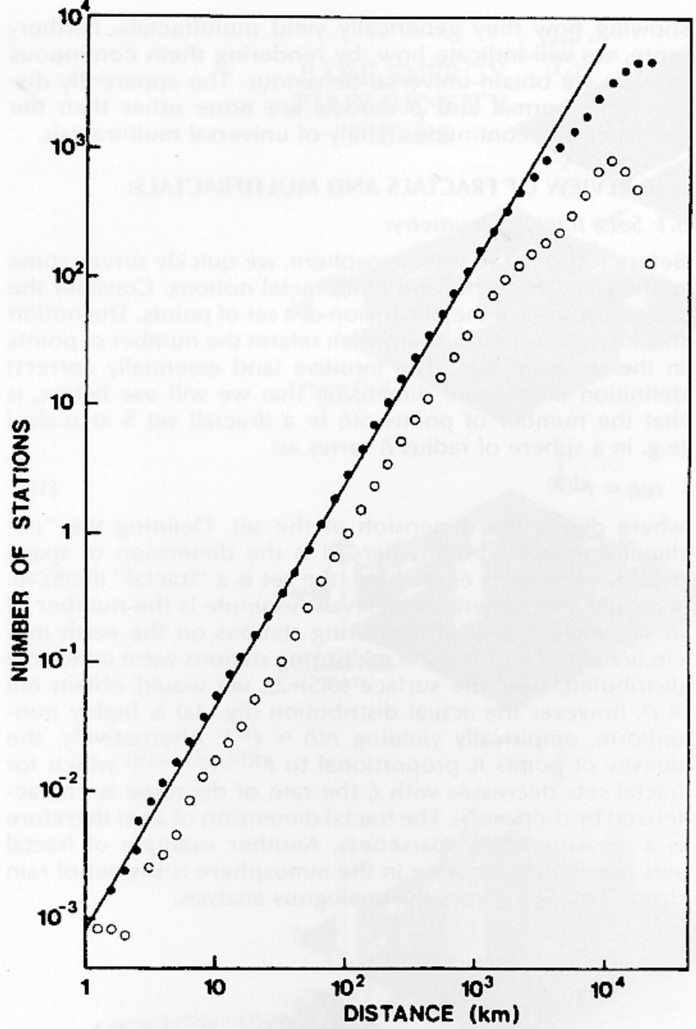


Fig. 3b: Open circles are the average number of stations within annuli of geometrically increasing radii, closed circles are the integral of the previous function, the function $\langle n(L)\rangle$ described in the text. The straight line has slope 1.75.

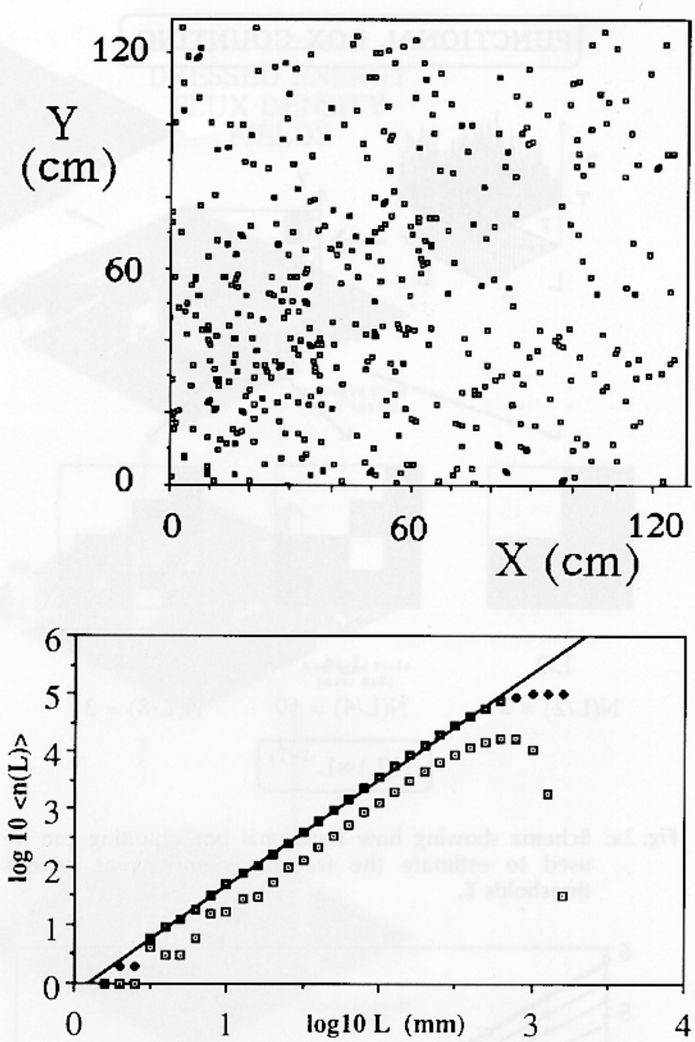


Fig. 4a,b: Raindrops on blotting paper: similar to 3a,b (see Lovejoy and Schertzer 1990d for more details).

conserved i.e. $\langle\mu\epsilon\rangle=1$. Just as in thermodynamics, we can distinguish between "micro-canonical" and "canonical" cascades in which ϵ is conserved respectively on each realization (in the above, the sum of the four random factors introduced at each step is constrained to be exactly =4) or only over the ensemble as in the above. Although in freqent use (e.g. Meneveau and Sreenivasan 1987), because of the scaling, the microcanonical conservation is actually quite restrictive (it could better be termed picocanonical — Schertzer and Lovejoy 1990a) and will not be discussed further. In the simplest canonical model, $\alpha=1$, $\alpha'=\infty$: eddies are either "dead" or "alive" (zero of finite), and in the limit of infinitely many cascade steps, the support of ϵ (the alive regions) is a fractal set, codimension C (see e.g. fib. 1b). This is the β model mentioned above. However, taking α, α' finite, we already obtain the much more interesting "alpha model" (fig. 1c, see Schertzer and Lovejoy 1983) which involves a whole hierarchy of intensity levels, each distributed over a different fractal set. Note that the α and α' used in this section should not be confused with those used elsewhere in this paper.

The notation we have adopted makes the multifractal nature of the process evident. Consider the above process iterated twice down to scales λ^{-2}. There are now three possible states with the following probabilities and values:

$$\begin{aligned}
&Pr(\mu\epsilon=(\lambda^{C/\alpha})^2) = (\lambda^{-C})^2\\
&Pr(\mu\epsilon=\lambda^{C/\alpha}\lambda^{-C/\alpha'}) = 2\lambda^{-C}(1-\lambda^{-C}) \qquad (3)\\
&Pr(\mu\epsilon=(\lambda^{-C/\alpha'})^2) = (1-\lambda^{-C})^2
\end{aligned}$$

where now $\mu\epsilon$ is the total random factor for the two steps. The above two steps of the binomial process (3), are clearly identical to a single step of a trinomial process with scale ratio $\lambda'=\lambda^2$ (=4 here):

$$\begin{aligned}
&Pr(\mu\epsilon=\lambda'^{C/\alpha}) = \lambda'^{-C}\\
&Pr(\mu\epsilon=\lambda'^{C(1/\alpha-1/\alpha')}) = 2\lambda'^{-C/2}-2\lambda'^{-C} \qquad (4)\\
&Pr(\mu\epsilon=\lambda'^{-C/\alpha'}) = 1+\lambda'^{-C}-2\lambda'^{-C/2}
\end{aligned}$$

repeating the process of replacing an n step binomial process with scale ratio λ per step, by a single step n+1 state process with scale ratio λ^n, we can associate each value of ϵ with specific exponents ($\epsilon\approx\lambda^{\gamma_i}$) each with probabilities:

$$Pr(\mu\epsilon=\lambda^{\gamma_i}) = \sum_j p_{ij}\lambda^{-c_{ij}} \qquad (5)$$

Where the p_{ij}'s are various weights (the "submultiplicities" Schertzer and Lovejoy 1987b) of the various nascent codimensions c_{ij} and λ is the total scale ratio of the n steps. In the limit $\lambda\rightarrow\infty$, the smallest c_{ij} will dominate the rest allowing us to associate specific codimensions ($c_{ij}=\min_j c_{ij}$) for every order of singularity γ_i. Denoting the multifractal cascade constructed down to scale λ^{-1} by ϵ_λ, we obtain a fundamental multifractal relation (Schertzer and Lovejoy 1987b):

$$\Pr(\epsilon_\lambda = \lambda^\gamma) \approx \lambda^{-c(\gamma)} \quad (6)$$

where the symbol "$\approx$" indicates equality to within factors of γ and slowly varying functions of λ (such as $\log\lambda$, $\log\log\lambda$, etc.). It turns out that the limit $\lambda \to \infty$ is mathematically quite singular, and ϵ can only be defined as a weak limit of measures: i.e. the limiting ϵ is no longer a function — it is rather a measure, hence to obtain meaningful values, it must be integrated over various sets. In fact, the singular nature of this limit enables us to define both "bare" and "dressed" cascade quantities at resolution λ (the left and right hand sides respectively of fig. 1). The "bare" quantities are those obtained after a finite number of steps down to scale λ, whereas the "dressed" quantities are those obtained after integrating a completed cascade over the same scale. As can be seen from the figure, the "dressed" quantities appear to be more variable — this is indeed the case — we return to this important distinction later (section 4.2).

3.3 Data analysis and the Probability Distribution/Multiple Scaling (PDMS) technique:

Before considering other multifractal properties, we briefly show how equation 6 can be empirically tested using satellite data. Consider a typical empirical data set such as a satellite cloud picture, obtained by a sensor with resolution l where $L > l > \eta$, L is the outer scale of the variability, and η the inner (e.g. viscous) scale. Define the scale ratio $\lambda = L/l > 1$ and the field smoothed at scale λ by:

$$f_\lambda = \frac{\int_{S_\lambda} f(\underline{x})d\underline{x}}{\int_{S_\lambda} d\underline{x}} \quad (7)$$

where S_λ is an averaging set (e.g. for a satellite resolution element, the denominator is just the area of the element) of scale λ. In the simplest case called "self-similarity", S_λ is just a reduced copy of the large scale region S_1 (see section 5 for a discussion of more complex "reductions"). We have also assumed that f has been non-dimensionalized (normalized) for example, by dividing the original field by its climatological average value so that the statistical average $\langle f_\lambda \rangle = 1$. As $\lambda \to \infty$, f_λ is an increasingly finer resolution function which we expect to behave as a (dressed) ϵ_λ; if it represents a satellite image with a sensor of increasing resolution, we find that structures in the fields are more and more sharply defined, occupy smaller and smaller fractions of the image while simultaneously brightening (increasing in value) to compensate.

We can now use eq. 6 to obtain empirical estimates of $c(\gamma)$ (Lavallée et al 1990). Taking logs and rearranging, we obtain:

$$c(\gamma) = -\frac{\log \Pr\left((\log f_\lambda)/(\log \lambda) > \gamma\right)}{\log \gamma} \quad (8)$$

Hence, plotting the normalized log probability distribution ($-\log \Pr/\log\lambda$) against the normalized intensity ($\log f_\lambda/\log\lambda$) we obtain the resolution (λ) independent function $c(\gamma)$. Fig. 5a,b shows the results when this technique is applied to 5 visible and 5 infra red GOES (Geostationary Operational Environment Satellite) pictures respectively over Montreal. Before the analysis was performed, the raw data (in an inconvenient satellite map projection) were remapped on a regular 8x8km² grid (conic projection) over a region of 1024x1024km². As can be seen, all the distributions are nearly coincident, in accord with the multifractal nature of the fields. To judge the closeness of the fits, we calculated the mean $c(\gamma)$ curves as well as the standard deviations for 8, 16, 32, 64, 128, 256 km, finding that the variation is very small, being typically about ± 0.02 in $c(\gamma)$ (Lovejoy and Schertzer 1990a,b, see Lavallée et al. 1990 for theoretical discussion and some numerical results).

3.4 Multiple scaling properties:

Equation 6 has an equivalent statement in terms of the statistical moments of ϵ_λ:

$$\langle \epsilon_\lambda^h \rangle = \lambda^{K(h)} = \int \lambda^{h\gamma} \Pr d\gamma \approx \int \lambda^{h\gamma - c(\gamma)} d\gamma = \int e^{(h\gamma - c(\gamma))\log\lambda} d\gamma \quad (9)$$

Where "$\langle\rangle$" indicates ensemble (statistical) averaging and $\Pr d\gamma$ is the probability density taken from eq. 6. Since $\lambda^{K(h)} = e^{K(h)\log\lambda}$, as $\log\lambda \to \infty$, for each moment h, there is a corresponding singularity γ_h which dominates the average: $h = c'(\gamma_h)$ (the method of "steepest descents") and we obtain another (equivalent) multifractal relation:

$$\lambda^{K(h)} = \lambda^{\max(h\gamma - c(\gamma))}$$
$$K(h) = \max_\gamma (h\gamma - c(\gamma)) \quad (10)$$

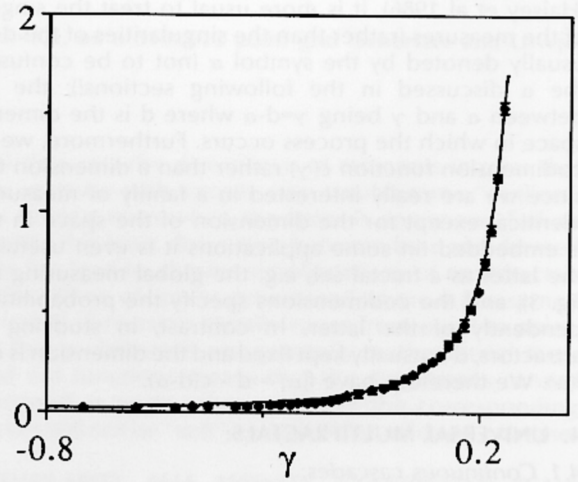


Fig. 5a: Estimates of the function $c(\gamma)$ obtained from five visible satellite images with resolution 8km over an area 1024x1024km. The points indicate the mean of the six individual $c(\gamma)$ functions obtained at 8, 16, 32, 64, 128, 256km. The solid line is the best fit regression to the universal form (eq. 10), and yields α=0.6.

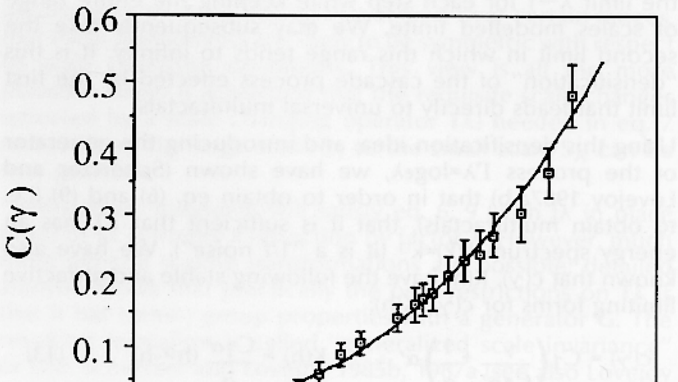


Fig. 5b: The same scenes viewed at infra red wavelengths. α=1.7.

The above (Legendre) transformation is equal to its inverse, hence we also obtain:

$$c(\gamma) = \max_h (h\gamma - K(h)) \qquad (11)$$

showing the complete equivalence between a description in terms of moments (characterized by K(h)) or probabilities (characterized by c(γ)). It is also possible to define another ("dual") codimension function (Hentschel and Procaccia 1983, Grassberger 1983, Schertzer and Lovejoy 1983; see eq. 16 below for its physical significance) associated with moments of various orders:

$$C(h) = \frac{K(h)}{h-1} \qquad (12)$$

For those who are familiar with multifractals, it is worth noting here that we have denoted the orders of singularities by the symbol γ because the atmospheric quantities of interest are modelled by densities of multifractal measures (such as ε) and γ gives the orders of these singularities directly. In other systems such as phase space portraits of strange attractors (Halsey et al 1986), it is more usual to treat the singularities of the measures (rather than the singularities of the densities) usually denoted by the symbol α (not to be confused with the α discussed in the following sections!); the relation between α and γ being $\gamma = d - \alpha$ where d is the dimension of space in which the process occurs. Furthermore, we use the codimension function c(γ) rather than a dimension function since we are really interested in a family of measures each identical except for the dimension of the space in which it is embedded (in some applications it is even useful to take the latter as a fractal set, e.g. the global measuring network fig. 3), and the codimensions specify the probabilities independently of the latter. In contrast, in studying strange attractors, d is usually kept fixed and the dimension is denoted f(α). We therefore have $f(\alpha) = d - c(d-\alpha)$.

4. UNIVERSAL MULTIFRACTALS:

4.1 Continuous cascades:

The cascade discussed above was based on iterating a scale invariant cascade step each of which increased the range of scales by a discrete (integer) ratio λ. In these processes, the limit of an infinite number of cascade steps is simultaneously the limit of infinite scale ratios. It turns out that if these limits are taken simultaneously, universal behaviour is generally not obtained — this is the source of some misplaced claims that universal behaviour is either absent or is very limited. However, rather than involving discrete ratios, physical models should be continuous, i.e. we should take first the limit $\lambda \to 1$ for each step while keeping the entire range of scales modelled finite. We may subsequently take the second limit in which this range tends to infinity. It is this "densification" of the cascade process effected by the first limit that leads directly to universal multifractals.

Using this densification idea, and introducing the generator of the process $\Gamma\lambda = \log\epsilon\lambda$, we have shown (Schertzer and Lovejoy 1987a,b) that in order to obtain eq. (6) and (9) (i.e. to obtain multifractals), that it is sufficient that Γλ has an energy spectrum $E(k) = k^{-1}$ (it is a "1/f noise"). We have also known that c(γ), K(h) have the following stable and attractive limiting forms for c(γ), K(h):

$$c(\gamma) = C_1\left(\frac{\gamma}{C_1\alpha'} + \frac{1}{\alpha}\right)^{\alpha'} \qquad K(h) = \frac{C_1\alpha'}{\alpha}(h^\alpha - h) \qquad (13)$$

with $C_1 \le d$, $0 \le \alpha \le 2$, $1/\alpha + 1/\alpha' = 1$, $\alpha \ne 1$ (for α=1, see below). C_1 and α are the fundamental parameters: unlike the geometric parameter d(S) which describes the sparseness of scaling sets, C_1 and α are dynamical, characterizing the (probability) generator Γλ of the process. C_1 characterizes the codimension of the mean of the process, hence the sparseness of the mean field, α (see below), the distance from monofractality. Note that with the unique exception α=2, the formula for K(h) is valid only for h>0, when h<0, K(h)=∞.

The derivation of eq. (13) relies on the (general) central limit theorem for the addition of random variables since the generator Γλ is the sum of elementary noises; α=2 is the familiar case, Γλ is gaussian. α<2 corresponds to the less well known infinite variance cases of the central limit theorem: rather than gaussians, we obtain (stable) "Levy" distributions, index α (an interesting technical point is that only the maximally asymmetric or "extremal" Levy variables are possible here, since otherwise K(h) would not be finite for any h — this appears to be the first application of extremal Levy variables, see Schertzer et al 1988, Schertzer and Lovejoy 1990 in particular appendix A).

There are five principle universality classes: α=0, 0<α<1, α=1, 1<α<2, α=2 (see fig. 6 for an illustration). We have already seen that the case α=2 yields the log-normal multifractal: this is actually a multifractal presentation of the process discussed by Kolmogorov 1962, Obukhov 1962. As we decrease α (1<α<2), we still have a regime with unbounded singularities (α′>2) but the generator becomes discontinuous characterized by extreme regularities and "holes". In the limit $\alpha \to 1$, we obtain exponential behaviour ($c(\gamma) = C_1\exp(\gamma/C_1 - 1)$, $K(h) = C_1 h\log h$)). In the case α<0<1, we have bounded singularities (the upper bound is $-C_1\alpha'/\alpha, \alpha' < 0$), and finally in the limit $\alpha \to 0$, we obtain the (monofractal, single fractal dimensional) β model, the other early cascade model corresponding to a birth/death process.

When α=2, the multifractals filtered at a finite scale of homogeneity have lognormal probabilities (see however the caveate below about "dressed" quantities corresponding to the small scale limit), and when α is not too much less than 2, they will be not too far from lognormal (the extreme fluctuations will be less violent). The multifractal nature of the atmosphere is therefore consistent with the widespread atmospheric lognormal (or "roughly" lognormal) phenomenology (see however fig. 5 where we find empirical values α≈1.7, α≈0.6 which are <2). Geophysical quantities generally are extremely variable, and the geophysical literature abounds with empirical fits to log-normal distributions. We suspect that closer inspection will reveal that the corresponding fields are actually multifractals and that the distributions are not really log-normal.

The above c(γ), K(h) functions are for conserved (stationary) quantities. For other quantities such as the velocity field, or the concentration of a passive (inert) admixture (see fig. 7 for an illustration) which are related at least for the "bare" singularities to these by either dimensional and/or power law relations, the corresponding c(γ) functions can be obtained by the linear transformation $\gamma \to a\gamma + b$.

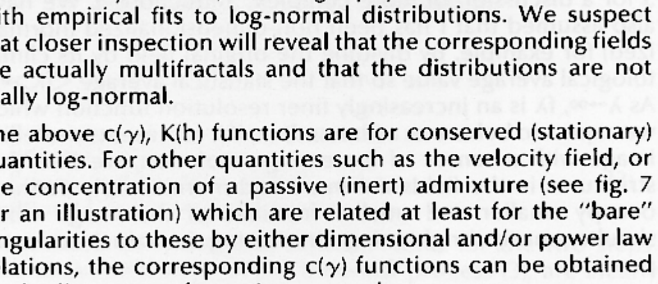
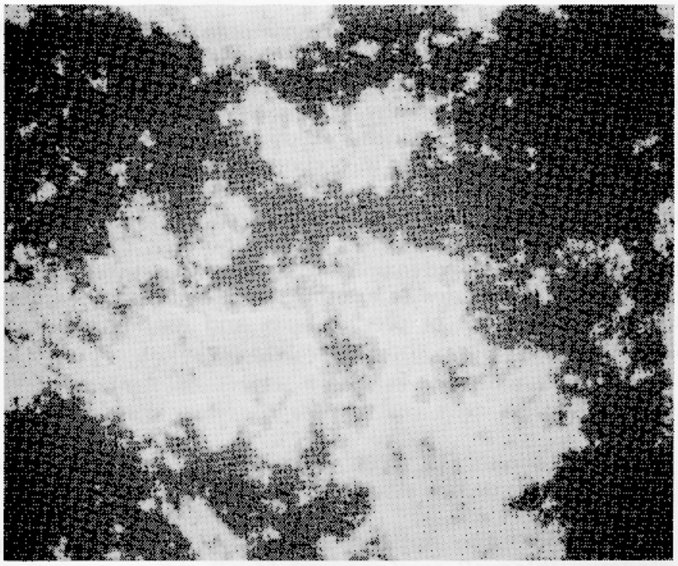


Fig. 6a: universal (bare) singularities codimension $c(\gamma)/C_1$ corresponding to the five classes; here α=2, 1.5, 1, .5, 0.

Fig. 6b: universal (bare) second characteristic function $K(h)/C_1$ (≡ $h.F(h)/C_1$, F(h) being the "free energy"), corresponding to the five classes; here α=2, 1.5, 1, .5, 0.

Fig. 6: The five basic universality classes.



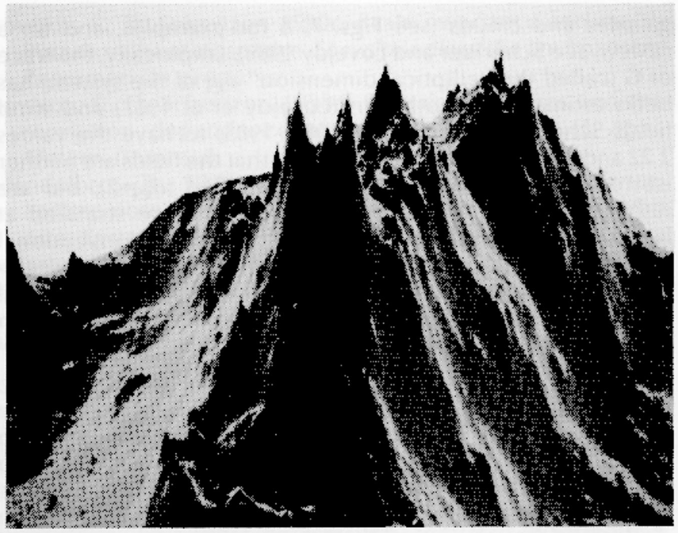


Fig. 7: Multifractal passive scalar cloud from a continuous cascade process with α=1.6, on a 512x512 point grid (Schertzer and Lovejoy 1990a, in collaboration with Jean Wilson, Gadjendra Sarma).

4.2 Bare and dressed multifractal properties, statistical outliers and the measurement of atmospheric quantities:

Before turning to the problem of anisotropy, we must first discuss a complication which arises because of a basic distinction between "bare" and "dressed" multifractal properties. The "bare" properties are essentially theoretical: they are typically obtained after a cascade process has proceeded only over a finite range of scales (see the left hand side of fig. 1c); strictly speaking, the above equations for $c(\gamma)$, $K(h)$ apply only to these quantities. The experimentally accessible quantities are different; they are obtained by integrating cascades (e.g. with a measuring device) over scales much larger than the inner scale of the cascade (i.e. in the atmosphere over scales >> 1mm). The properties of such spatial (and/or) temporal averages are approximated by those of the "dressed" cascades i.e. those in which the cascade has proceeded down to the small scale limit and then integrated over a finite scale (see the right hand side of fig. 1c). The small scale limit of these multiplicative processes is singular and is responsible for this basic distinction (it is also because of this that measuring atmospheric quantities is far from trivial!).

Unlike the bare cascade, the dressed cascade displays the interesting phenomenon of divergence of high order statistical moments, that is:

$$\langle \epsilon^h \rangle \rightarrow \infty \text{ for all } h \geq h_d \quad (14)$$

Where h_d is the critical exponent for divergence. The precise condition for divergence is quite simple (Schertzer and Lovejoy 1987a,b):

$$C(h_d)=d(S) \quad (15)$$

where S is the averaging set (e.g. line, plane, or fractal in the case of typical measuring networks), over which the process is averaged. The phenomenon of divergence of high order statistical moments arises directly from the fact that $C(h)$ is often unbounded (see eqs. 12, 13), and hence for large enough h, $C(h)>d(S)$. Conversely for a fixed h, divergence will still occur if the set S is sufficiently sparse so that $d(S)$ is small enough. The dressed codimension function is the same as that of the bare function for $\gamma<\gamma d$ where $c'(\gamma_d)=h_d$. For $\gamma>\gamma_d$, it is a straightline, slope h_d. Empirical values of h_d include 5, 5, 1.7 for wind, temperature, and rain respectively (see Lovejoy and Schertzer 1986 for a review). In empirical data sets the divergence of moments implies that moments increase with sample size; troublesome "outliers" continue to exist even when the latter is enormous. Failure to appreciate this basic distinction between bare and dressed quantities has lead to abusive simplifications of multifractals such as the notion of "point dimension" where it is assumed that the fractal dimension literally varies from point to point. The fact that multifractals are generally mathematical measures and not functions means that the dimensions cannot be understood as point values, and that the corresponding multifractals will not be "soft" or "calm" but wild and extreme.

5. SELF-SIMILARITY, SELF AFFINITY, GENERALIZED SCALE INVARIANCE:

We have considered the example of scaling in fluid turbulence where isotropic scaling ideas have been developed over a considerable period of time. However, the atmosphere is not a simple fluid system, nor is it isotropic; gravity leads to differential stratification, and the rotation of the earth to the Coriolis force and to differential rotation; radiative and microphysical processes (e.g. cloud/raindrop dynamics) lead to further complications. However even though the exact dynamical equations are unknown (as is generally the case in geophysics), we have argued that at least over certain ranges, that these phenomena are likely to be symmetric with respect to scale changing operations. This view is all the more plausible when it is realized that the requisite scale changes (effected by a scale changing operator $\mathbf{T}\lambda$) needed in eq. 7 to transform the large scale S_1 to the small scale S_λ can be very general.

"Self-similar" measures will satisfy eq. 6, 9 (with $f\lambda$ in place of $\epsilon\lambda$) of $\mathbf{T}\lambda$ is simply a reduction by factor λ. However, much more general scaling transformations are possible; detailed analysis shows that practically the only restrictions on $\mathbf{T}\lambda$ is that it has (semi-) group properties with a generator $\mathbf{G}$. The resulting formalism is called "generalized scale invariance" or GSI: Schertzer and Lovejoy 1985b, 1987a (see also Lovejoy and Schertzer 1985 for anisotropic fractal cloud simulations based on GSI). For example "self-affine" measures involve reductions coupled with compression along one (or more) axes; $\mathbf{G}$ is a diagonal matrix. If $\mathbf{G}$ is still a matrix ("linear GSI") but has off-diagonal elements, then $\mathbf{T}\lambda$ might compress an initial circle S_1 into an ellipsoid as well as rotate the result. Linear and non-linear GSI have already been used to model galaxies and clouds (see Figs. 7, 8 for examples, and for a review, see Schertzer and Lovejoy 1989). Empirically, the trace of **G** (called the "elliptical dimension" d_{el} of the system) has been estimated in both rain (Lovejoy et al 1987) and wind fields Schertzer and Lovejoy 1983, 1985) to have the values 2.22 and 2.55 respectively, indicating that the fields are neither isotropic (d_{el}=3), nor completely stratified (d_{el}=2), but are rather in between, becoming more and more stratified at larger and larger scales. This contrasts with the conventional view that arbitrarily splits the atmosphere into a large scale-similar two-dimensional (flat) regime with d_{el}=2, and a small scale (isotropic) three dimensional self-similar regime (with d_{el}=3) seperated by an elusive "dimensional transition".

Other relevant applications of anisotropic scaling include work (in progress) to estimate **G** empirically from satellite pictures in order to determine the scale changing group associated with differential rotation in the horizontal and to study its relation to cloud texture and classification. GSI is also the natural framework for introducing dynamics (time evolution) into the cascade (Schertzer and Lovejoy 1985a, Lovejoy and Schertzer 1985). For example preliminary results in rain indicate that d_{el}=2.62 for (x,y,t) space (two horizontal coordinates and time), showing that structures in (x,y,t) space are compressed along the t axis. Combining this result with the above for spatial stratification indicates that the full (x,y,z,t) rain process has d_{el}=2.84.

Defined by both the function $c(\gamma)$ (or equivalently, by the probability generator), and the scale changing generator **G**, anisotropic multifractals display a tremendous variety of behaviour: scaling systems therefore form a very broad class. Although in meteorology, there are good theoretical reasons to expect multifractal behaviour, we are just beginning to explore these systems, and there is no consensus about the exact limits. Systematic multifractal analysis of atmospheric fields as well as their numerical simulation (which produces surprising multifractal images — see fig. 7), will undoubtedly help us understand atmospheric dynamics, predictability and its limits, as well as contribute towards quantitative uses of remotely sensed and in situ data.

6. CONCLUSIONS

In the past, physicists have often shunned the atmosphere as an object of study, partly because it is practically impossible to control the parameters of this "laboratory", but even more importantly because of the extreme variability and strong anisotropy which easily leads to the impression that the statistical properties are quite non-stationary, and hence that appropriate statistical ensembles are impossible to define. This impression was all the more cogent since the necessary theoretical frameworks and corresponding data analysis and simulation techniques were not available until recently (and are currently undergoing rapid development).

In this paper, we have argued that both objections are not as relevant as they once seemed. On the one hand, the sheer quantity of data available helps to compensate for the lack of laboratory controls. More fundamentally, laboratory situations even approaching the range of scales in the atmosphere are not possible, so that in practice such data may be indispensible. On the other hand, by outlining some recent developments in non-linear dynamics — particularly multifractals — we have shown that extreme variability (very similar to that which is observed) arises quite naturally as a result of the multiplicative cascade processes actually at work. We showed that the apparent non-stationarity of atmospheric statistics (including the frequent occurence of statistical "outliers", which are often regarded as symptoms of non-stationarity), is in actual fact totally consistent with stationary multifractal cascades processes. In principle, it is possible that most atmospheric "situations" can be regarded simply as different realizations of the same ensemble (produced by the same dynamical stochastic process). We illustratd these ideas by analysing various data sets and producing simple multifractal cloud and topography models which, already displayed many realistic features. Further study of the various universal multifractals, in particular in conjunction with (anisotropic) "Generalized Scale Invariance" ("GSI"), may help reduce the existing meteorological "zoo" of structures ("fronts", "highs", "lows", "troughs", "clusters", "cells", "supercells", "jets", etc.) to a few manageable dynamically significant statistical parameters. It also provides an exciting new framework in which to study the issues of predictability and its limits, order, disorder, data analysis and stochastic simulation.

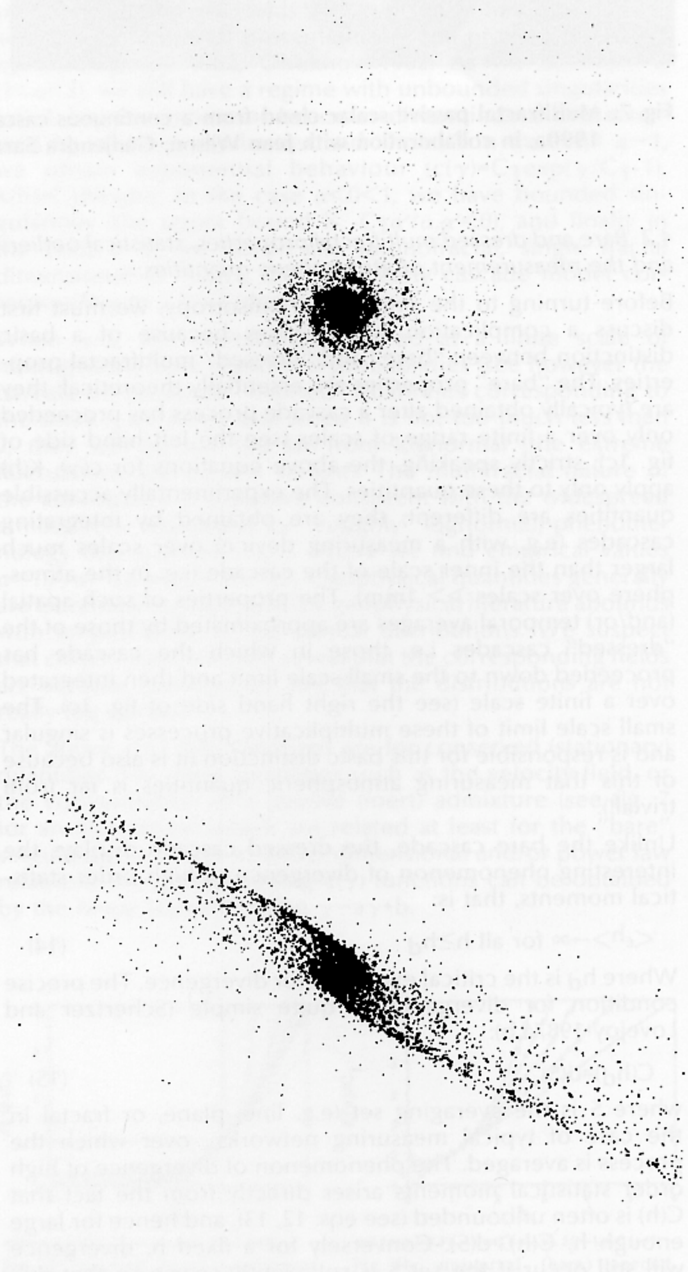


Fig. 8: Galaxy simulations and examples of differential stratification and rotation.

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