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GENERALISED SCALE INVARIANCE AND ANISOTROPIC INHOMOGENEOUS FRACTALS IN TURBULENCE[1]

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A generalisation of scaling is presented to deal with anisotropy and (multidimensional) intermittency. Implications, especially for meteorological fields, are discussed.

1. INTRODUCTION

Many geophysical fields are extremely variable over a wide range of time and space scales. The variability of the atmosphere is large over at least 9 orders of magnitude ($\sim$1mm to $\sim$1000 km) and creates strongly intermittent and anisotropic structures : the energy spectrum (E(k)) of the horizontal wind in the horizontal is $\sim k^{-5/3}$ whereas it is (roughly) the much steeper $\sim k^{-11/5}$ in the vertical. This difference is the spectral counterpart of the (large) vertical stratification.

For both analysing and simulating these structures, it is necessary to generalise both the notion of scale invariance and intermittency, through the introduction of anisotropic metrics and dimensions, and scale invariant measures characterised by multiple (fractal) dimensions. Interesting consequences are that multidimensionality is directly connected with the divergence of high statistical moments of average cascade quantities, multiplicative processes and new questions on detectability and predictability.

2. GENERALISED SCALE INVARIANCE (G.S.I.)

2.1 Motivations

To avoid the untenable dichotomy 2D/3D for large/small scales, we have proposed an alternative scaling model[1-3] (see also [4-5] for non-mathematical reviews) of atmospheric dynamics: the anisotropy introduced by gravity via the buoyancy force results in a differential stratification and a consequent modification of the effective dimension of space (from the isotropic value D=3 to 23/9=2.5555...).

In order to take into account this and other effects such as the differential rotation introduced by the Coriolis force, a general formalism of scaling is required. The fundamental problem is that of finding a family of "balls" representing the statistical properties of eddies at different scales, via (mathematical) random measures, such as the flux of energy through structures of a given scale.

2.2 Generalised notion of scale

Close examination of the phenomenology of

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turbulent cascades outlined the basic properties associated with the notion of scale and leads[6] to the following abstract definition in terms of a group (the "scaling group") of operators T_λ acting on a topological space M :

(i) T_λ is a multiplicative group ($\lambda \in R_+^*$) of transformations from M to M:

(1) $T_{\lambda\lambda'} = T_\lambda \circ T_{\lambda'} \qquad \forall \lambda, \lambda' \in R_+^*$

(in particular : $T_1 = 1$ = the identity and $T_\lambda^{-1} = T_{\lambda^{-1}}$)

(ii) there exists a family $\mathcal{B}_1$ of "balls" (open sub-sets of M) such that $\mathcal{B} = T_\lambda \mathcal{B}_1$ is a basis for the topology of M

(iii) there exists an increasing function ϕ from $\mathcal{B}$ to R_+, bounded on $\mathcal{B}_1$ and which factorizes in ($D \in R_+$) :

(2) $T_\lambda \phi = \lambda^D \phi \quad , \quad \forall \lambda$

(T_λ is naturally defined by : $T_\lambda \phi(B) = \phi(T_\lambda B), \quad \forall \lambda, B$)

Note the expression λ^D results from the group property of T_λ since it would be implied by the assumption of the existence of a continous function $g(\lambda)$ in (2).

As is easily shown, in case of a metric space, D plays the role of a dimension and ϕ can be taken as the radius of the balls defined by the distance, T_λ is the usual dilatation in case of isotropy. More generally we can use the measurability property of the balls. For instance on R^d, we can take ϕ as the Lebesgue (d- volume) measure, by supposing that the B's are Lebesgue measurable, and D equals d if the balls are the usual spheres or cubes. This is no longer true with strongly anisotropic balls (such as self-affine, but not self-similar, ellipsoïds). Even more anisotropic (and/or irregular) balls can be dealt with: (anisotropic) fractal sets. In all these cases ϕ can be taken as the measure which is finite and positive on the balls (Lebesgue or Haussdorff) and the scale is given by $\phi^{1/D}$.

2.3 Linear GSI case

The group T_λ is generated by a (bounded) linear application G according to :

(3) $T = \exp(G \log \lambda) = \sum_{n=0}^{\infty} (\log\ )^n G^n / n!$

The following conditions[6] are necessary and sufficient to obtain a scaling group:

a) measurable case: $D_{el} = \mathrm{Trace}(G) > 0$

D_{el} can be considered as the effective dimension of the space, or its elliptical dimension [1-3]. Non-linear examples are given in [6].

b) metric case: $\inf \mathrm{Re}\, \sigma(G) \geq 1$

where $\sigma(G)$ is the spectrum of G. If the unit ball is defined by the ellipsoïd generated by a symmetric operator A, the following condition is obtained:

(5) $\inf \sigma(\mathrm{sym}\,(AG)) \geq 1$

(Sym (AG) denotes the symmetric part of AG)

Particularly simple examples of linear GSI are obtained by the use of quaternions[6], and used to exploit[7] the FSP model[8-9] to give examples of (mono-dimensional) fields respecting linear metric GSI.

3. GSI AND MULTIDIMENSIONAL INTERMITTENCY

3.1 Introduction

Usual stochastic processes (such as Brownian motion) are obtained by the (weighted) addition independent identically distributed (i.i.d.) random variables (e.g. integrals of white noise). Conversely, the multiplicative group T_λ suggests that in GSI the most natural type of process to use are those obtained by multiplication, corresponding also to the non-linear breaking of eddies.

The former case is mono-dimensional, while the latter generally leads to multiple dimensions. Many efforts have been made to relate the most obvious aspect of intermittency-its "spottiness"[10]- to a turbulent support with a single fractal dimension[11,12]. However, as pointed out in[1,2], phenomenological models of intermittency[13,14,11] lead, generally to multiple dimensions, corresponding to the different (tensorial) powers of the measure of the flux of energy. Indeed, a sequence of dimensions is easily obtained[2,3,15,16] by considering the divergence of high moments of the density ε_A of this flux with respect to different D_A-dimensional Hausdorff measures:

(6) $< \varepsilon^h{}_A > = \infty$ for $D_A < C(h) = D_{el} - D(h)$

$C(h) = \log_\lambda < W^h > /(h-1)$

where W is the random variable which distributes the density during a step of the cascade.Note that the condition stated in Eq.6 corresponds to the one of non-intersection of sets A and S(h) of co-dimension C(h) (since, usualy for sets A, B: $D(A \cap B) = D(A) - C(B)$, D and C indicating the dimension and co-dimension of the referenced sets). Increasing h corresponds to studying the more intense regions. C(h) is an increasing function of h, or the most intense regions are the most sparsely distributed.

3.2 GSI and multiplicative processes-multiplicative chaos

Instead of adding random increments of finer and finer resolution along the cascade (as in the FSP[8-9]), one may multiply by random increments of finer and finer resolution. This multiplicative procedure corresponds to the non-linear break-up of eddies into sub-eddies (Mandelbrot's cascade model of intermittency on a rigid grid corresponds to a discrete product).

The limit of such processes represents a mathematical problem -called multiplicative chaos- where some results have been recently obtained[17]. Nevertheless, due to the multiplicative property of both T and the way the process is constructed, we may introduce the co-dimension function C(h) (f being a multiplicative density increment):

(8) $< T_\lambda^{-1} f^h > = \lambda^{(h-1)C(h)} < f^h >$

and generalise thus ealier results (Eq.6).

3.3. Implications of multidimensionality

We introduce the "structure integral" S(h, A), instead of the classical structure function, to study the behavior of a stochastic measure m:

(9) $S(h, A) = < m^h(A) > / < m(A) >^h$

where A is a D_A-dimensional measuring-set (e.g.:on which the averages are taken). Generalised scale invariance implies :

(10) $S(h, T_\lambda A) = \lambda^{-p(h, D_A)} S(h, A)$

For the simple case where the phenomenon is mono-dimensional with dimension D_s (co-dimension $= D_{el} - D_s = C_s$) $p(h, D_A)$ is linear in h:

(11) $p(h, D_A) = C_s.(h-1)$, if: $D_A > C_s$

In other cases, C_s has to be replaced by the co-dimension function C(h) (hence $p(h, D_A)$ is no longer linear in h) and the (physical) measures become sensitively dependent on D_A (especialy due to the the condition of intersection). Such a study on radar-data[18] (with D_A= 1, 1.5, 2, 3, 4), supports multidimensional behaviour for the rain.

A dimensional detectability condition results from the condition of non-degeneracy of the statistics (i.e.:non-zero and finite) that is (according to Eq.6 or 11) the intersection of the set of observation (A) and the support sets S(h). Thus scale resolution is not sufficient to estimate the detectability of the phenomena, e.g. the most intense phenomena will be lost with a sparse set of observations. It has been shown[19] that indeed ground networks have lower dimensions than 2 (e.g. the world meteorological surface network has D_A 1.75). Due to non-linear interactions, it turns out to raise new questions on predictability, which until now has been studied only in terms of scales.

4. CONCLUSION

Motivated by the strong anisotropy and intermittency of the atmosphere, we have developed a formalism called generalised scale invariance. The formalism is based on two sets of elements and may be regarded as an extension of earlier work on cascade processes (especially[10,3]).

The first is a group of general scale changing operators, whereas the second are the intermittent measures invariant under the operators. In a turbulent cascade, the scale changing operator transforms eddies into sub-eddies, while leaving the physically significant energy flux invariant (here represented by a scaling measure). It may be worth noting that explicit geometry is not always required, since measurable properties are sufficient.

We stressed that multidimensionality is theoretically the rule for multiplicative processes, and such a behaviour has been tested directly on radar determined rain field[18]. It raises new questions on detectability and predictability of turbulent phenomena because of dimensional dependance.

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