GENERIC MULTIFRACTAL PHASE TRANSITIONS AND SELF-ORGANIZED CRITICALITY

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ABSTRACT

We discuss statistical mechanisms leading to phase transitions in high dimensional multifractal processes. Second order transitions naturally arise from finite sample sizes, whereas first order transitions are consequences of the scale and dimension of the observations. These transitions are elaborated and their genericity are underscore. We point out the close first order, low temperature phase transitions relation to self-organized criticality and its implications for nonlinear physical systems, in particular in astrophysics.

1. Introduction

Multifractal variability crops up in fields ranging from strange attractors[1,2], turbulence[3-5], statistical physics[6,7], high energy physics[8], astrophysics[9] and geophysics[10]. The basic scaling behavior of these dissipative nonequilibrium systems is determined by exponent functions which have analogues in thermodynamics (for reviews see refs.[11], we will rather follow[12,13]). Two distinct statistical mechanisms lead to phase transitions (discontinuities of the free energy and thermodynamic potential analogues). "Frozen free energy" (second order) transitions[7,8] arise from finite sample sizes[14], whereas much more wild first order transitions are consequences of the scale and dimension of the observations on large samples[3,15]. These are totally different from the high temperature transitions found in (low dimensional) deterministic chaos[16,17] which are basically created by breaks in the scaling symmetry of the probability measure in phase space. In our case these phase transitions are generically produced by the scaling and have direct implications for extreme, catastrophic, events in physical space. They can explain recent results in turbulence[18] as well as the appearance of self-organized critical phenomena[19]. Finally, we give empirical evidence from astrophysical catalogues that the integrated luminosity function does indeed have a first order multifractal phase transition.

2. Multiplicative processes and multifractal codimension formalism

To study these transitions, we rely on multiplicative processes[3,20,21,22] produced by scaling random multiplicative modulations of larger structures by smaller ones which yield highly intermittent space/time fields (see Fig.1a). Each realization corresponds to a finite D-dimensional cut of a process in an infinite dimensional probability space.

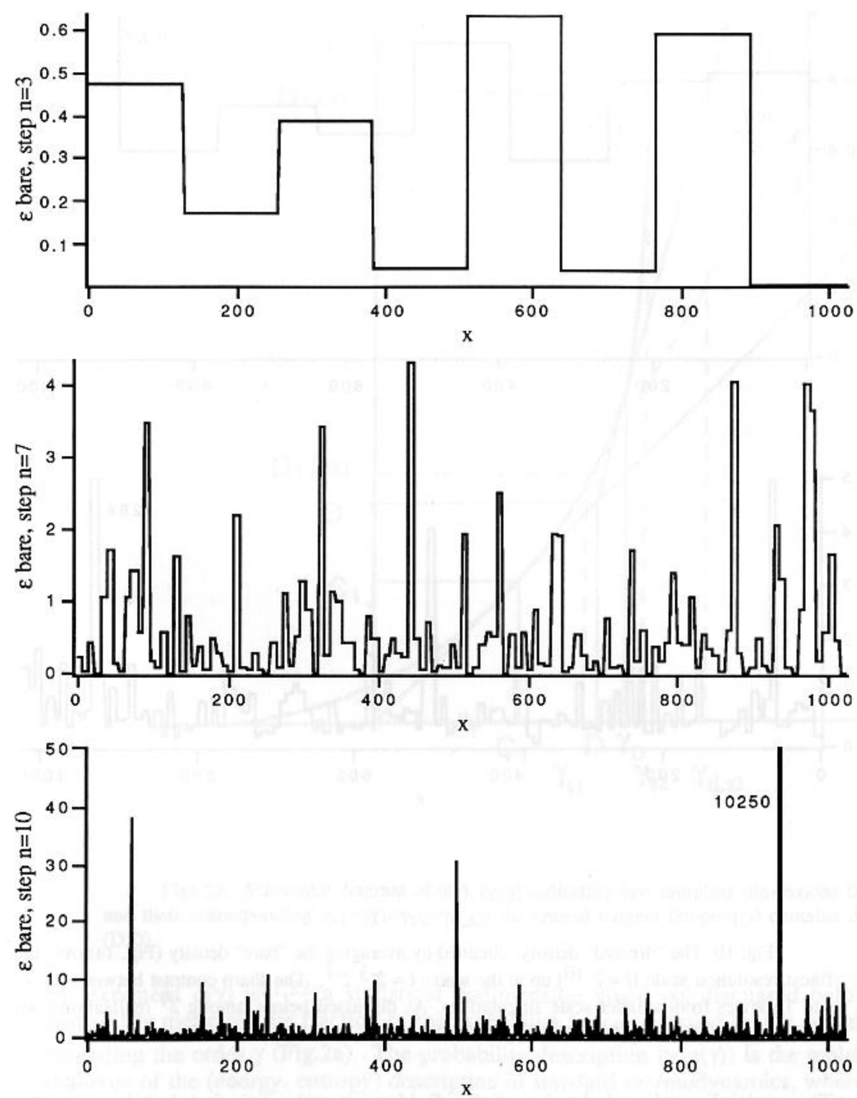


Fig 1a: Construction of a "bare" multifractal field by a multiplicative cascade process ("lognormal" model with C_1=0.9, see discussion below) starting with an initial uniform unit density. At each step the homogeneity scale l is divided by 2 and we display from top to bottom $l = 2^{-3}, 2^{-7}, 2^{-10}$.

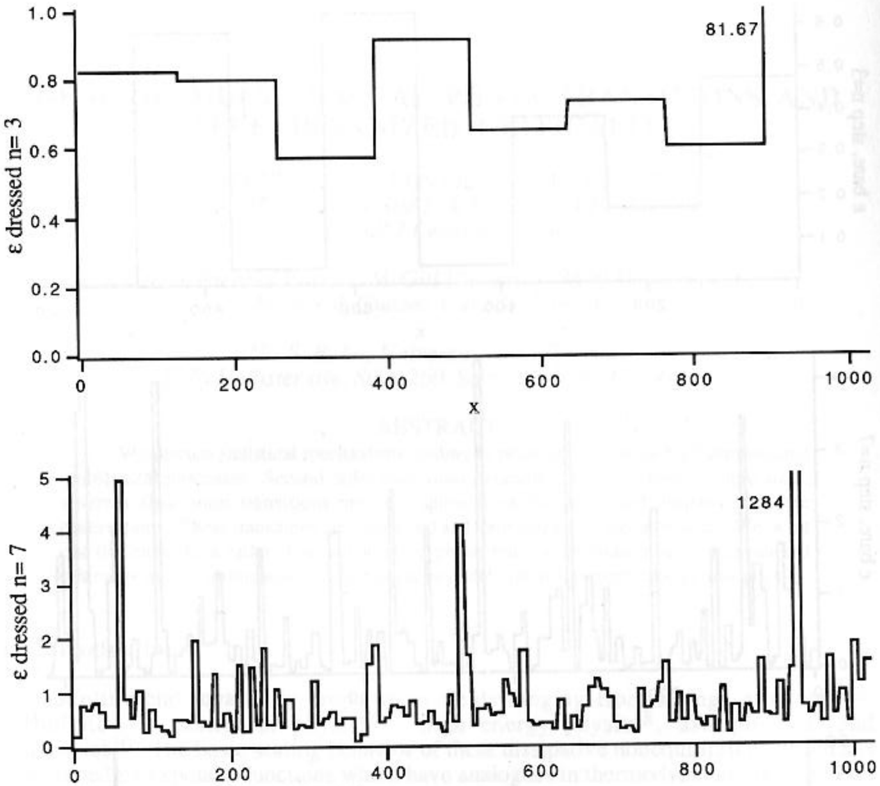
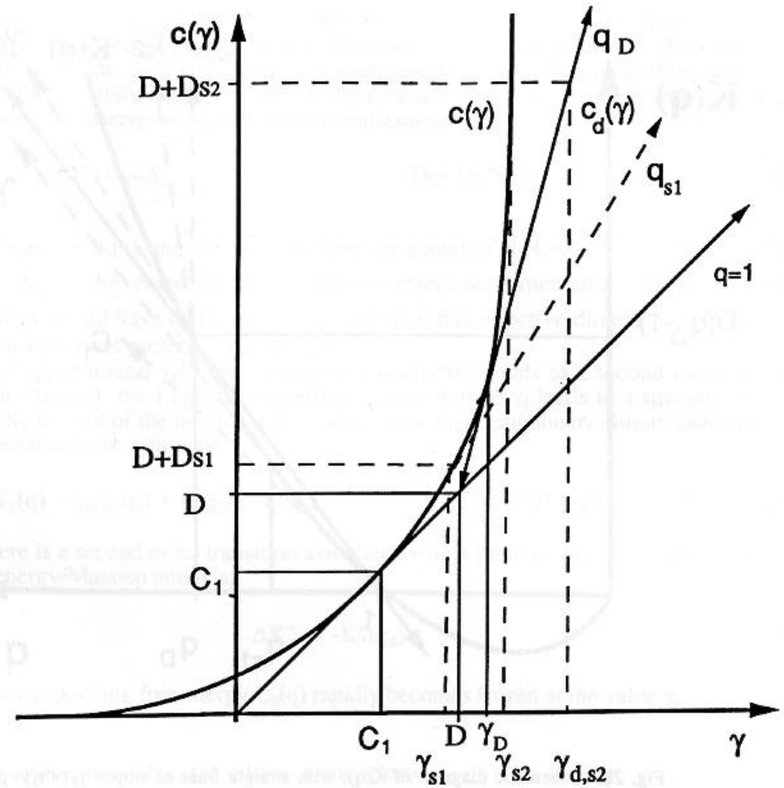


Fig. 1b The "dressed" density, obtained by averaging the "bare" density (Fig. 1a) over the finest resolution scale ($l = 2^{-10}$) up to the scales $l = 2^{-3}\ 2^{-7}$. The sharp contrast between the 1a and 1b arises from smaller scale singularities. As discussed below, among 2^9 realizations, we chose the one displaying the "hardest" singularity.

The multiple scaling behavior of this field ε_λ at scale ratio λ (=L/l the ratio the largest scale L to the scale l), can be either characterized by its probability distribution or by the statistical moments which are obtained via a Laplace transform (here and below the sign $\approx$ means equality within slowly varying or constant factors):

$$\Pr(\varepsilon_\lambda \geq \lambda^\gamma) \approx \lambda^{-c(\gamma)}; \qquad \langle \varepsilon_\lambda{}^q \rangle \approx \lambda^{K(q)} \approx \int \lambda^{q\gamma} \lambda^{-c(\gamma)} dc(\gamma) \qquad (1)$$



Figs 2a: Schematic diagram of $c(\gamma)$, $c_d(\gamma)$ indicating two sampling dimensions D_{s1}, D_{s2} and their corresponding $\gamma_{s1}<\gamma_D<\gamma_{s2}<\gamma_{d,s2}$; the critical tangent (slope q_D) contains the point (D,D).

the exponent $c(\gamma)$ is[21,22] a statistical codimension since -as discussed below- the probability measures the fraction of the probability space occupied by the singularities exceeding the order γ (Fig.2a). The probability description (γ, $c(\gamma)$) is the multifractal analogue of the (energy, entropy) description of standard thermodynamics, whereas the moment description (q, $K(q)$) is the analogue of the (inverse temperature, Massieu potential) description. Since a free energy analogue is $K(q)/(q-1)$, discontinuities (phase transition analogues) will be apparent in either the free energy or Massieu potential description. (Entropy, Massieu potential) and (c ,K) are Legendre transform[4] pairs:

$$K(q) = \max_\gamma (q\gamma - c(\gamma)) \qquad\qquad c(\gamma) = \max_q (q\gamma - K(q)) \qquad (2)$$

these relations establish the one to one correspondence $q = c'(\gamma)$, $\gamma = K'(q)$ (Figs.2a,b).

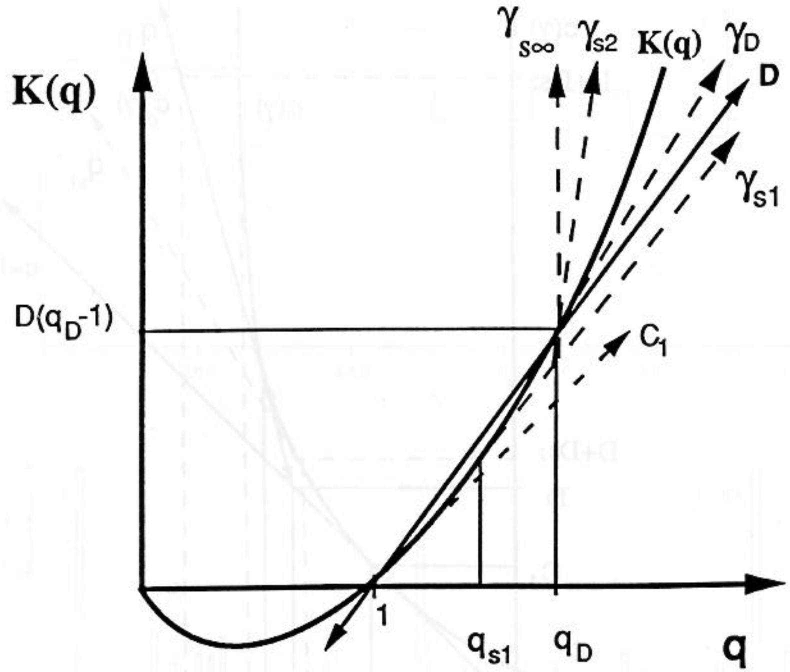


Fig. 2b: Schematic diagram of K(q), with straight lines of slopes $\gamma_{s1}<\gamma_D<\gamma_{d,s2}<\gamma_{d,s\infty}$ ($=\infty$) indicating the behavior for increasing sample size N_s ($N_{s\infty}=\infty$). The line of slope D defining q_D is also shown.

At scale ratio λ, the probability can be estimated as the ratio of the number $(N_\lambda(\gamma))$ of structures with singularities $\geq\gamma$ to the total number of structures (N_λ): $\Pr(\varepsilon_\lambda \geq \lambda^\gamma) \approx N_\lambda(\gamma)/N_\lambda$. Whenever $D\geq c(\gamma)$, $c(\gamma)$ also has a geometrical interpretation over a D-dimensional observing set A. In this case, not only $N_\lambda \approx \lambda^D$, but also, on almost any single realization, $N_\lambda(\gamma) \approx \lambda^{D(\gamma)}$, with a positive $D(\gamma)$ which is then a geometric fractal dimension. This restrictive geometric interpretation corresponds to the starting point of ref.[4]. However, the "hard" singularities (see below) which are the most interesting have $c(\gamma)>D$ and are only present in "canonical" multifractals[15,21], which invariants (e.g. turbulent energy flux) are conserved in the "canonical" sense, i.e. on the ensemble average. One may note that recently the need of a codimension formalism has been implicitly acknowledged by Mandelbrot[23].

3. Second order multifractal phase transitions

Consider a sample consisting of N_s independent realizations, each of dimension D,

each covering a range of scales λ. Increasing N_s we gradually explore the entire probability space; encountering extreme but rare events that would be almost surely missed on any finite sample (Fig 2a). The sampling dimension[24] D_s quantifies the extent to which the probability space is explored by estimating the highest order singularity (γ_s) we are likely to observe on N_s independent realizations:

$$c(\gamma_s)=D+D_s=\Delta_s\ ; \qquad D_s \approx \log N_s/\log\lambda \tag{3}$$

follows from the Eq. 1 and the fact that there are a total of $N\cdot N_s=\lambda^{D+D_s}$ structures in the sample. Δ_s is the corresponding (overall) effective dimension. More extreme singularities would have codimensions greater than this effective dimension $(c\geq\Delta_s)$ and are almost surely not present in our sample.

The upper bound γ_s[24] for observable singularities leads to a second order phase transition. Indeed, the Legendre transform of $c(\gamma)$ with $\gamma\leq\gamma_s$ leads to a spurious linear estimate K_s instead of the nonlinear K for $q>q_s$; $q_s=c'(\gamma_s)$ being the maximum moment that can accurately be estimated :

$$K_s(q) = \gamma_s(q-q_s) + K(q_s)\ , \quad q\geq q_s; \qquad K_s(q) = K(q) \quad q\leq q_s \tag{4}$$

hence there is a second order transition associated with a jump in the second derivative of the free energy/Massieu potential:

$$\Delta K''_s = -K''(q_s) \tag{5}$$

and the corresponding free energy $C_s(q)$ rapidly becomes frozen at the value γ_s.

4. First order multifractal phase transitions

We now discuss the more violent first order transitions which may occur for a multifractal process typically observed by spatial and/or temporal averaging on scales $l>>\eta$ (the inner size of the process) , i.e. with corresponding ratios $\lambda=L/l$, $\Lambda=L/\eta$ with $\Lambda>>\lambda$. The variability at the observation scale ratio λ may be wilder (Fig. 1b) than the corresponding field obtained by stopping the cascade process at the same scale ratio. Borrowing some renormalization jargon, we may say that the observation is "dressed"[22] by the small scale activity, whereas the process stripped of its small scale activity is called "bare". All the interactions and the resulting fluctuations of the field on scale ratios between λ and Λ are "hidden" from direct observation; nonetheless they are entirely responsible for the much more violent dressed variability.

We have thus to consider the statistical behavior of integrals of multifractals, in particular, the D-dimensional integral over A:

$$\Pi_\lambda(A) = \int_A \varepsilon_\lambda d^D\underline{x}, \tag{6}$$

corresponding to the energy flux (turbulence) or the multifractal probability (strange attractors). Π_λ has multiple scaling behavior corresponding in the usual strange attractor notation to the multifractal measure exponents α, f, τ, but due to the D dimensional

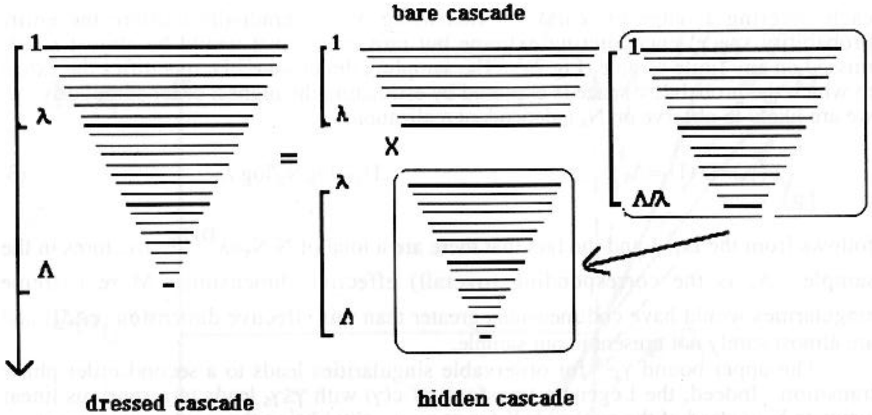


Fig.3: A schematic diagram showing a cascade constructed down to scale ratio Λ, dressed (averaged) up to ratio λ. This is equivalent to a bare cascade constructed over ratio λ, multiplied by a hidden factor obtained by reducing by factor λ a cascade constructed from 1 to Λ/λ.

integration there are extrinsic D dependences[15] (rendered explicit by the subscript "D"):

$$\Pr(\Pi_\lambda(B_\lambda)\geq \lambda^{-\alpha_D}) \approx \lambda^{f_D(\alpha_D)}/\lambda^D;\ \ \alpha_D=D\text{-}\gamma\ ;\ f_D(\alpha_D)=D\text{-}c(\gamma) \tag{7}$$

on a ball B_λ of size L/λ. The "trace moments"[22] generalize the partition function used in strange attractors[2] by combining ensemble with spatial averaging ("superaveraging"):

$$Tr_{A_\lambda}(\varepsilon_\lambda{}^q) \approx <\sum_i \Pi_\lambda(B_{\lambda,i})^q> \approx \lambda^{-\tau_D(q)};\ \ \tau_D(q)=(q\text{-}1)D\ \text{-}K(q) \tag{8}$$

summing over A at resolution λ, i.e. on a covering of $N_\lambda(A)\approx\lambda^D$ disjoint balls $B_{\lambda,i}$ $(\Pi_\lambda(B_{\lambda,i})\approx\varepsilon_\lambda\lambda^{-D})$. While the intrinsic quantities γ, c(γ), K(q) are independent of the dimension D of the observing space A, unfortunately α_D, $f_D(\alpha_D)$, $\tau_D(q)$ diverge as we increasingly explore the infinite dimensional probability space (D→∞; contrary to the finite phase space dimension D for strange attractors).

Because of its multiplicative construction, the dressed field (ε_d) factors (see Fig. 3) into a bare ε (large scale) and a hidden ε_h (small scale) component ($\varepsilon_d=\varepsilon\varepsilon_h$). The hidden small scale component may contribute as a highly variable prefactor having occasional avalanche-like effects on the large scale as soon as a singularity of order greater than D occurs (see Fig. 1a-b for an example). This is because the D-dimensional integration cannot smooth it down to the scale of observation; the observation scale (l) is no longer effective and the scale of homogeneity η prevails). Since Λ/λ>>1, the dressed singularity (γ_d) computed using the observation scale will be much larger. These events will remain statistically negligible until a critical singularity order γ_D, for $\gamma<\gamma_D$ the dressed codimension (c_d) coincides with the bare codimension (c), but for $\gamma>\gamma_D$ c_d will be determined by maximizing the probability, i.e. minimizing c, *with the only constraint*

being the convexity, c_d thus follows the tangent (Fig. 2a):

$$c_d(\gamma_d)=q_D(\gamma_d\text{-}\gamma_D)+c(\gamma_D)\quad \gamma_d\geq\gamma_D;\quad c_d(\gamma_d)=c(\gamma)\quad \gamma_d\leq\gamma_D \tag{9}$$

$q_D=c'(\gamma_D)$ is the slope of the algebraic fall-off of the dressed probability distribution, and is the critical order of divergence of statistical moments ($q\geq q_D$, $<\varepsilon_h{}^q>=\infty \Rightarrow <\varepsilon_d{}^q>=\infty$) since a Legendre transform of a linear function diverges ($K_d(q)=\infty$, $q\geq q_D$). It was shown[3,22] that q_D is the solution of:

$$D=q_DD\text{-}K(q_D)=q_D(D\text{-}\gamma_D)+c(\gamma_D) \tag{10}$$

i.e. the critical tangent (Eq. 9) contains the point (D,D) (Fig. 2a). The latter property can be directly derived by showing that the (statistical) singularity of the trace moment[3,22] density is q(γ-D) -c(γ) (the opposite of the tangency), and it should reach the critical value D for $\gamma=\gamma_D$, $q=q_D$, hence a simple and direct determination of $c_d(\gamma_d)$:

$$c_d(\gamma_d)=q_D(\gamma_d\text{-}D)+D\quad \gamma_d\geq\gamma_D \tag{11}$$

Note that these singular statistics of the hard behavior[15] of the dressed field have been taken as a basic feature of self-organized criticality[19] and are effective as soon as the sample size is large enough ($\Delta_s=D+D_s\geq c(\gamma_D)$). Following the argument for Eqs. 3-4 (see Fig. 2a), the maximum observable dressed singularity ($\gamma_{d,s}$) is given by the solution of $c_d(\gamma_{d,s})=\Delta_s$ and by taking the Legendre transform of c_d with the restriction $\gamma_d<\gamma_{d,s}$ we obtain the finite sample dressed $K_{d,s}(q)$:

$$K_{d,s}(q)=\gamma_{d,s}(q\text{-}q_D)+K(q_D)\qquad q>q_D;\ K_{d,s}(q)=K(q)\quad q<q_D \tag{12}$$

In the limit $N_s\to\infty$, $\gamma_{d,s}\to\infty$, and for $q>q_D$, $K_{d,s}(q)\to K_d(q)=\infty$ as expected. For N_s large but finite, there will be a high q (low temperature) first order phase transition, whereas the scale breaking mechanism proposed for phase transitions in strange attractors[16,17] is fundamentally limited to high and negative temperatures (small or negative q . This transition corresponds to a jump in the first derivative of the K(q):

$$\Delta K'(q_D)\equiv K'_{d,s}(q_D)\text{-}K'(q_D)=\gamma_{d,s}\text{-}\gamma_D=\frac{\Delta_s\text{-}c(\gamma_D)}{q_D} \tag{13}$$

On small samples ($\Delta_s\approx c(\gamma_D)$), this transition will be missed, the free energy simply becoming frozen and we obtain: $K_{d,s}(q)\approx(q\text{-}1)D$, which was already discussed with help of some turbulence experiments[3,14].

5. Numerical simulations

The numerical study of the first order transition is particularly demanding since we must not only dress the bare cascade over a large enough range of scales to obtain convergence to the hard behavior (Λ>>1), but we also require a large enough number of realizations ($\gamma_{d,s}>>\gamma_D$, $D_s>>D$). With these constraints in mind, we chose the "log-normal" multifractal which is the extreme of the family of universal multifractals[15,22,25,26] whose degree of multifractality (0≤α≤2) is a maximum (α=2).

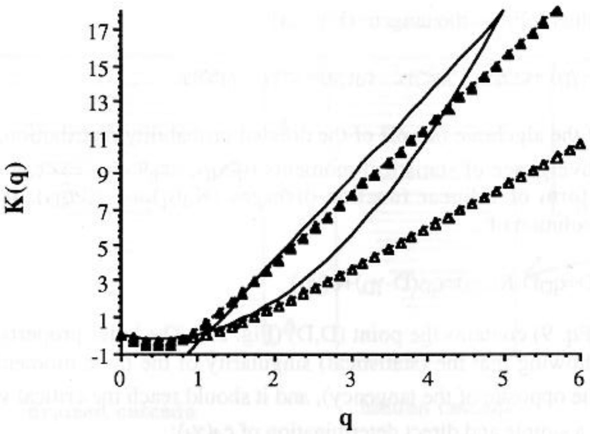


Fig. 4: A numerical simulation of phase transitions: K(q) (theoretical, parabola), $K_s(q)$ (observed second order transition, bare, open triangles), $K_{d,s}(q)$ (observed first order transition, partially dressed[14], closed triangles), $K_{d,s}(q)$ (theoretical first order transition, fully dressed, straight line).

The bare K(q) function is:

$$K(q) = \frac{C_1}{\alpha-1}(q^{\alpha}-q) \tag{14}$$

where $0 \le C_1 \le D$ for nondegenerate[22] multifractals. When $\alpha<2$, Eq. 14 is only valid for $q \ge 0$, when $q<0$, both the bare and dressed moments will diverge leading to other (negative temperature) phase transitions to be discussed elsewhere. For simplicity, the cascades were discrete; the unit interval was iteratively divided into lengths 1/2, 1/4, 1/8...1/1024, each interval, and every step being multiplied by appropriately normalized independent lognormal random variables (see Fig. 1a). The individual random variables had unit mean and were statistically independent. The absence of restrictions on the sum of their values allows[15,24] far much larger fluctuations than possible in the usual multifractals and is essential for the first order phase transition.

With $\alpha=2$, $q_D = D/C_1$ and low q_D is obtained with either C_1 large or D small (see bellow). For simplicity, we took D=1 and C_1 large ($=9/10 \Rightarrow q_D=10/9$). Fig. 4 shows the result when $N_s=2^9$. The cascades were constructed over a ratio $\Lambda=2^{10}$, dressed over a ratio 2^7 (Fig. 1 shows the realization giving the highest dressed singularity at this scale), so that the remaining bare range (λ) was only $2^3 \Rightarrow D_s=9/3=3$, $\Delta_s=4$. Finite size effects ($\Lambda<\infty$) must be carefully considered in the statistical estimates which must be performed scale by scale (Fig. 4).

6 Empirical evidence

The existence of multifractal phase transitions may explain long-standing

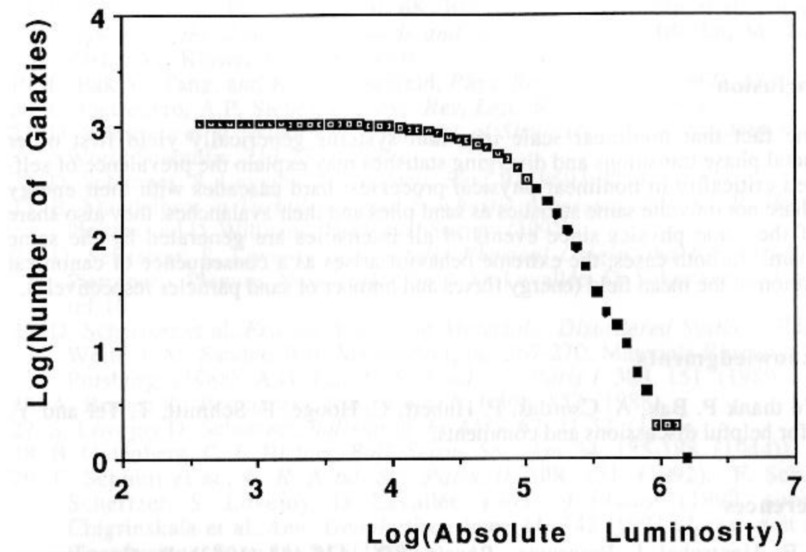


Fig. 5: The data used are from the Center for Astrophysics redshift Catalog[31], apparent luminosity magnitude < 14.5 and velocity<4000km/s. The sample contains 1682 galaxies, of which 436 responsible for the observed q_D. The cutoff at Log(lum)= 4.8 can be physically understood from the missing galaxies due to the r^2 dependence of optical detection of galaxies[30].

geophysical evidence[27] for hard behavior, e.g. the well-known Gutenberg-Richter law in earthquakes [29] as well as more recent evidence in turbulence[18]. Indeed, an alternative method of obtaining a small q_D is having a small D. D<1 may in fact be relevant in turbulence where evidence of hard multifractals has existed for some time. Recall the Kolmogorov relation between the observed velocity fluctuations (Δv_λ), and the energy flux (ε): $\Delta v_\lambda \approx \varepsilon^{1/3}\lambda^{-1/3}$. A simple interpretation of the linear scaling exponent -1/3 is that it represents a fractional integration of order D=1/3 of $\varepsilon^{1/3}$ and as elaborated elsewhere the estimates[16,29] of universal multifractal exponents support a low $q_{\varepsilon D} \approx 2$, consistent with empirical estimates going back ten years[3].

A first order multifractal phase transition mechanism may also explain rather straightforwardly the observed luminosity distribution of galaxies, i.e. the algebraic fall-off of the integrated Luminosity Function. Indeed, although various authors[29] proposed fits to this algebraic fall-off regime of this integrated histogram of observable galaxies exceeding a given level of luminosity, until now no physical interpretation was made. On the contrary, a first order phase multifractal transition with a critical moment order[30] $q_D \approx 2.2$ may explain this regime as shown in Fig. 5. It is likely the straight line part of the graph would extend to much lower luminosities if we had access to all the non-visible galaxies.

7. Conclusion

The fact that nonlinear scale invariant systems generically yield first order multifractal phase transitions and diverging statistics may explain the prevalence of self-organized criticality in nonlinear physical processes: hard cascades with their energy fluxes share not only the same statistics as sand piles and their avalanches, they also share some of the same physics since events of all intensities are generated by the same mechanism. In both cases, the extreme behavior arises as a consequence of canonical conservation of the mean field (energy fluxes and number of sand particles respectively).

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