INTERNATIONAL SYMPOSIUM ON GENERATION OF LARGE-SCALE STRUCTURES IN CONTINUOUS MEDIA

# NONLINEAR DYNAMICS OF STRUCTURES

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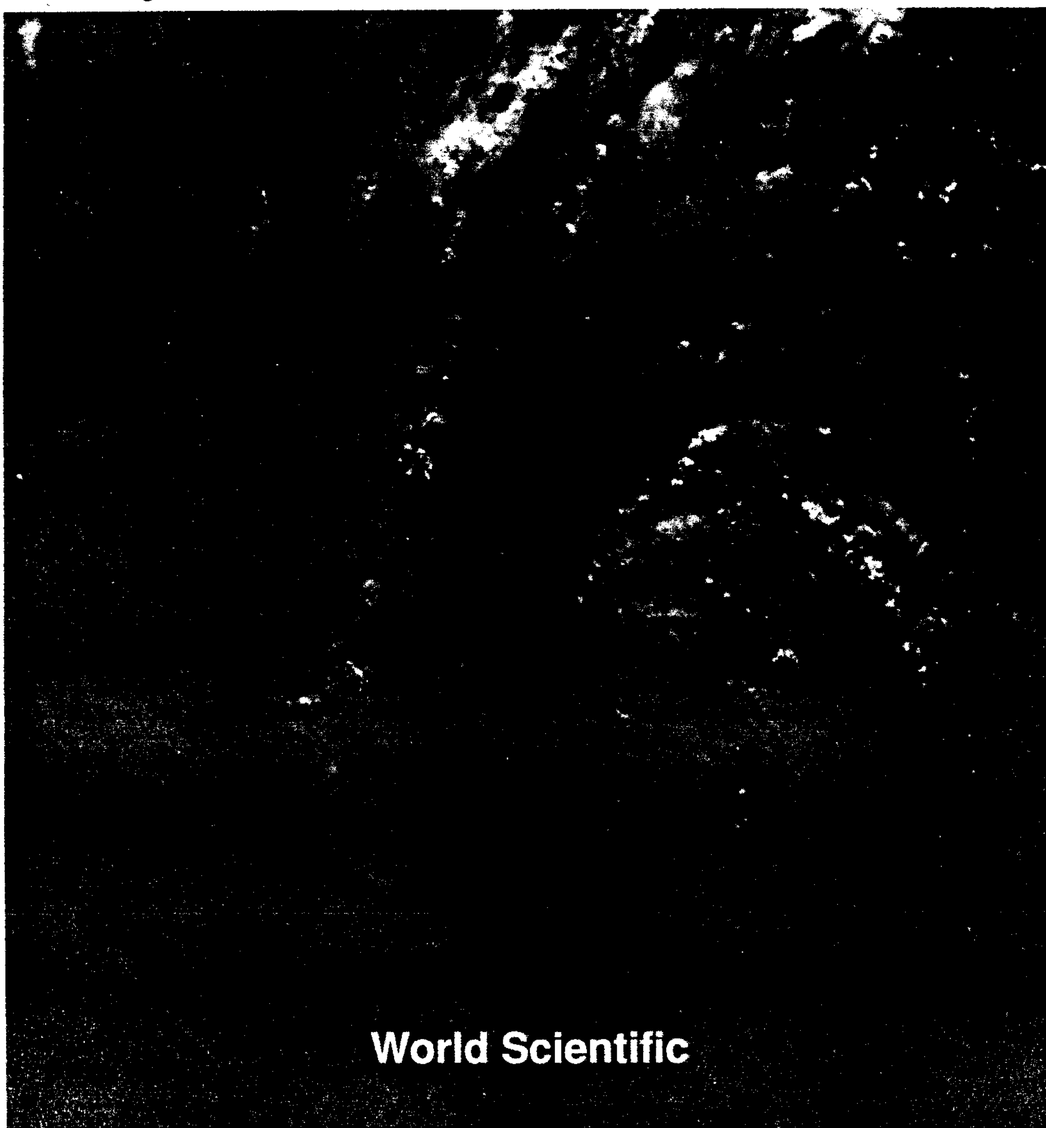

# UNIVERSAL HARD MULTIFRACTAL TURBULENCE: THEORY AND OBSERVATIONS

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ABSTRACT

Using a turbulent multifractal formalism, we review various results on multifractals, and develop these in several directions. We show how multiplicative cascades lead to highly intermittent multifractal measures with singularities distributed over fractals with various codimensions. We show how to classify different types of multifractals according to the highest order singularities they possess. The calmest multifractals we consider are the geometric (Parisi Frisch) multifractals which have by definition only completely localized and calm singularities. A less calm process is the (already delocalized) microcanonical cascade which has some events sufficiently rare that they are missed by the geometrical description. The most general case corresponding to canonical conservation of flux involves "wild" singularities that are sufficiently violent that they necessarily violate conservation of flux on individual realizations; these are of course out of reach of microcanonical cascades. In general, they even involve "hard" singularities which although very rare are sufficiently violent to cause the high order statistical moments to diverge.

We next consider the turbulent mixing of cascades and show how this leads to universal multifractals which are characterized by three fundamental exponents H, $C_1$, $\alpha$. The latter determine the degree of nonconservation of the mean, the mean inhomogeneity, and the degree of multifractality respectively; it is simultaneously the fundamental characterization of the stochastic generator. The fundamental parameter is $\alpha$; when it is zero (its minimum), the process is monofractal (the so-called $\beta$ model), as $\alpha$ increases to its maximum value 2, the hierarchy of singularities develops. For $\alpha<1$, the process is only conditionally hard; it might even be not only soft, but calm, whereas for $\alpha\geq 1$, it is unconditionally hard.

The existence of universal multifractals greatly simplifies the analysis and simulation of multifractals. For example, it enables us to obtain a robust method ("double trace moments") of determining the hierarchy of singularities by estimating the universal exponents directly. We then apply this technique to turbulent velocity and temperature data obtaining the first empirical estimates of their universality parameters finding $\alpha_T\approx 1.2\pm 0.1$, $\alpha_v\approx 1.3\pm 0.1$. In both cases, since $\alpha>1$, the turbulence is found to be unconditionally hard. This evidently is a fundamental characterization of the solutions of the Navier-Stokes equations.

## 1. Intermittency and Scale Invariance

### 1.1 Fractals and Multifractals:

It has long been realized that turbulent velocity and temperature fields are intermittent in the sense that active regions occupy tiny fractions of the space available. Early attempts used scaling (hence fractal) notions to characterize the intermittency[1,2], particularly the $\beta$-model[3]. The intermittency was thus characterized by a unique fractal dimension, $D_s$, the support of the turbulence.

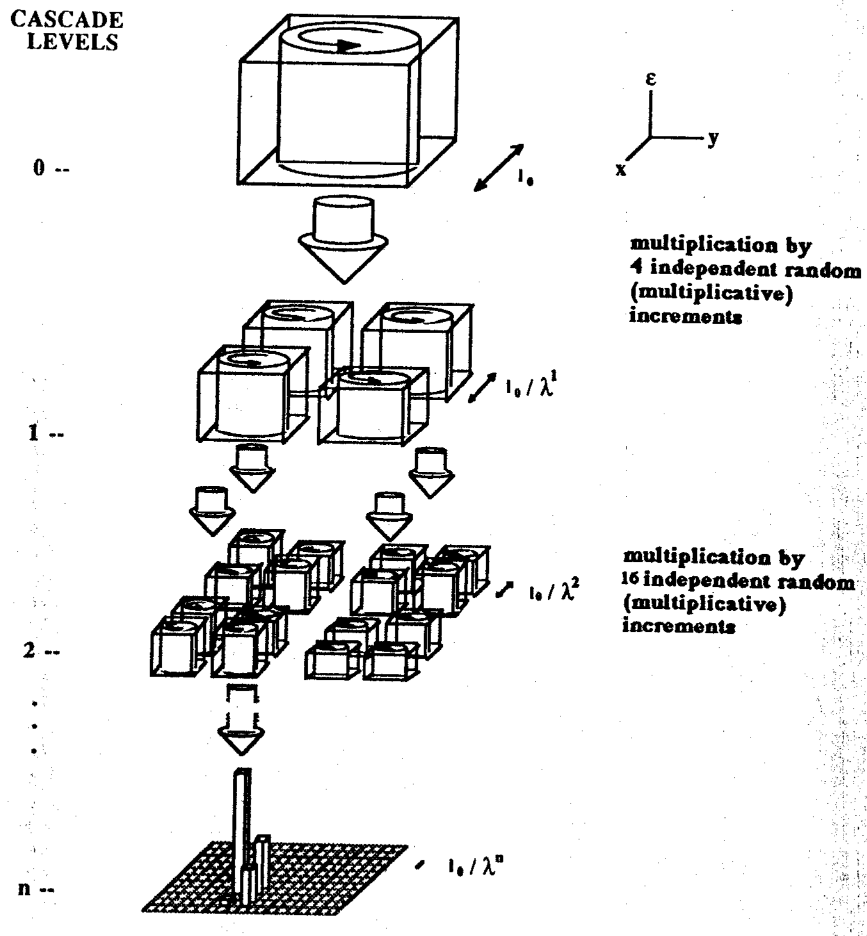



Fig. 1 A schematic diagram showing the first steps of a cascade process.

Unfortunately, this suggestive geometric description was found to be an unstable approximation when the caricatural dead/alive sub-eddy in a $\beta$ model cascade is replaced by the more realistic weak/strong sub-eddies in "$\alpha$-model" cascades[4,5]. In the $\alpha$ model, we are no longer dealing with the geometry of sets, but rather with a scale invariant field (measure) involving an entire hierarchy of fractal dimensions. The strength of an $n^{th}$ generation eddy will now depend on all of its ancestors and is no longer absolute. We must fix a "survival" threshold, and the dimension of the support associated with the threshold depends on the threshold... The result is a hierarchy of singularities of the



density of the flux, since (by construction) the latter is conserved even though being concentrated on a support with zero volume!

1.2 An explicit multifractal process, the $\alpha$ model:

As an example, consider a cascade produced by dividing a unit cube (the initial eddy with energy flux $\varepsilon=1$ everywhere), into sub-cubes (subeddies) each of scale $\lambda^{-1}$ where $\lambda$ (=2 here) is the scale ratio (see the schematic diagram Fig. 1). The fraction of the energy flux transfered from a large eddy to one of its sub-eddies is given by independent random factors ($\mu\varepsilon$) given by the Bernoulli law indicated in Eq. 1.

$$\Pr(\mu\varepsilon=\lambda^{\gamma^+})=\lambda^{-c}, \qquad \Pr(\mu\varepsilon=\lambda^{\gamma^-})=1-\lambda^{-c} \tag{1}$$

The parameters $\gamma^+, \gamma^-$, c are usually constrained so that the ensemble average $<\mu\varepsilon>=1$, $\lambda^{\gamma^+}>1$ ($\gamma^+>0$) corresponds to strong subeddies, $\lambda^{\gamma^-}<1$ ($\gamma^-<0$) to weak subeddies. This model was intoduced[4] and called the $\alpha$ model because of the divergence of moment exponent $\alpha$ it introduced (in the notation used below, the corresponding divergence parameter is $h_D$). The dead/alive $\beta$ model is recovered with $\gamma^-=-\infty$, $\gamma^+=c$, c being the codimension of the support (=$D-D_s$, D is the dimension of space in which the cascade occurs). As the cascade proceeds, the pure orders of singularities give rise to an infinite hierarchy of mixed orders of singularities ($\gamma^-<\gamma<\gamma^+$), after n steps these singularities are given by a binomial law:

$$\gamma=(n^+\gamma^+ + n^-\gamma^-)/n; \quad n^+ + n^- = n; \qquad \Pr(n^+=k)=\binom{k}{n}\lambda^{-ck}(1-\lambda^{-c})^{n-k}$$

$$\Pr(\varepsilon_{\lambda^n}\geq(\lambda^n)^{\gamma})\approx N_n(\gamma)/N_n\approx(\lambda^n)^{-c_n(\gamma)} \tag{2}$$

$N_n\approx(\lambda^n)^{-D}$ is the total number of eddies at scale $\lambda^{-n}$, D the dimension of space and and $\binom{k}{n}$ indicates the number of combinations of n objects taken k at a time. In the large n limit, $c_n(\gamma)\approx c(\gamma)$ and we are lead[6] to the multiple scaling probability distribution law:

$$\Pr(\varepsilon_{\lambda^n}\geq(\lambda^n)^{\gamma})\approx(\lambda^n)^{-c(\gamma)} \tag{3}$$

The corresponding law for the statistical moments is obtained via Laplace transforms of the probability distribution:

$$<\varepsilon_{\lambda^n}{}^h>\approx(\lambda^n)^{K(h)}=\int(\lambda^n)^{h\gamma}(\lambda^n)^{-c(\gamma)}d\gamma\approx(\lambda^n)^{\max_\gamma(h\gamma-c(\gamma))} \tag{4}$$

where the right hand side is obtained by the saddle point approximation, generally valid since we are interested in the large n limit[1]. The Laplace transformation relation between probabilities and moments, thus lead to the following Legendre transforms[7] of c(γ) between c(γ), K(h):

[1]For universal multifractals with parameter $\alpha<2$ the Legendre transformation only yields the exponential part of the probability distribution.

$$K(h) = \max_{\gamma} (h\gamma - c(\gamma))$$
$$c(\gamma) = \max_{h} (h\gamma - K(h)) \quad (5)$$

Finally, we can now introduce the dual codimension function C(h) which characterizes the $h^{th}$ order moment (in the next subsection we see its significance):

$$C(h) = \frac{K(h)-hK(1)}{(h-1)} \quad (6)$$

1.3 General properties of $c(\gamma)$, $K(h)$:

We now establish several general properties $c(\gamma)$, $K(h)$. We start with the Legendre relations Eq. 5. These relations establish a one to one correspondence between orders of singularities and moments:

$$h = c'(\gamma) \quad (7a)$$
$$\gamma = K'(h) \quad (7b)$$

A basic property of K(h) is that it is convex ($K''(h)>0$ for all h); this follows from the fact that it is defined as the (base $\lambda$, Laplace) second characteristic function of log $\varepsilon_\lambda$ (Eq. 4); relations 7a,b show that $c(\gamma)$ is also convex. We now consider the mean of the process:

$$\langle\varepsilon_\lambda\rangle = \lambda^{-H}; \qquad K(1) = -H \quad (8)$$

H will therefore express the degree of nonconservation (nonstationarity) of the mean; for example, H=0 for the energy flux (it is conserved by the nonlinear terms of the Navier Stokes equations), whereas, it is of the order of 1/3 for the wind and temperature fields[8,9,10]. In the following, it will be convenient to consider an associated, normalized multifractal with H=0 obtained e.g. by multiplying the unnormalized multifractal by $\lambda^H$. Denoting normalized quantities by the subscript "N", using Eqs. 3-6 we obtain the following relations:

$$\varepsilon_{N\lambda} = \lambda^H \varepsilon_\lambda$$
$$K_N(h) = K(h) + hH$$
$$\gamma_N = \gamma + H$$
$$c_N(\gamma_N) = c(\gamma)$$
$$C_N(h) = C(h) \quad (9)$$

Introducing the normalized quantities corresponds to removing the mean scaling (H), and using the above relations, we can then easily infer the corresponding properties of the unnormalized processes. We can also note that most of the properties are easier to derive on normalized quantities. A simple example is the demonstration that C(h) ($\equiv C_N(h)$) is always increasing; recalling that $K_N(1)=0$, $C_N(h)$ is the slope of the chord connecting the points $(1,K_N(1))$ and $(h,K_N(h))$ since $K_N(h)$ is convex, this is increasing, we therefore conclude that $C'(h)>0$ for all h.



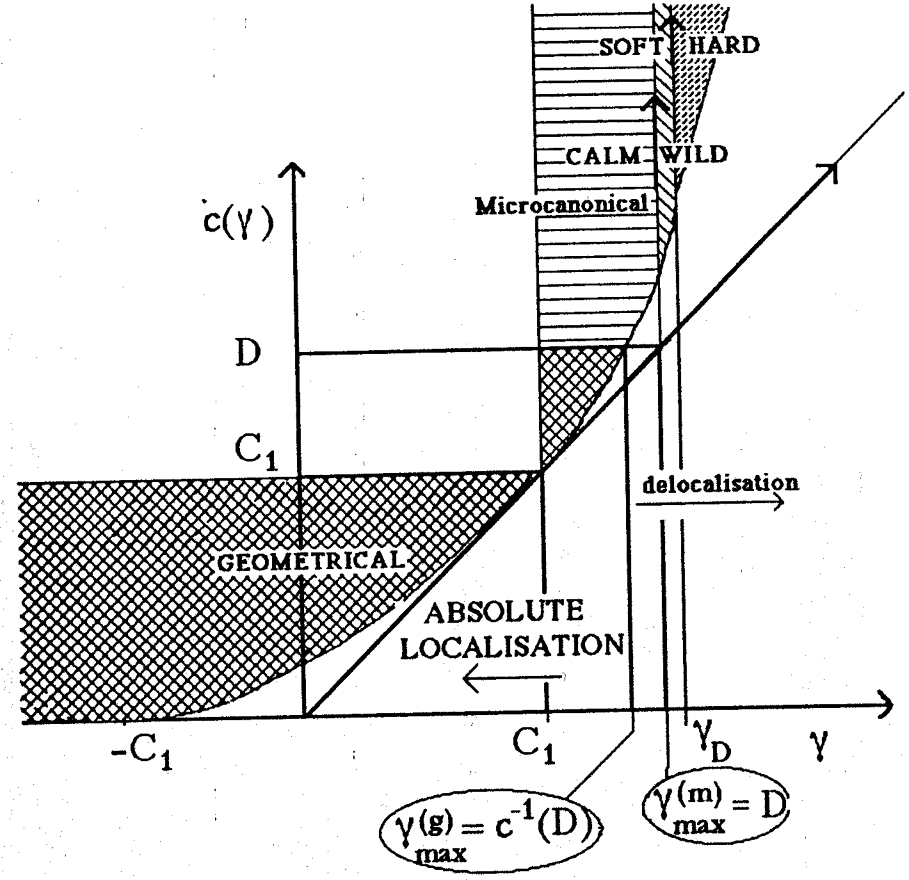



Fig. 2-a: Phase diagram showing the attainable singularities for a (normalized) multifractal, showing the fixed point $C_1=c(C_1)$, the maximum orders of singularities of geometrical (Parisi Frisch) multifractals $\gamma^{(g)}_{max}$ ($=c^{-1}(D)$), and microcanonical multifractals $\gamma^{(m)}_{max}$($=D$). Wild singularities are those with $\gamma \geq \gamma^{(m)}_{max}$ and necessarily involve realizations which do not respect flux conservation. Wild singularities with $\gamma > \gamma_D$ ($>\gamma^{(m)}_{max}$) lead to the appearance of hard multifractals with divergent high order statistical moments. Note that the dimension D refers to the dimension of space in which the various multifractals are defined and that the represented singularities are delimited by the two extreme universal cases (as discussed in section 2.4) the so called β-model and lognormal model.

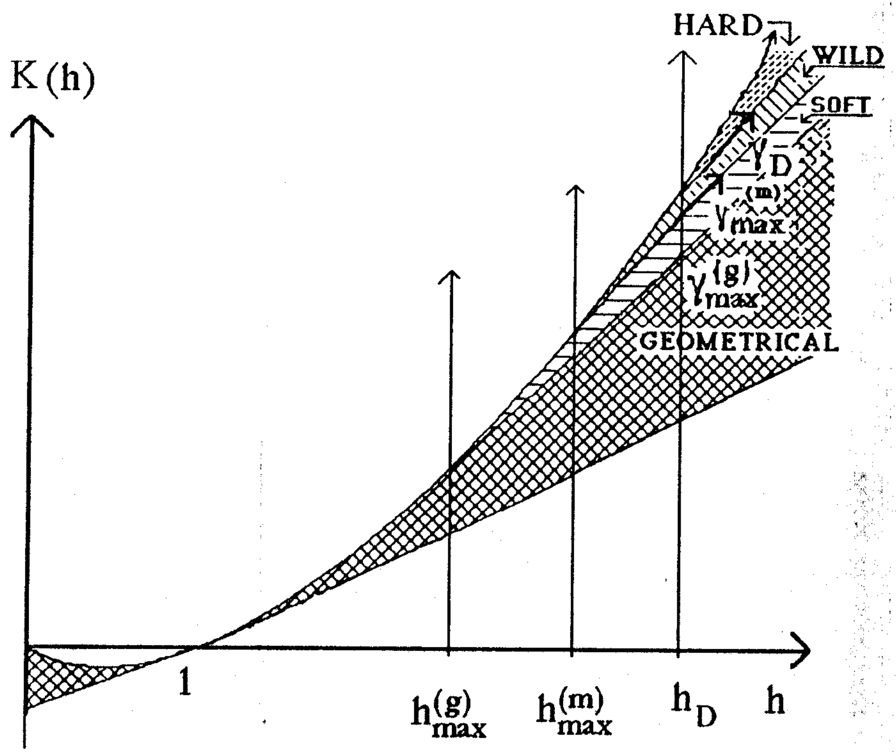



Fig. 2b: The dual phase diagram in moments representation instead of singularities. The (normalized) moment scaling function K(h) corresponds to the codimension function $c(\gamma)$ in fig. 2a. Recall that $h_{max}=c'(\gamma_{max})$.

Some elementary properties of $c(\gamma)$ can be obtained by applying relations 6, 7a to the moment h=1 and the functions $c_N(\gamma)$, $K_N(h)$. Equation 7a shows that the singularity corresponding to h=1 ($\gamma_1$) satisfies: $c'_N(\gamma_1)=1$, and Eq. 6 that $\gamma_1=c_N(\gamma_1)$ (recall $K_N(1)=0$) hence, $c_N(\gamma)$ is tangent to the line x=y at the point $\gamma_1$. Furthermore, using Eq. 7b, we find $\gamma_1=K_N'(1) = C_N(1) = C_1$; the codimension of the mean. If $C_1>D$, the mean is so sparse that on the observing space (dimension D), the process almost surely almost everywhere zero, it is "degenerate", hence we require $C_1 \le D$. We obtain the behaviour shown in fig. 2 a-b for the normalized quantities. Using Eq. 9, we can transpose them back to the unnormalized quantities. Henceforth, except when needed, we will drop the subscript N in order to simplify our notation.



1.4 Hard and Soft Multifractals; calm and wild singularities:

The dual codimension function C(h) is of particular significance in analyzing the "dressed" quantities that result when completed cascades (i.e. in the limit $\lambda^n \to \infty$) are integrated over sets of dimension D. In this case, the smoothing introduced by the integration is sufficient to allow some moments to converge. Specifically, the critical order of moment $h_D$ (>1) is given by $C(h_D)=D$; $h<h_D$ will have convergent dressed moments, for $h \ge h_D$ they will be divergent[6]. "Hard" multifractals involve some divergence, while in "soft" multifractals D is large enough for convergence to be obtained for all h.

To see this, introduce $\Pi_\lambda(A)$, the energy flux through the set A (dimension D) which is the integral of the density $\varepsilon_\lambda$:

$$\Pi_\lambda(A) = \int_A \varepsilon_\lambda d^D\underline{x} \qquad (10)$$

We are interested in various statistical moments of the flux $<\Pi_\lambda(A)^h>$. If h is a positive integer:

$$<\Pi_\lambda(A)^h> = <\int_A\int_A\ldots\int_A \varepsilon_\lambda(\underline{x}_1)\varepsilon_\lambda(\underline{x}_2)\ldots\varepsilon_\lambda(\underline{x}_h)d\underline{x}_1 d\underline{x}_2\ldots d\underline{x}_h> \qquad (11)$$

This expression is rather difficult to deal with. It can however be readily bounded by the trace moments[6] obtained by setting $\underline{x}_1=\underline{x}_2=\ldots\underline{x}_h$:

$$Tr_A\varepsilon_\lambda{}^h = <\int_A \varepsilon_\lambda{}^h d^{hD}\underline{x}> \le <\Pi_\lambda(A)^h>; \qquad h>1 \qquad (12)$$

Theoretically, the trace moments have the advantage of being Hausdorff measures, hence, in the limit $\lambda \to \infty$, they will be either zero or infinity (with at most one nonzero finite value); for h>1, their divergence will imply the divergence of $<\Pi_\lambda(A)^h>$ (for h<1, the above inequality is reversed, this is important in connection with the nondegeneracy of the process). Writing out the trace as a sum over boxes size $\lambda^{-1}$, volume $\lambda^{-hD}$:

$$Tr_A\varepsilon_\lambda{}^h = \sum_{A_\lambda} <\varepsilon_\lambda{}^h>\lambda^{-hD} \qquad (13)$$

where $A_\lambda$ is the set A at resolution $\lambda$; the sum is over the $N_\lambda(A) \approx \lambda^D$ covering boxes:

$$Tr_A\varepsilon_\lambda{}^h = \lambda^D \lambda^{K(h)} \lambda^{-hD} = \lambda^{(h-1)(C(h)-D)} \le <\Pi_\lambda(A)^h>; \quad h \ge 1 \qquad (14)$$

hence, for h>1, $<\Pi_\lambda(A)^h> \to \infty$ for all $h>h_D$ where $C(h_D)=D$.

It can be easily checked that the order of singularity $\gamma_D$ corresponding to $h_D$ via Eq. 7b is a "wild" singularity. By wild singularities we refer to those which break the microcanonical conservation of the flux ($\Pi$), i.e. its conservation on individual realizations. To see this, note that the convexity of K(h) implies that $\gamma_D$ (=$K'(h_D)$) is greater than $C(h_D)$ =D, since the slope of its tangent ($\gamma_D$) is greater than the slope (C(h)) of the chord connecting the points $(1,K_N(1))$ and $(h,K_N(h_D))$. Then use the fact that D is also the upper bound of the singularities respecting the microcanonical conservation that we denote by $\gamma^{(m)}_{max}$ (the superscript m corresponding to "microcanonical"), since this bound must satisfy:

$$\Pi_\lambda \geq \lambda^{-D} \lambda^{\gamma^{(m)}_{max}} \Pi_1 \qquad (15)$$

hence, this bound is reached only when for each step we concentrate all the density of the flux on a single subeddy (of volume $\lambda^{-D}$). As $\gamma_D$ is (strictly) larger than $\gamma^{(m)}_{max}$ not all wild singularities lead to hard multifractal, more precisely the width of the interval of soft but wild singularities is:

$$\gamma_D - \gamma^{(m)}_{max} = K'(h_D) - D = (h_D - 1)C'(h_D) \qquad (16)$$

this is always positive since $h_D > 1$, $C' > 0$.

Conversely, if for any reason, the orders of singularities are bounded above (by $\gamma_{max}$) with $\gamma_{max} < \gamma_D$, then the divergence will be suppressed, and the corresponding multifractals will be "soft". For $\gamma_{max} < \gamma^{(m)}_{max} = D$, they will soft and "calm". This is exactly the situation in the two most popular varieties of multifractals: the microcanonical multifractals just discussed, and the geometric multifractals[7]. In the geometric multifractals, there is no stochastic process, no probability space, the singularities are distributed over geometrical sets whose largest codimension is therefore equal to that of the embedding space (since geometric sets cannot have negative dimensions). We therefore obtain (see the graphical presentation in fig. 2a,b):

$$\gamma^{(g)}_{max} = c^{-1}(D) < \gamma^{(m)}_{max} \qquad (17)$$

hence the geometric multifractals are always softer than the microcanonical multifractals. Other cases involving bounded orders of singularities are α model multifractals and universal multifractals with $\alpha \leq 1$ (see below). Unlike the geometric and microcanonical multifractals, the latter are not generically soft; for any space D, their parameters can be varied to render them hard.

Finally, we may note that there is a moment $h_{max}$ corresponding to $\gamma_{max}$, given by $h_{max} = c'(\gamma_{max})$; when $h > h_{max}$, the Legendre transform indicates that K(h) becomes linear:

$$K(h) = h\gamma_{max} - c(\gamma_{max}); \qquad h > h_{max} \qquad (18)$$

this behaviour is shown in fig. 2b. The transition to straight line behaviour for $h > h_{max}$ could be considered a phase transition (see reference[11] for the "flux dynamics" formulation of turbulent cascades).

1.5 Bare and dressed properties; local/nonlocal multifractals:

We have shown that although the densities $\varepsilon_\lambda$ at scale λ have finite moments of all positive orders, that the limiting flux Π(A) will generally have diverging moments. If we consider Π(A)/Vol(A) with A = a box size $\lambda^{-1}$ as an estimate of $\varepsilon_\lambda$ over the box (see fig. 3); the two will therefore have totally different statistical properties. This surprising difference between the properties of the cascade constructed down to scale λ, and that of a completed cascade integrated (smoothed) over the same scale prompted Schertzer and Lovejoy[6,12] to denote the two quantities "bare" and "dressed" respectively. The difference arises because the limit is not defined in the sense of functions, but rather only as a weak limit of measures; denoting $\varepsilon = \lim_{\lambda\to\infty} \varepsilon_\lambda$, ε is a measure, whereas $\varepsilon_\lambda$ is a function. While the latter has a value at each point in space, the former is defined only on almost any neighborhood of almost any point.

Another subtlety inherent in random multifractal process is that contrary to



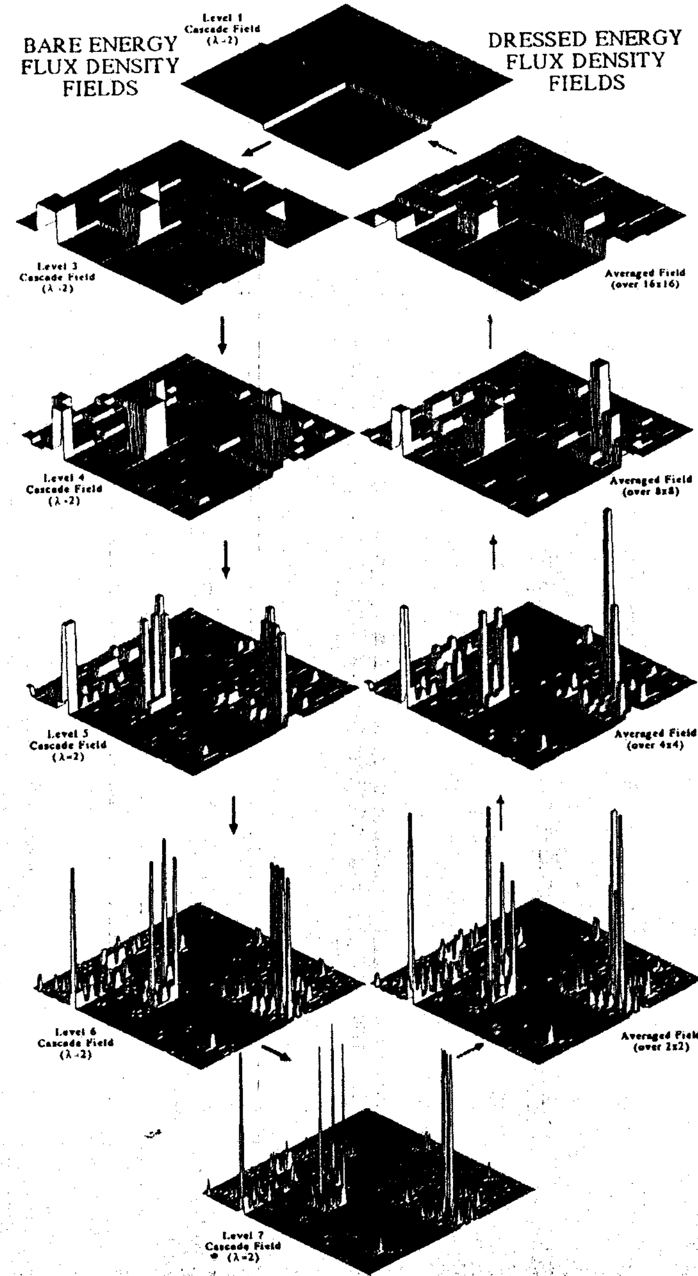



Fig. 3: An illustration of the first steps of an α model cascade showing the bare cascade process on the left, and the dressed process on the right.

geometric multifractal processes, their singularities are not localized. To understand this better, consider an "incipient" singularity about a point $\underline{x}$: $\gamma_\lambda(\underline{x})=\log\varepsilon_\lambda(\underline{x})/\log\lambda$, complete localization (such as that of geometric multifractals which are localized by definition), is obtained when $\gamma(\underline{x})=\lim_{\lambda\to\infty}\gamma_\lambda(\underline{x})$ is well defined[1]. On the contrary, in cascade processes. $\gamma_\lambda(\underline{x})$ will generally follow a random walk as $\lambda$ is increased not converging to any limit (nonetheless, at any scale $\lambda$ the (nonlocal) histogram of the incipient singularities (Eq. 3) still holds). In this sense, all cascade processes (except $\beta$ model process, see below) are non-local. However, there are various degrees of nonlocality. For example, in microcanonical cascades, the exact conservation of flux implies that a local fluctuation at scale $\lambda$ in density is necessarily localized to that scale regardless of the following cascade steps. This is a consequence of the combined scaling and microcanonical conservation of energy flux which prompted Schertzer and Lovejoy[13] to coin the expression "picocanonical" to stress the high degree of localization involved.

More generally, the degree of localization of a cascade singularity will depend on its order. For example, a large $\mu\varepsilon$ at some stage in a cascade (corresponding to a large fluctuation in energy flux over the corresponding scale, and a large boost in the local orders of singularities) will require many more cascade steps to "wash out" than a small fluctuation: the rapidity or likelihood of a large (negative) decrease in an incipient singularity being counteracted by succeeding increases will depend on the asymmetry of the probability distribution of $\mu\varepsilon$ (hence on the asymmetry of $c(\gamma)$ about its minimum). The extreme example, involving complete localization, is for $\beta$ model cascades, in which it suffices for a single decrease to completely wipe out any positive orders of incipient singularities. The result is that the active regions are distributed over a geometric (monofractal) set, hence the localization is total. In the next section, we shall see that there is natural parameter describing the degree of multifractality ($\alpha$), the $\beta$ model corresponds to the extreme asymmetry ($\alpha$=0, the minimum), complete localization, whereas $\alpha$=2 (the maximum) corresponds to complete symmetry, and the least localization, the localization decreases with $\alpha$.

1.6 On notation:

In parallel with the development of the turbulent multifractal formalism discussed here, another formalism has developed in order to deal with the multifractal probability measures associated with low dimensional chaotic attractors[14]. The fundamental quantity was considered to be the multifractal measure itself (p), rather than its density, its singularities being denoted $\alpha$=D-$\gamma$ where D is the dimension of the corresponding phase space[2]. The dimension corresponding to $\alpha$ is denoted $f(\alpha)$; $f(\alpha)=D-c(\gamma)$. Similarly, rather than studying the scaling of $h^{th}$ order moments of the density (K(h)), the scaling of the $q^{th}$ moments of the measure were used (denoted $\tau(q)$), h=q, $\tau(q)$ = (h-1)D-K(h)), similarly, the dual dimension and codimension functions are related by D(q)=D-C(h). See summary table 1 for the comparison.

[1]Localization is usually simply assumed without any justification; this is the source of many difficulties with existing multifractal analysis techniques: see Lavallée et al (15) for a discussion and critique.

[2]This is not to be confused with the $\alpha$ used below.



| Quantity | Strange attractor formalism | Turbulence formalism |
|---|---|---|
| Probability/ Energy flux | p | $\Pi(A)=\int_A \varepsilon d^D\underline{x}$ |
| Order of singularity | $\alpha$ | D-$\gamma$ |
| Dimension of singularities | $f(\alpha)$ | D-$c(\gamma)$ |
| Order of moments | q | h |
| Scaling of moments | $\tau(q)$ | (h-1)D-K(h) |
| Dimension of moments | D(q) | D-C(h) |
| Partition function/ Trace moments | $Z(p^q)$ | $Tr_A\, \varepsilon^h$ |

Table 1 comparing the strange attractor and turbulence formalism for multifractals.

The basic reason that we cannot use the $f(\alpha)$, $\tau(q)$ notation is that we are interested in stochastic cascade processes in which $D\to\infty$, hence $\alpha$, $f(\alpha)$, $\tau(q)$, $D(q)\to\infty$, while $\gamma$, $c(\gamma)$, K(h), C(h) all remain finite and still characterize the strength of the singularities, their sparseness the multiple scaling and the moment codimensions respectively. In passing, we may note that this turbulence notation avoids artificial problems of negative dimension ("latent dimensions" - Mandelbrot[16]), which occur when when multifractals are studied on low dimensional spaces e.g. with $D<\gamma_{max}$.

## 2. Universality

2.1 Can we describe the infinite hierarchy of fractals with only few parameters?

As only a convexity constraint intervenes on K(h) and $c(\gamma)$, the definition of the hierarchy requires a priori an infinite number of parameters... However, we may note that the following three parameters (H, $C_1$, $\alpha$) are of fundamental significance. We have already discussed H which characterizes the deviation from conservation, and $C_1$ which is the local trend of the normalized K(h) near the mean:

$$K(h) \approx C_1(h-1); \qquad h\approx 1 \tag{19}$$

H, $C_1$, thus define the best monofractal approximation to the mean of the process. Clearly, we can continue this local description and introduce the parameter $\alpha$ which characterizes the local radius of curvature $R_c$ of $c(\gamma)$, hence deviation from monofractality:

$$R_c(\gamma=C_1) = \frac{(1+c'(C_1))^{3/2}}{c''(C_1)} = 2^{3/2}\alpha C_1 \tag{20}$$

where the factor $2^{3/2}$ is introduced for convenience (as will be clear later). Using the fact that $c(C_1)=C_1$, $c'(C_1)=1$, we see that the above definition of $\alpha$ is equivalent to:

$$\frac{d^2c(C_1)}{d\gamma^2} = \frac{1}{\alpha C_1} \tag{21}$$

Using equations 7a,b in the neighborhood of the corresponding value h=1, we can establish the corresponding relations for the second derivative and local radius of curvature of K(h) near h=1. We obtain:

$$\frac{d^2K(1)}{dh^2} = C_1\alpha; \qquad R_K(1) = \frac{(1+C_1^2)^{3/2}}{C_1\alpha} \qquad (22)$$

The parameters H, $C_1$, $\alpha$ thus define the local multifractal hierarchy (around the average behaviour), but below we show that this characterization will become global for the universal multifractals (up to the order $\gamma_D$ where the divergence of moments intervenes).

2.2 Universality by mixing of processes:

It is rather easy to verify that the particularities of the discrete models remain as the cascade proceeds to its small scale limit. Although the limit is not universal it already poses very interesting and non trivial mathematical problems[17,18,19]. However, by keeping the total range of scale $\lambda$ fixed and finite, we may mix (by multiplying them) independent processes of the same type, preserving certain characteristics (e.g. variance of the resulting processes). When this is done, a totally different limiting problem is obtained. We will then seek the second limit $\lambda \to \infty$. We are indeed lead to extend results obtained[11] on the (already fairly general) densification of scales, i.e. introducing more and more intermediate scales in a given multiplicative process . Such a densification intervenes in building up a continuous process from a discrete process. The more general case corresponds to mixing of identical independent processes, eventually densifying the intermediate scales (an example, built up on mixing of $\alpha$-models, is given in fig. 4a-g).

To see how mixing of cascades leads to universal multifractals, consider the multiplicative mixing of N independent processes each with scale ratio $\lambda$, $\varepsilon_\lambda^{(i)}$:

$$\varepsilon_\lambda = \left(\prod_{i=1}^{N}\varepsilon_\lambda^{(i)}\right)^{a_N}\lambda^{-b_N} \qquad (23)$$

where we anticipate that some renormalization will be necessary; this is effected by $a_N$, $b_N$ that we determine below so that in the limit of large N, the resulting process $\varepsilon_\lambda$ is independent of N. In terms of the corresponding orders of singularities:

$$\gamma = a_N\sum_{i=1}^{N}\gamma_i - b_N; \qquad \varepsilon_\lambda=\lambda^\gamma; \qquad \varepsilon_\lambda^{(i)} = \lambda^{\gamma_i} \qquad (24)$$

We now use the fact that the $K_i(h)$, K(h) corresponding to $\gamma_i$, $\gamma$ are second characteristic functions; recall that adding random variables corresponds to adding second characteristic functions, we thus obtain:

$$K(h) = \sum_{i=1}^{N} K_i(a_N h) - b_N \qquad (25)$$

For simplicity, we assume that all the $\varepsilon_\lambda^{(i)}$ are statistically the same, i.e. $K_i=K_1$ for all i:



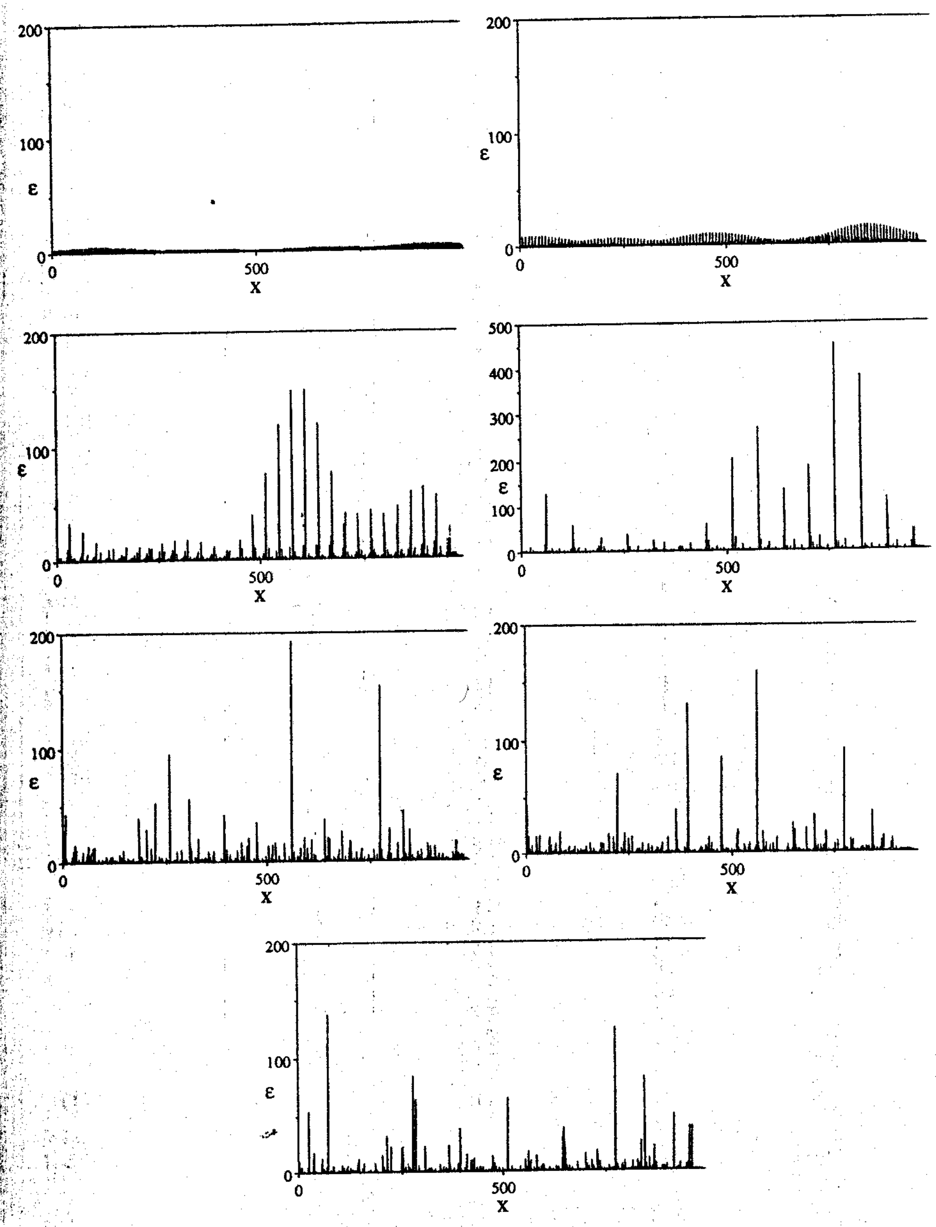



Fig. 4a-g: An example of the progressive mixing of a models with a fixed total range of scales, by progressively adding in α models with intermediate scales as explained in the text.

$$K(h) = NK_1(a_N h) - b_N h \qquad (26)$$

the renormalization constants $a_N$, $b_N$, are now chosen so that in the large N limit the left hand side is independent of N; this will require that $a_N$ decrease with N: the limit will only depend on the behaviour of $K_1(h)$ hear h=0; it will generally be of the form:

$$K_1(h) = m_\alpha h^\alpha + m_1 h; \qquad h \approx 0 \qquad (27)$$

$\alpha$ is the order of differentiability of $K_1$ at the origin; $m_1$, $m_\alpha$ are constants. A basic property of second characteristic functions is that their $m^{th}$ derivative at the origin is proportional to the $m^{th}$ order moment, hence, if $\alpha$ is nonintegral, it will be the highest order of convergent moments, i.e. $Pr(\gamma_1 > s) \approx s^{-\alpha}$, $s >> 1$, the large s behaviour corresponding to small h behaviour (h and s are Laplace conjugate variables). Using the above form for $h \approx 0$, we obtain:

$$a_N = N^{-1/\alpha}; \qquad b_N = m_1 N^{1-1/\alpha} \qquad (28)$$

With this choice, the following stable K(h) is obtained, the only aspects of $\varepsilon_\lambda^{(1)}$ which is relevant are the parameters $\alpha$, $m_\alpha$:

$$K(h) = m_\alpha h^\alpha \qquad (29)$$

These solutions are also attractive and correspond to the "extremal" or maximally asymmetric Levy distributions[20]. This $\alpha$ is the Levy index of the generator (see next subsection), the case $\alpha=0$ corresponds to the $\beta$ model, and $\alpha=2$, to the familiar gaussian generator for $\gamma$, $m_\alpha$ is the variance, all the moments will converge, hence the index $\alpha$ must be $\leq 2$. Overall, these solutions are stable and attractive and define the universality classes.

We can now demonstrate this mixing of turbulence using the $\alpha$ model discussed in section 1.2. Consider the mixing of N $\alpha$ models over a single cascade step with scale ratio $\lambda$ (considering more than one step makes no essential difference). We obtain:

$$K_1(h) = \frac{1}{\log\lambda} \log(\lambda^{h\gamma^+} \lambda^{-c} + \lambda^{h\gamma^-} (1-\lambda^{-c})) \qquad (30)$$

We can now either expand this about the origin (noting $K_1(0)=0$, and $K_1(h)$ is analytic near h=0, hence $\alpha=2$), or equivalently use the first cumulants of $\gamma$ obtaining:

$$m_1 = \langle\gamma\rangle = \gamma^+ \lambda^{-c} + \gamma^- (1 - \lambda^{-c})$$
$$m_2 = \frac{1}{2}\log\lambda\, \sigma_\gamma^2 = \frac{1}{2}\log\lambda\, (\langle\gamma^2\rangle - \langle\gamma\rangle^2)$$
$$= \frac{1}{2}\log\lambda (\gamma^+ - \gamma^-)^2 \lambda^{-c}(1-\lambda^{-c}) \qquad (31)$$

We thus find that the "mixed" $\alpha$ model defined by:

$$\varepsilon_\lambda = \lim_{N\to\infty} \left( \prod_{i=1}^{N} \varepsilon_\lambda^{(i)} \right)^{N^{-1/2}} \lambda^{-(\gamma^+ \lambda^{-c} + \gamma^- (1-\lambda^{-c}))N^{1/2}} \qquad (32)$$



has a log-normal distribution, the base $\lambda$ variance is $m_2$ as given above. This shows in a very concrete way the two very different limiting procedures discussed at the beginning of the section. On the one hand, the usual approach of iterating the cascade one step at a time, each time simultaneously introducing a new random factor and increasing the range of scales, leads to $K_1(h)$ unchanged in the limit. On the other hand, by keeping the range of scales fixed and mixing in more and more $\alpha$ model processes, the initial $K_1(h)$ tends to the universal K(h) determined only by the two parameters $\alpha$, $m_\alpha$ (see eqs. 27, 32).

2.3 Mixing, densification of scales, and continuous cascades:

In a general manner, we are lead to consider multiplicative processes, generated in an exponential way i.e. we define the generator $\Gamma_\lambda(\underline{x})$ and its Fourier transform $\hat{\Gamma}_\lambda(\underline{k})$:

$$\varepsilon_\lambda = e^{\Gamma_\lambda} \qquad (33)$$

which will be a noise concentrated on the band of wave numbers $[1,\lambda]$. As soon as every scale in the band is excited (the spectrum is a continuum in the band), we are dealing with continuous cascades. The various constraints on $\Gamma_\lambda$ so that $\varepsilon_\lambda$ is a normalized (bare) multifractal are detailed in [6,12]: i) $\Gamma_\lambda$ is limited to wavenumbers k in the interval $[1,\lambda]$. ii) the second characteristic function $K_\lambda$ of $\Gamma_\lambda$ has a log divergence: $K_\lambda(h) \approx Log(\lambda)\, K(h)$ iii) to allow convergence for positive order moments, the positive fluctuations of $\Gamma_\lambda$ must fall-off more than exponentially fast. iv) The generator must be normalized ($K_\lambda(1)=0$) to assure the (canonical) conservation of the flux. From properties i), ii), we deduce that the (generalized) spectrum $E_\lambda$ of $\Gamma_\lambda$ must be[6] of the form $k^{-1}$ ($\Gamma_\lambda$ is a "1/f" noise).

One way to obtain a continuous cascade is by mixing more and more discrete $\alpha$ models each of them involving intermediate scales over a fixed range. An example is given in figures 4a-g which shows the mixing in successive stages. First, wavenumbers 1 and 513 of $\Gamma_\lambda$ are excited by gaussian random variables. The mixing proceeds by "densifying" the interval in Fourier space between wavenumbers k=1 and $k=\lambda=513$ by adding in noises with components corresponding to intermediate wavenumbers. For example, the first step involves an $\alpha$ model with k=257, the in the second step we add modes with k=129, 385, in the third with k=65, 193, 321, 449 etc., this densification is continued until all the wavenumbers are excited. (fig. 4g). Each wavenumber respects the $k^{-1}$ spectrum and is normalized so that the total variance remains fixed.

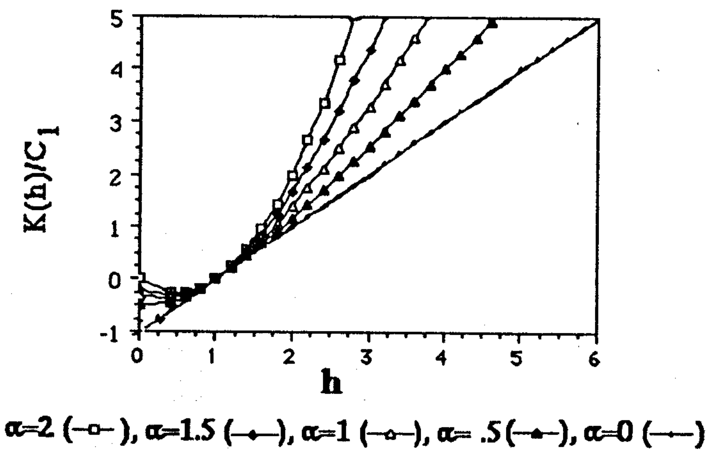
2.4 Universality classes:

We have already shown how to obtain universality by mixing, obtaining the (normalized) universal K(h) functions in terms of $\alpha$, $m_\alpha$. Recalling that $h^\alpha/\alpha$, $\gamma^{\alpha'}/\alpha'$ are Legendre transform pairs ($\alpha^{-1} + \alpha'^{-1} = 1$, $0 \leq \alpha \leq 2$, $-\infty < \alpha' \leq 0$ or $2 \leq \alpha' < \infty$), and using the definition in section 2.1 for $C_1$, $\alpha$, we obtain.

$$\alpha \neq 1:\ K(h) = \frac{C_1 \alpha'}{\alpha}(h^\alpha - h) \qquad \text{(only for } h \geq 0 \text{ when } \alpha < 2;\ = \infty \text{ for } h < 0);$$

$$\alpha = 1:\ K(h) = C_1\, h\, Log(h) \qquad (36)$$

which upon Legendre transformation yields the following codimension function $c(\gamma)$ (for the probability density rather than the probability distribution[13]):



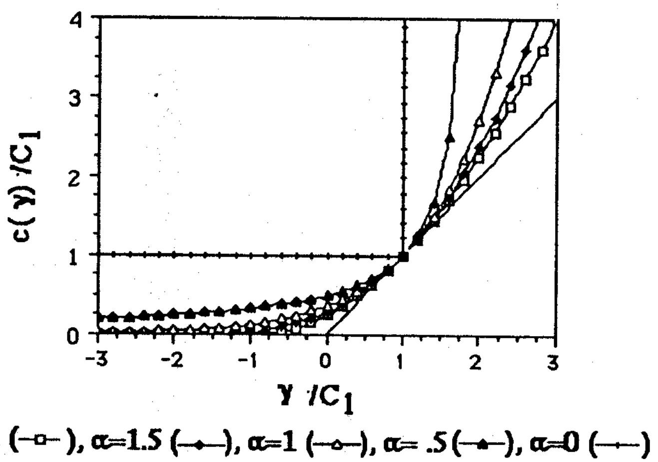




Fig. 5a,b shows representative $c(\gamma)$, $K(h)$ curves for the above universal multifractals. The five main qualitatively different cases ($\alpha=2$, $2<\alpha<1$, $\alpha=1$; $1<\alpha<0$, $\alpha=0$) are dealt with in detail in reference[11].



$$\alpha\neq 1: c(\gamma) = C_1\left(\frac{\gamma}{C_1\alpha'} + \frac{1}{\alpha}\right)^{\alpha'} \qquad \text{(only for } dc/d\gamma>0 \text{ when } \alpha<2)$$

$$\alpha=1: c(\gamma) = C_1 \exp\left(\frac{\gamma}{C_1}-1\right) \qquad (37)$$

Fig. 6a, b the corresponding basic white noises of continuous universal multifractals in the highly asymmetric case $\alpha<2$ (Fig. 6a) and the exceptional symmetric case $\alpha=2$ (Fig. 6b). Fig. 7a,b, shows how these processes can be used to simulate clouds and topography (see reference[21] for more details).

Here we only consider whether they yield soft or hard multifractals. From Eq. 36, it is easy to see that for $\alpha\geq 1$, $C(h)$ diverges with h, hence they are "unconditionally hard", whereas for $\alpha<1$, $C(\infty) = C_1/(\alpha-1)$ which is <D for large enough but finite D, hence these multifractals are "conditionally soft." Alternatively, from Eq. 37 (see fig. 4a), we see that when $\alpha\geq 1$, $\gamma_{max}\to\infty$ hence the corresponding multifractals are hard, whereas when $\alpha<1$, $\gamma_{max} = C_1/(\alpha-1)$ hence they are only conditionally hard. Considerable evidence on the divergence of various moments of turbulent atmospheric quantities has suggested for several years that turbulence really is hard; below (see also [23]), we show that it is unconditionally so.

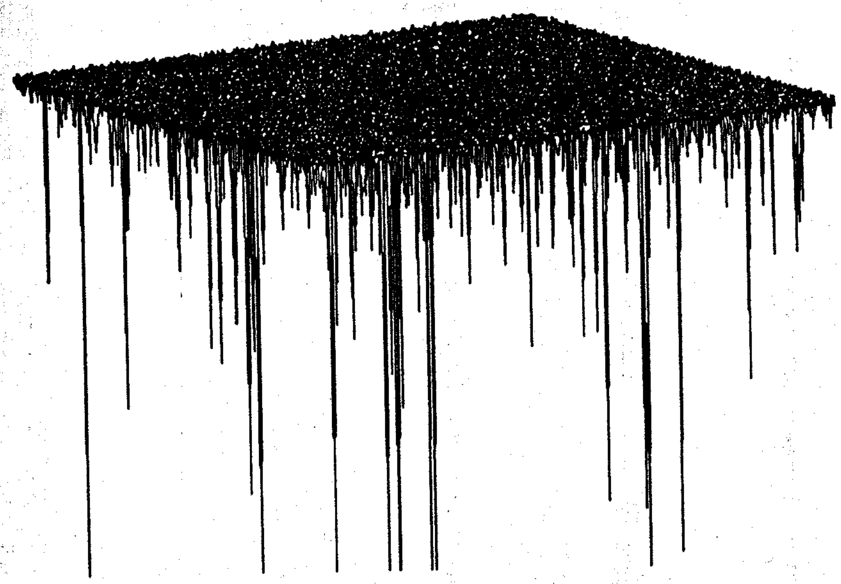


Fig. 6a: The Levy subgenerator (white spectrum) of a continuous cascade process with $\alpha=1.6$ clearly showing the strong asymmetry between the modest postive fluctuations and the extreme negative fluctuations.

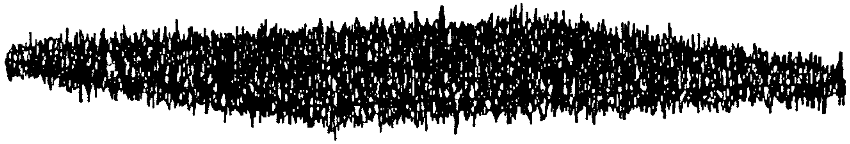


Fig. 6b: Same as 6a but for a (symmetric) gaussian generator.

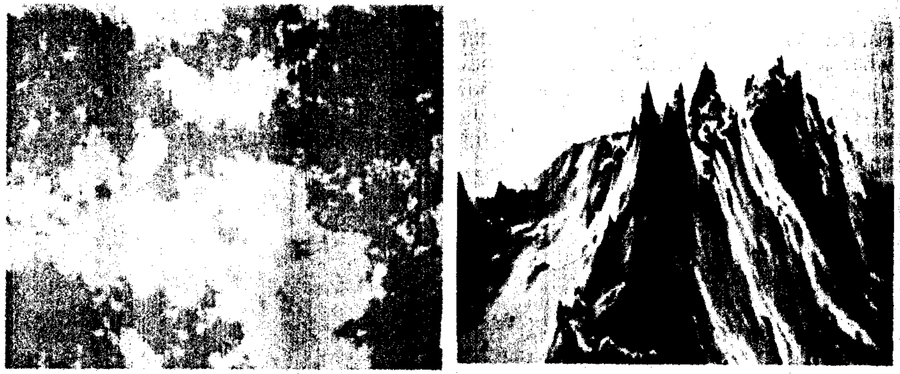


Fig. 7a: Multifractal cloud resulting from the exponentiation of the process above followed by a final power law filter so as to yield passive scalar scaling. Grey scale corresponds to the logarithm of the passive scalar density.

Fig. 7b: Multifractal mountain obtained with a similar continuous cascade combined with ray tracing. See Lavallée et al [22] for relevant data analysis.

## 3 Empirical results

3.1 Multifractal velocity and temperature indices:

Ever since the development of the first concrete cascade models for turbulent intermittency, there has been considerable effort devoted to testing the models and evaluating their parameters. Here we are interested in evaluating the fundamental exponents H, $C_1$, $\alpha$. In this case, H is known on dimensional grounds to have the value 1/3. The various quantities are related as follows (the subscript indicates the resolution $\lambda$):

$$\Delta v_\lambda \approx \varepsilon_\lambda^{1/3}\lambda^{-1/3} \qquad \Delta T_\lambda \approx \phi_\lambda^{1/3}\lambda^{-1/3} \qquad (38)$$



$$\langle\varepsilon_\lambda^h\rangle \approx \lambda^{K_\varepsilon(h,1)} \qquad \langle\phi_\lambda^h\rangle \approx \lambda^{K_\phi(h,1)} \qquad (39)$$

where $\phi_\lambda$ is the temperature cascade quantity[1] analogous to $\varepsilon$, and K(h,1) is the scaling exponent related to $C_1$ and $\alpha$ via Eq. 36.

3.2 Double Trace Moments:

We now describe a simple technique which is the first specifically designed for estimating $\alpha$, $C_1$, and that overcomes many of the above difficulties encountered in standard multifractal techniques[15,24]. Consider a stationary (conserved) multifractal process $\varepsilon_{\lambda'}$ (i.e. $\langle\varepsilon_{\lambda'}\rangle=1$), such as the turbulent energy flux at resolution $\lambda'$, the ratio of the outer (largest) scale of interest to the smallest scale of homogeneity. The h, $\eta$ double trace moment at resolutions $\lambda$ and $\lambda'$ is defined as:

$$Tr_\lambda(\varepsilon_{\lambda'}{}^{\eta})^h = \left\langle \sum_{A_\lambda} \left( \int_{A_\lambda} \varepsilon_{\lambda'}{}^{\eta}\, d^D\underline{x} \right)^h \right\rangle \propto \lambda^{K(h,\eta)-(h-1)D} \qquad (40)$$

$$K(h,\eta) = K(h\eta,1) - h\,K(\eta,1) \qquad (41)$$

Where the sum is over all the resolution $A_\lambda$ sets (dimension D) required to cover the multifractal, and $K(h,\eta)$ is the (double) scaling exponent given in Eq. 42, K(h,1) is the usual scaling exponent (second characteristic function of the generator denoted K(h) above). Using universal multifractals (Eqs. 36, 37) the following relation[15] can be derived:

$$K(h,\eta) = \eta^\alpha K(h,1) = \begin{cases} \dfrac{C_1}{\alpha-1}\eta^\alpha (h^\alpha - h) & \alpha \neq 1 \\ C_1\,\eta\, h\, \mathrm{Log}(h) & \alpha = 1 \end{cases} \qquad (42)$$

Therefore $\alpha$ can be estimated from a linear regression of $\log|K(h,\eta)|$ vs. $\log\eta$ for fixed h, and $C_1$ from the intercept with the $\log\eta=0$ axis. Eq. 42 holds for moments $h\eta$ of order low enough so that convergence occurs[2]. Previous methods for estimating $\alpha$ relied on either K(h,1) or its Legendre transform (the codimension function) which requires either special graphical techniques or (poorly conditioned) nonlinear regressions.

3.3 Data sets and experimental procedure:

The velocity measurements were made by Gagne at the ONERA wind tunnel, Modane with high resolution hot wire anemometers sampling at 10kHz; we analysed a small portion lasting only several seconds. The temperature measurements were made at McGill by Hooge and Bertrand at 5Hz for 20 minutes. The time series (which had empirical energy spectra of the form $E(\omega)\approx\omega^{-\beta}$, with $\beta_v\approx 1{,}70$ and $\beta_T\approx1.75$, $\omega$ is the frequency) were power law filtered in Fourier space to remove the $\lambda^{-1/3}$ scaling in Eq. 38, in order to yield the roughly stationary quantities $\varepsilon^{1/3}$, $\phi^{1/3}$. The amount of filtering required to yield an exactly stationary process is not exactly $k^{-1/3}$. It also depends on the exponent involved in the statistical space/time transformation required to transform from temporal to spatial statistics (in the atmosphere the usual Taylor hypothesis needs

[1] $\phi=\varepsilon^{-1/2}\chi^{3/2}$ where $\chi$ is the temperature variance flux to smaller scales, which is conserved independently of $\varepsilon$ if T is a passive scalar quantity.

[2] When $\eta$ and $h\eta > h_D$ (or $h_{max}$), then the $\log|K(h,\eta)|$ curve becomes independent of $\eta$, giving a direct estimate for the critical order of moments $h_D$.

anisotropic generalization), but also on the (initially unknown) values of $C_1$, $\alpha$, as well as on possible deviations from the theoretical behaviour[1]. Fortunately, the double trace moment technique is not affected by residual nonstationarity[22], as long as the spectrum is fairly flat (i.e. the filtered signal $\beta \approx 0$).

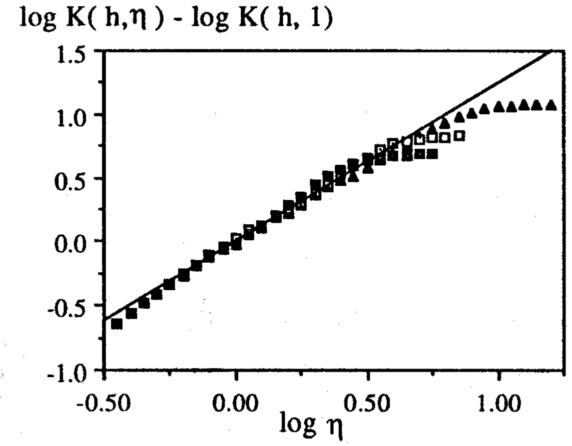



Fig. 8a: Plot of log $|K(h,\eta)|$ - log $|K(h,1)|$ v.s. log $\eta$ for the velocity data discussed in the text clearly showing the straight line region predicted for universal multifractals, slope $\alpha_v$=1.3. The high $\eta$ breakdown is due to the divergence of moments, and occurs as predicted at about $\eta$=5. The different curves shown are for h= 0.5, 1.2, 4.

3.4 Results of the Double Trace Moment Technique:

The double trace moment technique is applied to the series - transformed as indicated above - for various values of the parameters h, $\eta$. The results are shown in figs.8a, b. In both cases, as long as $\eta$ and $h\eta$ are below a value $\approx 5$, the plots of log $|K(h,\eta)|$ vs. log $\eta$ are very straight, as expected for universal multifractals, with slopes $\alpha_T \approx 1.2 \pm 0.1$, $\alpha_v \approx 1.3 \pm 0.1$, and intercepts yielding $C_{1T}$ =0.085, $C_{1v}$ = 0.130. The $C_1$ values corresponding to the conservative fluxes $\varepsilon$, $\chi$ are obtained using eqs. (38) and (39): $C_{1\varepsilon}=C_{1v}3^{\alpha_v}$, $C_{1\chi}=C_{1T}3^{\alpha_T}$, hence $C_{1\varepsilon} \approx 0.50$, $C_{1\chi} \approx 0.35$ in agreement with the values estimated with lognormal and $\beta$ models[23]. Furthermore, using the formula $K(h_D,1)=(h_D-1)D$ where D is the dimension of the averaging set; here =1 for the time[2], the following critical values are found: $h_{Dv}$ = 4.5±1 and $h_{DT}$ = 6±1. These values are compatible with the breakdown of linearity observed in figs. 8a,b, and on the direct empirical estimates of divergence of moments[5,25] of orders 5.

Because of the divergences, the measurements indicate that turbulence is a hard multifractal process. For $\alpha \geq 1$ it is unconditionally so, hence averaging over four dimensional (space/time) sets rather than just in time, will not be sufficient to ensure convergence of all moments. The only other empirically estimated multifractal indices are $\alpha \approx 1.7$, 0.6 for infra red and visible satellite cloud radiances respectively[26,27], and $\alpha \approx 1.8$ for the earth's topography[22].

[1] Especially for T which is not exactly a passive scalar due to buoyancy effects.

[2] Actually, we should use the elliptical dimension for the correct space/time transformation; we ignore this complication here.



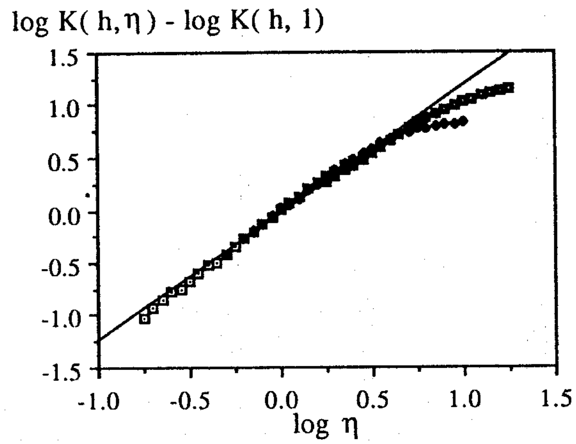



Fig. 8b: Same as fig. 8a except for the temperature data discussed in the text, the straight line has slope $\alpha_T$=1.2. The high $\eta$ breakdown is due to the divergence of moments, and occurs as predicted at about $\eta$=5. The different curves shown are for h= 0.5, 2, 4.

## 4. Conclusions

We have rendered more concrete and precise a series of multifractal notions. We first show how to classify multifractals according to their highest order of singularities. The calmest multifractals we considered were the geometric (Parisi Frisch) multifractals which have completely localized singularities. There is no associated stochastic process, nor probability space. We then consider microcanonical cascades in which by construction, the energy flux is exactly conserved on each realization, we find that it is not only delocalized (the orders of singularities and codimensions are no longer point values but statistical exponents), but already possesses rare violent singularities which are out of reach of the geometric multifractals. Finally, we turn to the most general canonical cascades involving wild singularities associated with the looser conservation of energy flux only over ensemble averages, rather than on each realization. They generally (although not necessary) also feature even more violent "hard" singularities which cause high order statistical moments to diverge. We underline the necessity of using a turbulent (rather than the more usual strange attractor) formalism for multifractals since all multifractals (except the calmest geometric variety) are defined on infinite dimensional probability spaces.

Although a priori, the specification of multifractals involves an entire (codimension) function, hence an infinite number of parameters, we show that stable, attractive universality classes exist greatly simplifying multifractal analysis and simulations. We illustrate this universality by showing how mixing of various $\alpha$ model cascades leads to a log-normal cascade. This result is quite different from that obtained in the usual (nonuniversal) limit process which involves iterating a single process over more and more cascade steps.

Finally, we outline a new "double trace moment" mulitfractal analysis technique which is very robust because it is the first specifically designed to determine the hierarchy of singularities by exploiting the universality classes. We apply this technique to the analysis of turbulent temperature and velocity data, obtaining the estimates $\alpha_T \approx 1.2 \pm 0.1$, $\alpha_v \approx 1.3 \pm 0.1$ respectively. Since $\alpha > 1$, this shows that turbulence is unconditionally hard, the high order moments (here, in one dimension, estimated to have order 5 or greater) will always diverge no matter how high a dimension of space is used to "smooth" them.

Our experimental confirmation of multifractal cascade models of turbulence has fundamental implications for solutions of Navier-Stokes equations for the high Reynolds number limit, since with the help of three fundamental exponents it characterizes the possible class of solutions, in particular the entire hierarchy of singularities!

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