

Stochastic chaos and multifractal geophysics

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1. Introduction

There is no doubt that the ancient debate about the origin of order from disorder (“chaos” versus “cosmos”) has been considerably invigorated by the rapid progress in our understanding of the nonlinear dynamics of deterministic systems with few degrees of freedom: the “chaos revolution”. Over the last twenty years, in field after field, it has changed our outlook. Regular, predictable planetary motion is no longer “typical”; irregular, random-like chaotic behaviour is now the norm. What is particularly striking is that unlike the Newtonian, quantum or relativistic revolutions, the deterministic chaos revolution has not been associated with any decisive new applications. This is a good indication that the revolution is only just beginning and that many surprises lie ahead.

In this paper, we argue that the principle reason for the paucity of applications is its restriction to systems with few degrees of freedom. Based on a series of developments in the last ten or fifteen years in fractals and multifractals, we develop a rather different framework for handling nonlinear systems with many degrees of freedom; “stochastic chaos”. While we believe this alternative to be quite broad - for example it potentially encompasses much of our atmospheric, geophysical and perhaps astrophysical environment - we do not claim exclusivity. Rather we view stochastic chaos as complementary to deterministic chaos with the former being valid in systems involving many interacting components and the latter being valid when only a few corresponding “degrees of freedom” are important. Both model types belong in the physicist’s toolbox. So far, the utility of stochastic chaos lies primarily in its ability to exploit a (nonclassical) symmetry principle called scale invariance, associated with fractals and multifractals. We will therefore argue that the ubiquity of fractals in nature is an indication of the wide scope for applying stochastic chaos models¹. More philosophical and historical discussion can be found in [Lovejoy and Schertzer, 1998].

2. Deterministic chaos

Although the basic problem was recognized by [Poincaré, 1892] the general property of nonlinear systems of having “sensitive dependence” on initial conditions only became widely known in the 1970’s somewhat after the work of Lorenz [Lorenz, 1963]. Popularly known

as the “butterfly effect” this term denotes the general property of nonlinear systems to amplify small perturbations. In the area of random-like behaviour of fluids, “turbulence”, this led to the idea that the latter arose through a series of instabilities, each one involving amplification of a tiny noise degree of freedom. “Fully developed [fluid] turbulence” thus came to be viewed ([Landau, 1944]) as the state obtained after an infinite number of such instabilities had occurred, i.e. when an infinite number of degrees of freedom had been excited in this way.

By itself, the butterfly effect did not make the “chaos” revolution. Two key developments were necessary. The first was the discovery that random-like behaviour of fluids *could in principle* be the result of only three instabilities hence the focus on systems with a small number of degrees of freedom. The second was the discovery of universal behaviour. Lack of universality was disastrous for both theory and experiment since not only would the behaviour of a nonlinear model depend (nontrivially) on every theoretical model detail but experimentalists had no way of knowing in advance what experimental conditions were important and which were irrelevant.

This lack of “universality” was partially overcome by [Metropolis *et al.*, 1973] qualitative idea of “structural universality” (followed shortly by the first experimental confirmation, in a fluid system [Gollub and Sweeney, 1975]). However, the real breakthrough came when [Feigenbaum, 1978] and independently [Grossman and Thomae, 1977] obtained quantitative (“metric”) universality making it possible to quantitatively test the theory empirically. This was soon done in certain fluid systems (e.g. [Gollub *et al.*, 1980]), and in several others. By the early 1980’s the rapid pace of developments led to what could properly be called the “chaos revolution”.

2.2 Later developments and problems.

Although the basic qualitative implications - that random-like behaviour is “normal”, not pathological - is valid irrespective of the number of degrees of freedom of the system in question, the focus on small number of degrees of freedom systems - led to a simultaneous ballooning of hype concomitant with a drastic restriction of the scope of “chaos” to meaning precisely deterministic systems with few degrees of freedom. This restriction, coupled with the development of new empirical techniques for “reconstructing the attractor” (i.e. quantifying the nature of the chaotic dynamics, notably via the Grassberger-Procaccia algorithm, [Grassberger and Procaccia, 1983]), led to a major focus on applications and to a number of curious - if not absurd - results.

It is perhaps easiest to understand these aberrations by considering the example of the climate system. It had been taken for granted that the climate involved a large (practically infinite) number of degrees of freedom (classical estimates of the number in the atmosphere alone are about 10^{27}). However, for certain mostly theoretical purposes (especially to determine whether or not the climate is stable), low number of degrees of freedom models (“zero dimensional” because they ignored all spatial structures/variations) were sometimes used (e.g. [Budyko, 1969], [Sellers, 1969], [Ghil, 1976]). However, with the impetus of chaos, detailed attempts were made (using standard frequency analysis) to compare characteristic model oscillations with those of climate series [LeTreut and Ghil, 1983]. Soon afterwards, new chaos tools were applied to the data; the (now classical) attempt was that of [Nicolis and Nicolis, 1984] in which it was quite seriously claimed, that only four parameters (degrees of freedom) were required to specify the state of the climate²! Somewhat later, attempts were made (e.g. [Kaplan and Glass, 1992]) to prove purely objectively from analysis of data - that in spite of appearances - random-like signals were in fact deterministic in origin.

These attempts were flawed on several levels. At a purely technical level, Grassberger himself [Grassberger, 1986] pointed out that Nicolis’ climate analysis was based on far too

few points (less than 200!) and on artificially smoothed data; one could not in fact technically substantiate any conclusion on the finiteness of the number of degrees of freedom. A more important (but still) technical point, was that it had already been forgotten that truly random ("stochastic") processes such as Brownian motion with infinite numbers of degrees of freedom would also give a low finite dimensional result if subjected to the Grassberger-Procaccia algorithm (e.g. [Osborne and Provenzale, 1989]). However what is paramount is that these attempts were predicated on a basic confusion: the supposition that nature *is* (i.e. ontologically) either deterministic or random. The best that any empirical analysis could ever hope for would be to demonstrate that specific deterministic models fit the data better (or worse) than specific stochastic ones. On the contrary, the terms "determinism" or "random" denote specific aspects of our theories/models of nature; as scientists it is our job to find the best type of model, not to dogmatically restrict the types of model to preconceived categories³.

3. Stochastic chaos

3.1 General considerations

The justification for using small numbers of degrees of freedom chaos systems as models for complex geophysical, astrophysical, or ecological systems - each involving nonlinearly interacting spatial structures (fields) - has two related aspects each of which we argue are untenable. The first is the illogical inference that because deterministic systems *can* have random-like behaviour, that random-like systems are - in spite of appearances - best modelled as *not* random after all. The second is that - again in spite of appearances - that the spatial structures which apparently involve huge variability and many degrees of freedom spanning wide ranges of scale⁴, can in fact be *effectively* reduced to a small finite number.

Before making the alternative explicit, let us note an additional scientific development that helps to make it more plausible: the axiomatization of probability theory by [Borel, 1908], and [Kolmogorov, 1933] which showed that probability is perfectly objective; statistics need not be an expression of ignorance. Following general usage we denote such objective randomness as "stochastic". The fundamental characteristic of stochastic theories/models which distinguishes them from their deterministic counterparts is that they are defined on (infinite dimensional) probability spaces, hence they are automatically approximations to systems with large but finite number of degrees of freedom. Formally, to determine whether or not a theory/model is deterministic or stochastic, it suffices to enquire as to the mathematical nature of the spaces upon which the primitive concepts of the theory/model are defined (i.e. are they random variables/functions in the strict mathematical sense or not?). Note also that purely formally, if only because a deterministic outcome is a special case of a stochastic outcome, stochastic theories/models are in any case more general than their deterministic counterparts.

The basic alternative for nonlinear dynamics with many degrees of freedom is now easy to state. The idea, simply put, is to model/theorize random-like systems with large numbers of interacting components by using objective random models: "stochastic chaos"!

3.2 Physical arguments for stochastic chaos; the example of turbulence:

Stochastic chaos is particularly advantageous with respect to classical approaches when a nonclassical symmetry is present: scale invariance. Although classical approaches are often impotent in handling such systems in comparison, stochastic approaches (involving scale invariant cascade processes) immediately yield a wealth of information.

Consider the example of fluid turbulence which must be counted amongst the most difficult problems in physics. The basic dynamical - and deterministic - ("Navier-Stokes") equations have been known for nearly 150 years, yet the fundamental problem remains whole: how to reconcile the (violent) nonclassical turbulent statistics/structures with the equations. Two features symptomatic of this difficulty are that a) advances have often been made using approaches having very little direct contact with the dynamical equations, b) the alternatives (analytic closures, renormalisation, cascades) precisely involve hypotheses about the possible stochastic behaviour of... the deterministic equations!

Perhaps one of the most successful of these alternatives is the paradigm of turbulent cascades which nonetheless respect the fundamental dynamical symmetry of invariance under changes of scale. It is already remarkable that a rather immediate development of this paradigm lead to the first quantitative laws of turbulence: the Richardson law of turbulent diffusion [Richardson, 1926] and the scaling law for the velocity field itself [Kolmogorov, 1941]. Basing himself on three statistical hypotheses, Kolmogorov postulated a "quasi-equilibrium" for turbulence. The rate of large scale forcing energy leads to a flux of energy flowing through the "inertial range" of intermediate scales towards small scales, where (at a small "Kolmogorov scale") it is dissipated. In the quasi-equilibrium regime the three quantities should be equal, at least for an appropriate average.

It is remarkable that for over fifty years very little progress has been made in improving the (nearly hand-waving) original Richardson and Kolmogorov arguments. This is true in spite of the development of powerful analytical tools, including various "closure" and Renormalization Group techniques. Without appeal to artificial ad hoc hypotheses⁵, these attempts have lead neither to satisfactory derivations of the Richardson, nor Kolmogorov laws⁶. The failure of these analytic approaches is even more striking since both are at best "mean field" laws i.e. even these lowest order laws are still beyond the reach of present analytical developments! Indeed, as first pointed out by [Landau, 1944] and [Batchelor and Townsend, 1949], these problems can be traced to the presence of a very strong type of inhomogeneity called "intermittency". Not only does the "activity" of turbulence induce inhomogeneity, but the activity itself is inhomogeneously distributed. The cascade paradigm provides a convenient framework to study this phenomenology yielding very concrete models and interesting conjectures. In particular, it is now increasingly clear that a very general outcome of stochastic cascades are the multifractal measures discussed in section 4.

3.3 Classical examples of stochastic chaos: random walks and fractals:

Perhaps the best known example of stochastic chaos is the random walk. It also gives us a simple example of stochastic universality - without which stochastic chaos - like its deterministic counterpart - would be unmanageable and irrelevant. In fact - in order to obtain the usual Brownian motion - the only hypothesis necessary is that the variance of each elementary step is finite. Nevertheless - and this not widely enough known - when the latter hypothesis is relaxed, universality still survives! Indeed, Lévy ([Levy, 1925]) showed that there exists a universal attractor, called Lévy stable laws, depending only on three parameters, the most important one, the Lévy index describes in fact how the variance divergences. As discussed below, these additive results have analogues in multiplicative cascade processes (hence for multifractals).

The drunkard's walk is a scale invariant process; which yields a mathematically simpler object than cascades; the walker's "trail" is a scale invariant geometric set; a fractal. In contrast to this stochastic fractal set, the first fractal set - the Cantor set - was deterministic and was originally proposed on purely mathematical grounds. Starting with ([Richardson, 1926], [Welander, 1955] and [Steinhaus, 1960]), there have been periodic suggestions that various physical systems resemble fractal sets. However it wasn't until [Mandelbrot, 1977]

that the explicit development and use of scale invariant geometric (fractals) sets as physical models became widely accepted. As we argue below, the major breakthrough in applications occurred in the 1980's with multifractals; the extension of scale invariance to fields (see section 4) rather than just geometric sets.

3.4 Self-Organized criticality

Contrary to standard deterministic chaos systems, self-organized critical (SOC; [Bak et al., 1987]) systems are cellular automata with high numbers of degrees of freedom. For certain classes of discontinuous rules and appropriate boundary conditions, the systems were found to evolve spontaneously to a "critical state" involving huge fluctuations. Viewed as a high number of degree of freedom extension of deterministic chaos, the scaling of the SOC could be considered to be "on the edge of chaos" (referring to power law rather than exponential decorrelations). However, for several reasons, SOC can be more profitably viewed as an example of stochastic chaos. First, since the initial conditions must be random in order to obtain any interesting result, the overall process is in fact stochastic (in spite of the deterministic evolution rule). Recently, a new generation of SOC models have been developed with stochastic evolution rules, thus the stochastic nature of the SOC paradigm is now quite explicit. However, perhaps the most significant point is that SOC models share two basic features with multifractal cascades: scale invariance and extreme variability pointing to a deep connection between the two [Schertzer and Lovejoy, 1997], see section 4.2.

4. Scale invariance symmetries and cascades

4.1 Cascades, and multifractals

Initially, Richardson's cascade was simply a conceptual scheme for explaining the transfer of energy from the planetary scales, down to the small scales (roughly 1mm) where it is dissipated by viscosity. However as mentioned above, a key feature is the time/space intermittency which motivated the development of explicit multiplicative cascade models⁷: the "pulse in pulse" model, [Novikov and Stewart, 1964], [Yaglom, 1966], "weighted curdling", [Mandelbrot, 1974], the " β model" [Frisch et al., 1978], the " α model", [Schertzer and Lovejoy, 1983], the "random β model", [Benzi et al., 1984], the "universal" and the "continuous" cascade models [Schertzer and Lovejoy, 1987], the "p model", [Meneveau and Sreenivasan, 1987] etc. The simplest (" β model") is obtained by making the simplistic assumption that at each cascade step the turbulence is either dead or alive. Since the same random mechanism is repeated unchanged scale after scale, the process is scale invariant; in the small scale limit the "active" regions form a geometrical fractal set of points.

Ignoring for the moment the artificiality of the straight construction lines and the factor of two break-up of eddies into subeddies, we can now make a step towards realism by introducing a slight modification: we continue to flip coins, but now we multiplicatively "boost" or "decrease" the energy flux density (ϵ_λ) than than boosting or killing the eddies (the " α model"; $\lambda=L/l$, the ratio of the large scale L to the small scale resolution l). The result is a multifractal field with an infinite number of levels of activity: the singularities γ .

$$Pr(\epsilon_\lambda > \lambda^\gamma) \approx \lambda^{-c(\gamma)} \quad (1)$$

where c is the codimension; "Pr" indicates "probability" and the " \approx " sign means equality to within slowly varying factors. For $c < D$ (the dimension of the observing space), the set of points exceeding a given intensity form geometric fractal sets with dimension⁸ $D(\gamma) = D - c(\gamma)$. For large λ structures with larger γ dominate those with smaller γ so that unlike classical sto-

chastic processes such as brownian motion or gaussian noise, we obtain the stochastic appearance of structures.

This “ α model” has all the essential ingredients of the more sophisticated models needed for realism. The main improvement is the use of continuous rather than discrete scale ratios. Not only does this eliminate the straight-line artifacts, but it generically yields “universal multifractals” i.e. special multifractals which - just like random walks discussed earlier - occur irrespective of the details of the basic dynamical mechanism and depending on only two basic parameters⁹ ([Schertzer and Lovejoy, 1987]):

$$c(\gamma) = C_1 \left(\frac{\gamma}{C_1 \alpha'} + \frac{1}{\alpha} \right)^{\alpha'} ; \quad \frac{1}{\alpha} + \frac{1}{\alpha'} = 1 \quad (2)$$

$0 < C_1 < D$ is the codimension of the mean and $0 < \alpha < 2$ is the “Levy index”, 0 corresponding to monofractality, 2 corresponding to the maximum degree of multifractality, the “log-normal” multifractal¹⁰.

Just as universality in low dimensional systems allowed for empirical tests; so too with multifractal universality in these high number of degrees of freedom systems. To date, over twenty geophysical, astrophysical and other fields and time series have been shown to be multifractal¹¹ over at least some range of scales (this includes cloud radiances, and liquid water content, mountains, ocean surfaces, the distribution of galaxies, wind, temperature and rain fields, river stream flow, lava flows, low frequency speech, music, finance, etc.; see [Lovejoy and Schertzer, 1995] for a survey).

4.2 The multifractal (stochastic) butterfly effect and self-organized criticality:

The significance of sensitive dependence on initial conditions, the “butterfly effect” is that if the system is sufficiently unstable then a small disturbance can grow, totally modifying the future state of the system. In the atmosphere - which down to the viscous scale of millimeters is scaling and presumably unstable - this would mean that the sequence of weather events (which would include -in Lorenz’s metaphor [Lorenz, 1993] Texas tornadoes) would be different. In our stochastic multifractal cascade model, we may identify an analogous “stochastic butterfly effect” by studying the small scale limit of the cascade and by determining under which conditions the small scale can dominate the large.

Contrary to “additive” stochastic chaos such as Brownian motion, as cascades proceed to smaller and smaller scales, the turbulent activity at a given point is not changed by a smaller and smaller amount, rather, it is modulated by random factors; this leads to the absence of (pointwise) convergence¹² of $\lim_{\lambda \rightarrow 0} \varepsilon_\lambda(\underline{x})$. However, there is nevertheless a kind of weak convergence of measures i.e. averages over finite regions of the completed cascade will converge. To consider this problem, introduce the flux through a region of scale λ :

$$\Pi_\Lambda(B_\lambda) = \int_{B_\lambda} \varepsilon_\Lambda d^D \underline{x} \quad (3)$$

where B_λ is an intermediate sized “ball” (scale ratio $\lambda < \Lambda$ of dimension D). We can now define the “dressed flux density”:

$$\varepsilon_{\lambda(d)} = \lim_{\lambda \rightarrow \infty} \frac{1}{\text{vol}(B_\lambda)} \Pi_\Lambda(B_\lambda) \quad (4)$$

These finite averages are called “dressed” quantities to distinguish them from the “bare” quantities obtained by stopping the cascade at the corresponding scale. The dressed cascade takes into account all the small scale activity; $\varepsilon_{\lambda(d)}$ is therefore more variable than ε_λ ; specifically, due to the multiplicative nature of the cascades, it can be shown [Schertzer and Lovejoy, 1987] that:

$$\varepsilon_{\lambda(d)} = \varepsilon_\lambda \Pi_\infty(B_1) \tag{5}$$

i.e. the bare density (ε_λ) is modulated a the random scale independent (“hidden” factor) which has an algebraic (long-tailed) probability distribution leading to divergence of statistical moments¹³ and SOC behaviour. In this way, the small scale activity occasionally generates violent (“hard”) events [Schertzer and Lovejoy, 1992] which on the contrary dominate the variability due to the large scale (i.e. in eq. 5, ε_λ is occasionally dominated by $\Pi_\infty(B_1)$); the “multifractal phase transition route to SOC” [Schertzer, 1994]. This is the multifractal/cascade version of the butterfly effect: most of the time, the flapping of the wings will lead to nothing special; $\Pi_\infty(B_1)$ will be of order 1 and the perturbation will be small compared to the existing large scale weather structures. However, in a probabilistically precise way, the overall effect of all the small scale dynamics (which includes those small enough to be perturbed by a butterfly’s flapping) will occasionally dominate the effect of the large scale dynamics. This specific cascade prediction has been verified empirically in a dozen or so geophysical fields, including in the atmosphere the all important velocity field¹⁴ [Schertzer and Lovejoy, 1985a], [Schmitt et al., 1994].

4.3 Nonclassical (anisotropic) zooms and Generalized Scale Invariance:

Up until now, we have considered scale invariance intuitively using the example of cascade processes in which a simple (coin tossing and multiplicative modulation) mechanism is repeated scale by scale. This mechanism is the same in all directions (it is isotropic); the resulting fractals and multifractals are therefore “self-similar” in the sense that a small piece when blown up statistically resembles the whole. With the minor exception of “self-affinity” (which involves squashing along a coordinate axis), self-similarity is the very special case discussed in [Mandelbrot, 1983] and in most of the fractal/multifractal literature. However, no natural system is exactly isotropic; many physical mechanisms exist which can introduce preferred directions, the most obvious being gravity which for example leads to a differentially stratified atmosphere, ocean and earth interior. Sources of anisotropy which can lead to differential rotation are the Coriolis force (due to the earth’s rotation) or stresses (in fluids or rock) induced by external boundary conditions. Contrary to conventional wisdom (which equates scale invariance with self-similarity, and hence with isotropy), scale invariance still survives, although the notion of scale undergoes a profound change. The resulting formalism of Generalized Scale Invariance (GSI; [Schertzer and Lovejoy, 1985b], [Schertzer and Lovejoy, 1989], [Schertzer and Lovejoy, 1991], [Pecknold et al., 1996]) involves essentially two ingredients. The first is the definition of a unit (reference) scale (all the vectors defined by the frontier of the ball B_1), while the second is a family of scale changing operators T_λ which describe how the unit scale is blown up or down. The fundamental restriction is that T_λ should only involve the scale ratio λ so that there is no absolute notion of size i.e. that T_λ is a group¹⁵ with generator G : $T_\lambda = \lambda^{-G}$.

4.4 Symmetries and the relation between stochastic and deterministic chaos:

We have argued that even for (an apparently) mathematically well defined deterministic problem such as hydrodynamic turbulence, that the basic obstacle is an adequate treatment

of the scale invariance symmetry: the “puffs within puffs” of turbulent activity. On the other hand, with practically no ingredients beyond this symmetry, stochastic cascades give immediate insights: universal multifractals in which all the statistics are characterized by only three fundamental exponents. In physics, due to the intimate connection between symmetries and dynamical equations (Noether’s theorem, ([Noether, 1918]) it is generally accepted, that symmetries are synonymous with dynamics. Applied to the turbulent cascade approach this suggests that the latter would be equivalent to the usual deterministic approach if the remaining symmetries (i.e. other than scale invariance) of the Navier Stokes equations were known. We could then - at least in principle - use them to restrict the cascades in the appropriate way - presumably for example - to determine the remaining two universal multifractal parameters.

Steps in this direction have been made using “shell-models” (e.g. [Gledzer, 1973]) as a kind of compromise between cascades and deterministic chaos and as tools for exploring intermittency in fully developed turbulence. They can be introduced as caricatures of the Navier-Stokes equations via dynamical systems with limited numbers of degrees of freedom: at each scale corresponds an ordinary differential equation with a few (quadratic) interactions with neighbouring scales. The main drawback is that it remains a bad compromise: the crucial spatial dimensionality is merely lost, along with the algebraic discretization of scales, the number of (possible) degrees of freedom grows only logarithmically with the Reynolds number!

However, these deficiencies can be avoided in the framework of more consistent caricatures of Naviers Stokes yielding space-time deterministic cascades [Grossmann and Lohse, 1993]: the number of eddies grow algebraically with scale, as well as the number of corresponding equations of evolution. By keeping only a (well-defined) subset of triad of wave-vectors and using a direct analogy [Arnold, 1966], [Obukhov and Dolzhansky, 1975] between Navier-Stokes equations and Euler equation of the gyroscope, we obtain the Scaling Gyroscopes Cascade (SGC). [Chigirinskaya et al., 1996] shows that while the full SGC behaves very closely to observations of fully developed turbulence. The SGC (which - except for its stochastic initial conditions - is deterministic) would therefore seem to be an interesting model in between the usual the cascades and the deterministic Navier-Stokes equations.

While providing a solid mathematical bridge between the classical and cascade approaches in turbulence would be a major scientific achievement, in our view it would be wrong to be overly obsessed with this rapprochement. The reason is that very few natural systems correspond to pure hydrodynamic turbulence (i.e. exactly satisfying Navier Stokes equations). Even the presence of gravity in a fluid with density variations (i.e. buoyancy effects) takes us beyond this theoretical case; models already involve unsatisfactory (e.g. “Boussinesq”) approximations¹⁶. In other words, any empirical test of the Navier-Stokes equations must therefore rely (at least implicitly) on the existence of universal properties! We have already seen that on the contrary, stochastic chaos when coupled with GSI can very easily deal with gravity and much more complex anisotropy which are outside the scope of the usual approaches. Physically, the use of GSI in this way implies that the very notion of scale/size is not imposed from without, but is rather determined by the underlying nonlinear dynamics itself. This connection between the notion of scale and dynamics is analogous to that of General Relativity between the distribution of matter and energy and the metric (except that in GSI, size is not necessarily a metric concept).

4.5 Interacting scaling fields: Lie cascades.

In addition to anisotropy, real turbulent systems (e.g. the atmosphere, oceans) involve many nonlinearly interacting fields, and the true equations are not known. The traditional

deterministic approach involves modelling the interactions with nonlinear partial differential equations, and then discretizing the latter on large grids which are then solved numerically. Due to limitations in computer capacity (both speed and memory), only rather narrow ranges of scale (of the order of a factor of 10^2) are directly accessible (compared to the actual range of roughly 10^{10} in the atmosphere). In global models, this means that no structures smaller than about 1000km can be directly taken into account¹⁷. The attempts to model the subgrid interactions are called "parametrisations" and are plagued with problems¹⁸. In the cascade approach, one directly takes into account arbitrarily large ranges of scale; what is less straightforward is to properly account for the interactions between different fields. This can be done by introducing a "state vector" at each space-time point; this specifies the overall state of the system. Scale invariant vectors (the result of vector cascades) are called "Lie cascades" ([Schertzer and Lovejoy, 1995]); their study is only just beginning.

5. Space-time multifractal processes and stochastic prediction

5.1 Causality.

Up until now, the cascades have been fundamentally deficient as physical models since they have been static. In order to produce a dynamical model (i.e. one evolving in time), we first note that a general property of dynamical geophysical processes is that larger structures evolve more slowly than smaller ones; there exists a statistical relationship between the spatial extent of fluctuations (eddies, size) and their duration (e.g. in turbulence, the "eddy turnover time"). In scale invariant systems we anticipate that the velocity relating the two is also scaling; i.e. that the overall process is an (anisotropic) space-time multifractal process which can be handled using GSI. However, space-time scaling is not enough; time is not simply a redimensioned, rescaled spatial coordinate. The model must also satisfy the condition of "causal antecedence", otherwise it would violate the (stochastic) causality discussed above because effects could precede their causes. The resulting (causal) space-time scaling, multifractal process can be considered as a (nonclassical, fractional, nongaussian) diffusion process for the generators, see [Marsan *et al.*, 1996], [Schertzer *et al.*, 1997] for theoretical and numerical details and examples.

5.2 Limits to predictability and stochastic forecasting.

Let us now consider the problem of predicting deterministic and multifractal (stochastic) chaos. There are two related problems. First, the theoretical limits to predictability; how far ahead we could *in principle* predict with essentially perfect information. The second, is to find the optimum forecasting technique given a certain quantity of initial (and - for stochastic forecasts - past¹⁹) information on the state of the system.

To compare and contrast the predictability of the two types of system, consider a system whose energy flux at resolution λ ($\varepsilon_\lambda(t)$) is known at time t . In deterministic chaos, we assume that there will be a tiny error (dispersion) in this value leading to (an exponentially growing) dispersion in the (deterministic) trajectories in phase space. In the stochastic chaos system, we assume perfect knowledge at time t ; the dispersion in the values of $\varepsilon_\lambda(t+\Delta t)$ arises because of the intrinsic stochastic nature of the system, not due to measurement errors. In order to quantify the predictability, first recall that each singularity (intensity) determines a unique order of statistical moment (q). Since we anticipate that the intense phenomena will be less predictable than the weak, we can quantify this by using the root mean square (RMS) error of the conditional q^{th} order statistical moment (conditioned on the measurement $\varepsilon_\lambda(t)$) as:

$$E_q^2(\Delta t) = \frac{\langle \varepsilon_\lambda^{2q}(t + \Delta t) \rangle_{\varepsilon_\lambda(t)} - 1}{\langle \varepsilon_\lambda^q(t + \Delta t) \rangle_{\varepsilon_\lambda(t)}^2} - 1 \quad (6)$$

In comparison, deterministic chaos systems are essentially defined by their predictability properties: the fact that errors grow exponentially fast (the “sensitive dependence” on initial conditions); the growth rate is the “Lyapunov exponent” τ_i .

$$E_q \approx e^{\Delta t / \tau_i} \quad (7)$$

Predictability is severely limited, (essentially to the reciprocal of the Lyapunov exponent), but is not necessarily a function of the *intensities* of phenomena²⁰. If we require an average error over many different initial conditions (corresponding in the stochastic case to ensemble averages), then the limits to predictability will be a corresponding average (inverse) Lyapunov exponent (note that this exponential growth only occurs for fairly small time intervals; eventually, the error “saturates” at a large value).

In the multifractals, the result will depend on the effective²¹ temporal resolution of the series²²; we consider forecasting a series at space-time resolution τ . We have:

$$E_q(\Delta t) \approx \left(\frac{\Delta t}{\tau} \right)^{s(q)} \quad (8)$$

where $s(q)$ is a generally convex exponent (e.g. in the simplest model, the “log-normal” multifractal, $s(q) = C_1 q^2$). Two fundamental differences can be noted²³. The first is that rather than exponential increase in error, due to the scale invariance, one obtains a power law increase²⁴. The other is the fact that intense events (which correspond to large q) are much less predictable.

6. Conclusions

We have argued that in the past twenty years deterministic chaos has become nearly synonymous with nonlinear models of systems with few degrees of freedom. In contrast, many (most?) applications of nonlinear dynamics involve large numbers of degrees of freedom and can be simply and naturally modelled by “stochastic chaos” i.e. objective random models involving probability spaces (an infinite number of degrees of freedom). We criticized dogmatic attempts to exclusively use deterministic models; both stochastic and deterministic chaos models should be parts of scientists’ toolbox of models.

To date, the primary utility of stochastic chaos is the facility with which it enables us to exploit a nonclassical symmetry: scale invariance. We argued that scale invariance is much richer than is usually supposed providing for example, a potentially unifying paradigm for geophysics²⁵. Probably the most familiar examples of scale invariant objects are geometric fractal sets, however mathematically and physically, fields such as the temperature, wind, cloud brightness/density etc. are much more interesting; these are multifractals, which are generically produced in cascade processes. Such cascades involve a dynamical generator which repeats scale after scale from large to small scale structures. In this way, it builds up tremendous (nonclassical) variability associated with Self-Organized Criticality. Contrary to conventional structureless (“white”) noise (randomness) the variability is so strong that there is insufficient “self-averaging” the resulting fields display intense “singularities” whose origin is purely random. Another insufficiently appreciated aspect of scale invariance

is that it need not be self-similar (isotropic, the same in all directions). On the contrary, physical systems always involve preferred (possibly scale dependent) directions (especially gravity) leading to differential stratification and rotation of structures with scales; such anisotropic scale invariance requires the formalism of Generalized Scale Invariance (GSI) which defines new (anisotropic) ways of “zooming”/“blowing up” structures. In GSI the system’s dynamics determines the notion of size, the latter is not imposed from without in an ad hoc way.

Stochastic chaos combined with the scale invariance symmetry (“multifractals”) may allow us to take the chaos revolution a step forward by finally bringing large numbers of degrees of freedom systems into its purview. It already is making rapid progress in overcoming longstanding basic problems including the nature of turbulence, the weather and climate.

Notes

¹Fractals also arise in deterministic chaos; strange attractors are fractal sets which are defined in abstract phase spaces. The observed (real space) trajectories are typically much lower dimensional subspaces and are often nonfractal (smooth). In the stochastic chaos systems discussed here, on the contrary, it is rather the real space structures which are fractal, the corresponding phase spaces are nonfractal.

²See also [*Fraedrich*, 1986] .

³Since 1988, the European Geophysical Society’s regular sessions on Nonlinear processes played an important role in catalyzing much of this discussion. In particular the provocative issue of “Chaos versus Stochasticity in Geophysical Sciences” was the theme of three lively sessions (1993 - 1995) (organized by the authors and A.R. Osborne, A. D. Kirwin, S. B. Hooker).

⁴Below, we show how this fraction can be quantified (scale by scale, intensity by intensity) with the codimension function; in the atmosphere, this depends on the intensity of the phenomena, but the effective number of degrees of freedom is still typically enormous.

⁵See especially the semi-analytical closure Eddy Damped Quasi Normal approximation or the rather involved Lagrangian History Direct Interaction approximation; for review see [*Lesieur*, 1987].

⁶Note that “calm” (gaussian) stochastic models have been used in turbulence since the 1960’s essentially as technical adjuncts to RNG type approaches; see [*Herring*, 1997] for a discussion.

⁷The study of multiplicative random processes (at first reduced to the product of random variables) has a long history (see [*Aitchison and Brown*, 1957]), going back to at least [*McAlsister*, 1879] who argued that multiplicative combinations of elementary errors would lead to lognormal distributions. This law has almost invariably been used to justify the use of lognormal distributions i.e. it was tacitly assumed that the lognormal was a universal attractor for multiplicative processes.

⁸Since c is generally unbounded, events with $c > D$ will be produced by the process. However, if we attempt a geometric interpretation we see that it will involve impossible negative (“latent”) fractal dimensions; hence the latter is quite restrictive. Aside from avoiding

this “latent dimension paradox”, the codimension formalism has the further advantage of being independent of D , the dimension of the observing space.

⁹Most observables (such as the velocity shear Δv , in turbulence) are not the direct outcome of cascade processes; they are not conserved scale by scale the way the energy flux is conserved in turbulence; they involve a third parameter (H) characterizing the degree of nonconservation e.g. the Kolmogorov scaling gives: $\Delta v_\lambda \approx \varepsilon_\lambda^{1/3} \lambda^{-H}$ with $H=1/3$.

¹⁰Note that the term “lognormal” is not accurate since the divergence of moments leads to “multifractal phase transitions” i.e. to deviations from eq. 2 for $\gamma > \gamma_D$ where γ_D is a critical order of singularity; see below.

¹¹Most of these do indeed appear to be universal multifractals - at least to within the accuracy with which they have been estimated.

¹²In spite of this, most definitions of multifractals implicitly assume precisely such convergence, being based on point singularities; “Holder” exponents.

¹³Specifically, $\langle \Pi_\infty(B_1)^q \rangle = \infty$; $q \geq q_D$ where q_D is a critical order of moments which depends on the dimension of the integration set B_λ ; this implies $\Pr(\Pi_\infty(B_1) > s) \approx s^{-q_D}$; $s \gg 1$.

¹⁴[Schmitt *et al.*, 1994] found $q_D \approx 7$ for the horizontal component of the wind, but the size of the sample used (ten million measurements) was only barely enough to quantify the effect.

¹⁵Actually, T_λ need only be a semi group since inverse operations need not be defined.

¹⁶Indeed while the observed scaling stratification of the atmosphere is almost trivial to handle with GSI (G need only be a diagonal matrix not proportional to the identity), it is not obvious that the observed anisotropic scaling is compatible with any known deterministic dynamical equations!

¹⁷The nominal resolution of global weather models is somewhat better than this, but due to the artificial dissipation mechanisms (e.g. “hyperviscosity”), a factor of roughly four is lost.

¹⁸Indeed, if the recent empirical estimates of low, finite q_D for various atmospheric fields including the wind, temperature and rain fields are correct, then the multifractal butterfly effect is indeed operative and shows that consistent deterministic parametrisations of the small scale activity cannot be achieved even in *principle*.

¹⁹The basic idea of stochastic forecasting is precisely to exploit the stochastic “memory” of the system (i.e. correlations/interrelations between past and future). The most trivial example of a stochastic forecast is persistence; e.g. that tomorrow’s weather will be the same as today’s.

²⁰However, it does depend on the *trajectory*, and different intensities will be associated with different trajectories. Here we ignore this possible dependence of τ_i on q .

²¹For spatially averaged data, τ is the typical lifetime of the structures which are averaged out. If there is only temporal averaging ("point" spatial measurements), then τ is the averaging time/resolution.

²²In the atmosphere, this appears to be valid up to the "synoptic maximum" which is the time scale of planetary sized structures; roughly two weeks.

²³In turbulence closures, scaling errors of the form of eq. 8 are obtained, but due to their inability to handle intermittency, $s(q)$ is independent on q ; strong and weak events are equally predictable.

²⁴Since in the appropriate limit, power laws can be considered exponentials with zero Lyapunov exponents, this scaling could be considered the "edge of chaos".

²⁵Perhaps also for astrophysics where notably the large scale structure of the universe appears to be multifractal, see [Coleman and Pietronero, 1992], [Garrido et al., 1996; Labini and Pietronero, 1996], [Lovejoy et al., 1998].

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