Constraints on Higher Spin CFT_2

Kale Colville

McGill University Hep-th Journal Club

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Overview

Introduction

Motivation   Virasoro $\mathcal{W}$ algebras

Constraints from Unitarity by Explicit Calculation

Structure of Null States

Interpretation in AdS/CFT

Motivation

- Goal: Completely classify the space of unitary Conformal Field Theories in two dimensions
- In two dimensions the algebra of local conformal symmetries is infinite dimensional
- Rational theories with $c < 1$ have been classified
- Irrational theories with $c > 1$ still open work
- This talk: what happens if we add higher spin symmetries generated by currents of spin $s > 2$

Current Standing of Higher Spin CFTs

- For $d > 2$ the constraints on higher spin symmetry are quite powerful
- At $d = 3$, a theory with a conserved current of spin $s > 2$ must have an infinite tower of higher spin currents [Maldacena and Zhiboedov 1112.1016]
- Result has been extended for $d > 3$ [Boulanger et al. 1305.5180, Alba and Diab 1510.02535]
- In $d = 2$ adding higher spin currents give a $\mathcal{W}$ algebra

Virasoro

- The symmetry algebra of a two dimensional conformal field theory is the Virasoro algebra
- The algebra is given by

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0}$$

- Energy eigenstates (eigenstates of L_0) will fall into representations of Virasoro

Representations of Virasoro

- Construct highest weight representation of Virasoro algebra, analogous to $\mathfrak{su}(2)$ in QM
- Have a highest weight vector, $|h\rangle$, which is eigenvector of L_0
- L_n $(n > 0)$ act as lowering operators, $[L_0, L_n] = -nL_n$

$$L_0 |h\rangle = h |h\rangle , \qquad L_n |h\rangle = 0, n > 0$$

- Other states are obtain by acting with L_{-n}, $n > 0$
- Take as basis

$$\{L_{-k_1} L_{-k_2} ... L_{-k_n} |h\rangle\}, \qquad 1 \leq k_1 \leq k_2 \leq ... \leq k_n$$

- State has level N if it's L_0 eigenvalue is $h + N$
- E.g. $L_{-1}L_{-2} |h\rangle$ has level 3, in general $N = \sum_i k_i$

$\mathcal{W}$ Algebras

- Extend Virasoro algebra with additional higher spin primary fields
- Expand the primary fields in terms of modes

$$W(z) = \sum_k W_k z^{-k-h}$$

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0}$$

$$[L_m, W_n] = \big((h-1)m - n\big)W_{m+n}$$

$$[W_m, W_n] = \dots$$

- Demand the algebra closes with the specified fields

Example $\mathcal{W}_3$

- Add a single spin 3 current, which we will denote with W

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m+n,0}$$

$$[L_m, W_n] = \big(2m-n\big)W_{m+n}$$

$$[W_m, W_n] = p(m,n)L_{m+n} + q(m,n)C_{WW}^{\Lambda}\Lambda_{m+n} + \frac{c}{360}m(m^2-1)(m^2-4)\delta_{m+n,0}$$

$$\Lambda_n = \sum_{p\leq -2} L_p L_{n-p} + \sum_{p\geq -1} L_{n-p}L_p - \frac{3}{10}(n+2)(n+3)L_n$$

$\mathcal{W}$ Algebras

- Notation: A $\mathcal{W}$ algebra with primaries of spin $s_1, ..., s_n$ is denoted $\mathcal{W}(2, s_1, ..., s_n)$
- Specifying spins is not enough to uniquely determine the algebra
  - May not exist or may only be valid for certain values of central charge
  - May be more than one algebra with the same set of primaries
- We will focus on $\mathcal{W}_N$ algebras which correspond to $\mathcal{W}(2, 3, ..., N)$
- For $\mathcal{W}_N$ there is a unique algebra valid for generic values of c
- Analogue of minimal models for $\mathcal{W}_N$ have $c < N - 1$
- We will be concerned with the irrational regime, $c > N - 1$

Representation of $\mathcal{W}_N$ Algebras

- Works largely the same way as that of Virasoro
- Now have $N-1$ eigenvalues to describe the primary states

$$|h, q_3, ..., q_N\rangle$$

$$L_0 |h, q_3, ..., q_N\rangle = h |h, q_3, ..., q_N\rangle$$

$$W_0^s = q_s |h, q_3, ..., q_N\rangle \,, \quad s = 3, 4, ..., N$$

- States annihilated by lowering operators of all fields

$$L_k |h, q_3, ..., q_N\rangle = 0 = W_k^s |h, q_3, ..., q_N\rangle \qquad k > 0$$

Example $\mathcal{W}_3$

- Once again denote the spin 3 field as W
- The highest weight representation is given by $|h, q_3\rangle$ where

$$L_0 |h, q_3\rangle = h |h, q_3\rangle , \qquad W_0 |h, q_3\rangle = q_3 |h, q_3\rangle ,$$
$$L_k |h, q_3\rangle = 0 = W_k |h, q_3\rangle , \qquad k > 0$$

- Basis of states

$$\{L_{-k_1} ... L_{-k_n} W_{-\ell_1} ... W_{-\ell_m} |h, q_3\rangle\}$$
$$1 \leq k_1 \leq ... \leq k_n, \qquad 1 \leq \ell_1 \leq ... \leq \ell_m$$

- Once again the level is given by the eigenvalue of L_0 and is given by

$$N = \sum_i k_i + \sum_j \ell_j$$

The general procedure

- Now want to use unitarity to constrain the representations
- Given a basis of states $|i\rangle$ we construct the matrix

$$M_{ij} = \langle i|j\rangle$$

- The norm of any state is expressible in terms of this matrix,

$$|X\rangle = \sum_i X_i\,|i\rangle \quad \Rightarrow \quad \langle X|X\rangle = \sum_{i,j} X_i^* M_{ij} X_j$$

- The matrix M is Hermitian and can be diagonalized

$$\langle X|X\rangle = \sum_i \lambda_i |Y_i|^2$$

- To exclude negative norm states we must then have all eigenvalues of M to be non-negative

Virasoro

- Use highest weight representation
- The matrix M is known as the Kac matrix
- States with different level are orthogonal
- At level 1 only have one state: $L_{-1}|h\rangle$

$$\langle h| L_1 L_{-1} |h\rangle = 2h$$

- State is unitary for $h \geq 0$
- No new constraints at higher level for $c \geq 1$
- For $0 < c < 1$ higher levels do give constraints and leads to the classification of minimal models

$\mathcal{W}_3$

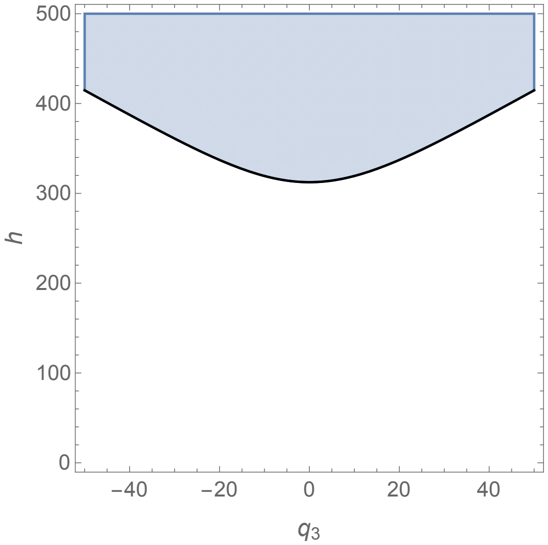
- Use highest weight representation
- States with different level are still orthogonal
- Now have two states at level 1:

$$|1\rangle \equiv L_{-1}\,|h, q_3\rangle\,, \qquad\qquad |2\rangle \equiv W_{-1}\,|h, q_3\rangle$$

- Result:

$$M = \begin{pmatrix} \langle 1|1\rangle & \langle 1|2\rangle \\ \langle 2|1\rangle & \langle 2|2\rangle \end{pmatrix} = \begin{pmatrix} 2h & 3q_3 \\ 3q_3 & \frac{h(2-c+32h)}{22+5c} \end{pmatrix}$$

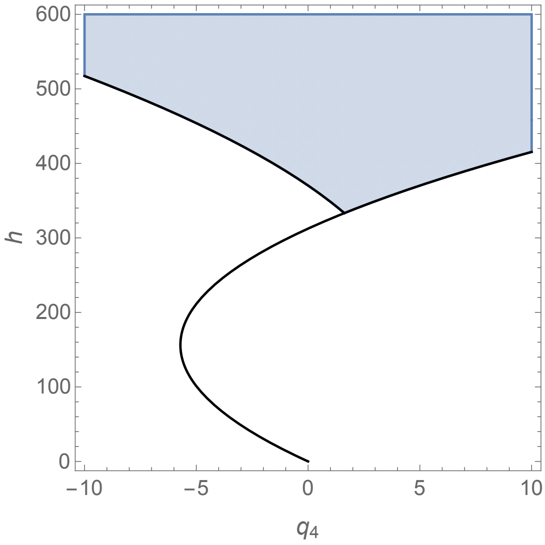
$\mathcal{W}_3$



- Figure is exclusion plot at $c = 10000$ (shaded region allowed)
- Lower bound on h as a function of q_3 (and c)
- Overall lower bound

$$h \geq \frac{1}{32}(c-2)$$

$\mathcal{W}_4$



- At fixed central charge have three parameters: h, q_3, q_4
- Figure is exclusion plot at $c = 10000$ and $q_3 = 0$
- Overall lower bound on positive norm states

$$h > \frac{1}{30}(c-3)$$

- Have null states that exist below this bound

Summary of Explicit Results

- Possible to continue the procedure for $N = 5, 6$ as well

$$\begin{aligned} \mathcal{W}_2 : &\qquad h > 0 \\ \mathcal{W}_3 : &\qquad h > \frac{1}{32}(c-2) \\ \mathcal{W}_4 : &\qquad h > \frac{1}{30}(c-3) \\ \mathcal{W}_5 : &\qquad h > \frac{3}{80}(c-4) \\ \mathcal{W}_6 : &\qquad h > \frac{4}{105}(c-5) \end{aligned}$$

- Bound is always of the form

$$h > \#\big(c - (N-1)\big)$$

Structure of Null States

- Want to figure out how this bound works for general N
- Algebra becomes more complicated as N is increased
- N ranges over all integers ≥ 2
- Idea: use the fact that boundary of the regions we are looking at correspond to intersections of null states

Structure of Null States

- Much like Virasoro, the determinant of the Kac matrix is known level by level for $\mathcal{W}_N$ algebras
- Determinant is expressed in terms of $N-1$ parameters L_i
- Charges (including h) are polynomial in the L_i
- The determinant vanishes for null states
- The regions of positive norm are bounded by null states
- The bounds we have found occur at intersections of null states

Example $\mathcal{W}_3$

- The level one determinant is expressed in terms of two parameters L_1, L_2:

$$M^{(1)} \propto L_1 L_2 (6L_1 - \sqrt{6(c-2)})(6L_2 - \sqrt{6(c-2)}) \times (4L_1 + 4L_2 - \sqrt{6(c-2)})(12L_1 + 12L_2 - \sqrt{6(c-2)})$$

$$h = \text{Second order polynomial}(L_1, L_2)$$

$$q_3 = \text{Third order polynomial}(L_1, L_2)$$

- Fix L_2 such that $M^{(1)}$ vanishes, then extremize h with respect to L_1

$$h_{\text{crit}} = \frac{1}{32}(c-2)$$

General $\mathcal{W}_N$

- For $\mathcal{W}_N$ everything now depends on $L_1, L_2, ..., L_{N-1}$
- Set some combination of L_i to values that make the determinant vanish, then maximize over the remaining L's
- When matching with the explicit results already obtained, a pattern emerges for which L_i to fix
- Conjecture that the pattern holds for higher values of N
- Result differs for even and odd N but is expressible as

$$h \geq \frac{c-(N-1)}{24}\left(1-\frac{6}{N(N^2-1)}\left\lfloor\frac{N}{2}\right\rfloor\right)$$

Spectrum of CFTs with Gravitational Duals

- For a holographic CFT with semiclassical dual we need to take $c \to \infty$ with N fixed
- In a holographic dual h is related to the mass of the corresponding state
- Heavy states have h which scales with the central charge

$$\begin{aligned} &\text{Black Holes} && h \geq \frac{c}{24} \\ &\text{Other Heavy Particles} && h = \alpha c, \quad \alpha < \frac{1}{24} \end{aligned}$$

- Light states

$$h \text{ finite as } c \to \infty$$

Implication of the Results

- For $\mathcal{W}_N$ the constraint is $h \geq \#(c - N + 1)$
- Linearity in c implies that there can be no light states in a holographic CFT
- Can only be dual to pure theories of gravity
- Agrees with other analysis [1602.08272 Perlmutter]

Other Constraints

- Can look at higher level Kac matrix
- Charged modular bootstrap

$$\mathrm{Tr}\left(W_0^2 q^{L_0-c/24}\bar{q}^{\bar{L}_0-\bar{c}/24}\right)$$

- Explicitly done for $\mathcal{W}_3$, yields new constraints but no contradictions

Conclusion

- Examined positivity of Kac matrix to derive constraints on spectrum of $\mathcal{W}_N$ theories
- States satisfy

$$h \geq \frac{c-(N-1)}{24}\left(1 - \frac{6}{N(N^2-1)}\left\lfloor \frac{N}{2} \right\rfloor\right)$$

- Holographically these $\mathcal{W}_N$ theories are dual to pure theories of gravity