Our Evolving Universe
Assignemnt 6 Solutions

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I'll present ***brief*** solutions to the wonderful questions for "Our Evolving Universe".

Problem 1

Suppose we observe no pulses of radiation from a neutron star. Is it possible that a civilization in some other star system would see this neutron star as a pulsar?

Sol: Yes, it is possible! Pulsars emit their light in a cone-shaped region, and so it is possible that the emission from the cone may miss earth, but another civilization could be in the emission region.

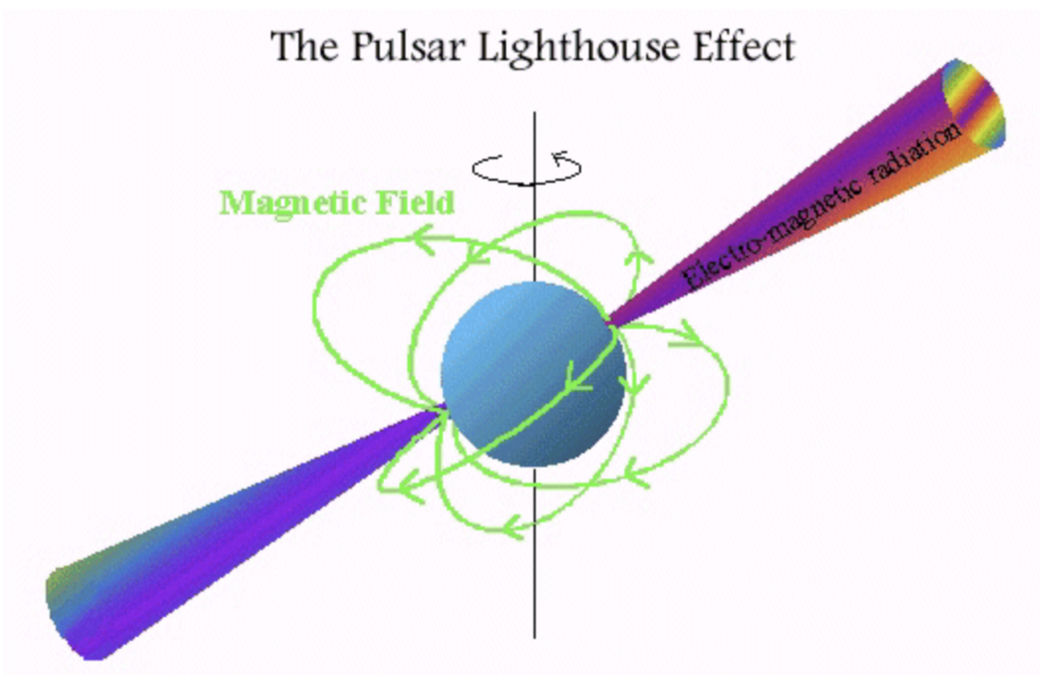

Figure 1: Source: http://hosting.astro.cornell.edu/academics/courses/astro201/pulsar.htm

Alternatively, if the pulsar was very far away, its radiation could be so far redshifted that we could not detect it, while a civilization close by would have an easier time detecting the higher energy photons.

Problem 2

Why is the joint detection of the merging neutron star binary GW170817 in gravitational wave and optical channels a very important event for modern astronomy?

Sol: There are numerous reasons! A few relevant ones in the context of cosmology are

- Allowed for a detection of electromagnetic radiation from the event, as well as gravitational waves. In black hole binary systems, there is no electromagnetic radiation released, and so this is what is known as a new age of "multi-messenger astronomy"
- Sets a strong constraint on the speed of propagation of gravitational waves. They are very likely moving at the speed of light
- Allows for the localization on the sky of the merger event. Gravitational waves on their own do not provide much information as to where the merger took place on the sky, but with additional observations of electromagnetic radiation we can determine exactly where the event took place
- Allows for a fairly unbiased determination of H_0, the number which sets the age and the expansion rate of the universe

Problem 3

Is the radius of the event horizon of the black hole resulting from the merger of a black hole binary system larger or smaller than the sum of the event horizon radii of the two original black holes? Explain.

Sol: The radius of the event horizon of a black hole (the Schwarzschild radius) is

$$r_s = \frac{2GM}{c^2}$$

and so if we have two black holes merging, say with masses m_1 and m_2, then the mass of the new black hole is just the sum of the two masses, say $m_f = m_1 + m_2$ where m_f stands for the final objects mass. This implies that the final objects Schwarzschild radius is

$$\begin{aligned} r_f &= \frac{2Gm_f}{c^2} \\ &= \frac{2G(m_1+m_2)}{c^2} \\ &= \frac{2Gm_1}{c^2} + \frac{2Gm_2}{c^2} \\ r_f &= r_1 + r_2 \end{aligned}$$

So the event horizon of the final object will just be the sum of the event horizons of the individual black holes before the merger

Problem 4

Did the very first high-mass stars in the history of the universe produce energy through the CNO cycle? Explain.

Sol: No, the first high mass stars will not produce energy through the CNO cycle. This is because the CNO cycle requires carbon, nitrogen, and oxygen to already be present as they act as catalysts in the fusion process of a star.

In order for carbon, nitrogen, and oxygen to be present, there must have been stars who have gone up the fusion chain and produced these elements in their cores. After producing these elements, these stars must have gone supernova in order to spread these heavier elements to other star forming regions. If we are talking about the very first high mass stars, there simply hasn't been time to populate these star forming regions with CNO elements yet, and so the CNO cycle cannot take place until a later time.

Problem 5

A typical white dwarf has a mass of about that of the sun, and the radius of the earth (about 6400km). Calculate the density. How does it compare with the density of familiar objects?

Sol: The parameters we are given are

$$m = 2 \cdot 10^{30}\ \text{kg} \qquad\qquad \text{r} = 6.4 \cdot 10^{6}\ \text{m}$$

The volume of a sphere is given by

$$\begin{aligned} V &= \frac{4}{3}\pi r^3 \\ &= \frac{4}{3}\pi (6.4 \cdot 10^6\ \text{m})^3 \\ V &= 10^{21}\ \text{m}^3 \end{aligned}$$

The density of an object is just the mass divided by the volume

$$\begin{aligned} \rho &= \frac{m}{V} \\ &= \frac{2 \cdot 10^{30}\ \text{kg}}{10^{21}\ \text{m}^3} \\ &= 2 \cdot 10^{9}\ \text{kg/m}^3 \end{aligned}$$

This is very dense! For reference, the density of water is $\rho_w = 997\ \text{kg/m}^3$, and the average density of the earth is $\rho_e = 5 \cdot 10^3\ \text{kg/m}^3$. A white dwarf is about a million times as dense as the earth!

Problem 6

Use the parallax formula to calculate the distance to each of the following stars (in light years): a) Alpha Centauri, $p = 0.74"$**; b) Procyon:** $p = 0.286"$

Parallax (short for parsec-arcsecond) has a very simple formula

$$d \approx \frac{1}{p}$$

This formula works when the parallax is given in arcseconds, and returns a value of d with the units of parsecs.

a) For Alpha Centauri, the distance is

$$d = \frac{1}{0.74"} = 1.35 \text{ pc}$$

Now recall that 1 parsec is 3.26 light years, to find the distance we ant

$$d = 1.35 \text{ pc} \, \frac{3.26 \text{ ly}}{1 \text{ pc}} = 4.40 \text{ ly}$$

b) This computation for Procyon is

$$\begin{aligned} d &= \frac{1}{0.286} \\ &= 3.50 \text{ pc} \\ &= 11.4 \text{ ly} \end{aligned}$$

Problem 7

The spectral lines of two stars in a particular eclipsing binary system shift back and forth with a period of 6 months. The lines of both stars shift by the same amount, and the Doppler shift indicates that each star has an orbital speed of 80,000 m/s. What are the masses of the two stars (assuming circular orbits about their centre of mass).

Sol: First of all, a 6 month shift in the lines means that the orbital period of the system is 6 months. Since we also have the speed of the stars, we can use this to compute the circumference (and the radius) of this orbit. The circumference of the circular orbit is given by the kinematic formula $d = v \cdot t$

$$v = 80,000 \text{ m/s} \qquad \text{t} = 6 \text{ months} = 1.58 \cdot 10^7 \text{ s}$$

Using these, the circumference is

$$d = v \cdot t = 1.26 \cdot 10^{12} \text{ m}$$

The circumference is related to the radius by $d = 2\pi r$ so the radius is

$$r = 2.01 \cdot 10^{11} \text{ m}$$

Then, by equating the gravitational force to that of circular motion, we can find the mass of the stars (assuming they are of equal mass)

$$\begin{aligned} F_c &= F_g \\ \frac{m_1 v^2}{r} &= \frac{G m_1 m_2}{r^2} \\ v^2 &= \frac{G m_2}{r} \\ m_2 &= \frac{v^2 \cdot r}{G} \\ &= 1.9 \cdot 10^{31} \text{ kg} \end{aligned}$$

ALTERNATIVELY we may use Kepler's laws to get this answer in a simpler way. The useful formula here is

$$\begin{aligned} T^2 &= \frac{4\pi^2}{G(m_1 + m_2)} a^3 \\ m_1 + m_2 &= \frac{4\pi^2}{G T^2} a^3 \end{aligned}$$

Where T is the period of the orbit, and a is the distance between the objects and the centre of mass of the system (r in what we derived above). Plugging in the numbers gives

$$m_1 + m_2 = 1.92 \cdot 10^{31} \text{ kg}$$

This would imply that if the masses are equal, $m_1 = m_2 = 9.63 \cdot 10^{30}$ kg. This is a factor of 2 different from our previous computation using forces, but I'll leave you to ponder where that came from.

Problem 8

Sirius A has a luminosity of 26$L_\odot$, where $L_\odot$ is the absolute luminosity of the sun, and a surface temperature of about 9400K. What is the radius?

Sol: The luminosity of an object is given by

$$L = 4\pi R^2 \sigma_{SB} T^4$$

Where σ_{SB} is the Stefan-Boltzmann constant that is unimportant for our current calculation. What is important is that there is a direct correlation between the luminosity of an object and its radius (as you would expect, a larger star has more surface area to radiate photons). Lets put this into ratios. The ratio of the luminosity of Sirius A (L_S) to that of the sun ($L_\odot$) is

$$\begin{aligned}\frac{L_S}{L_\odot} &= \frac{4\pi R_S^2}{4\pi R_\odot^2} \cdot \frac{\sigma_{SB}}{\sigma_{SB}} \cdot \frac{T_s^4}{T_\odot^4} \\ &= \frac{R_s^2}{R_\odot^2} \cdot \frac{T_S^4}{T_\odot^4}\end{aligned}$$

Rearranging this, we can solve for the radius of Sirius A

$$R_S = \left(\frac{L_S}{L_\odot}\right)^{1/2} \cdot \left(\frac{T_\odot}{T_S}\right)^2 \cdot R_\odot$$

So by looking up standard sun values we find

$$T_\odot = 5778 \text{ K} \qquad \mathrm{R}_\odot = 6.96 \cdot 10^8 \text{ m} \qquad \mathrm{L_S} = 26\mathrm{L}_\odot \qquad \mathrm{T_S} = 9400 \text{ K}$$

This gives us the radius

$$R_S = 1.34 \cdot 10^9 \text{ m}$$

Problem 9

In what ways are brown dwarfs similar to Jupiter-like planets? In what ways are they like stars?

Sol: Brown dwarfs are similar to Jupiter-like planets as they generally have masses similar to that of Jupiter (or 10-100 times the mass of Jupiter). Because of their low mass, most of these brown dwarves are unable to undergo nuclear fusion in their core. This is similar to planets in general, as the density and temperatures are not near high enough to start fusion (thankfully!)

They are similar to stars in their formation origin. Like other stars, they form via gravitational collapse in star-forming regions. This means that overdensities in the matter distribution collapse into a clump and form an object. In this case, the overdensities are relatively small, and so fusion is not started. Planets are not thought to have formed this way. Some leading theories on planet formation have them forming through the collision of many smaller asteroids over a very long time-frame.

Problem 10

Small black holes have a higher mass density than that of larger black holes (assume here that the mass is distributed uniformly within the horizon). How large does the black hole have to be in order for the density to equal that of regular water?

Sol: Recall first that the density of water is

$$\rho_w = 997 \text{ kg/m}^3$$

Recall also the density formula, and the expression for the Schwarzschild radius (event horizon) of a black hole

$$\rho = \frac{M}{V} = \frac{M}{\frac{4}{3}\pi r_s^3} = \frac{3M}{4\pi r_s^3} \qquad\qquad r_s = \frac{2GM}{c^2}$$

For simplicity, we will keep everything in standard SI units. Inputting the value of r_s into the expression for the density yields

$$\rho = \frac{3c^6}{32\pi G^3 M^2}$$

Now we are looking for how large the black hole has to be to give such a density. First, lets compute the mass of such a black hole with $\rho = 997 \text{ kg/m}^3$. We begin by rearranging the above formula for the mass M

$$M = \sqrt{\frac{3c^6}{32\pi G^3 \rho}}$$

Recalling that the constants are

$$c = 3 \cdot 10^8 \text{ m/s} \qquad\qquad \text{G} = 6.67 \cdot 10^{-11} \text{ m}^3 \text{ s}^{-2} \text{ kg}^{-1}$$

Gives us

$$M = 2.7 \cdot 10^{38} \text{ kg}$$

Which is about 100 million solar masses! Quite large indeed. The radius of the event horizon of such a black hole can be found be plugging back into the other formula

$$\begin{aligned} r_s &= \frac{2GM}{c^2} \\ &= 4.02 \cdot 10^{11} \text{ m} \end{aligned}$$

This is about the distance from us to the sun.