4. Oscillations

4.1 Linear Differential Equations

Ex: $m\ddot{x} - kx = 0$ linear time-indep. coeffs
ordinary (ODE)
second order
homogeneous

General: $f_n(t)x^{(n)} + f_{n-1}(t)x^{(n-1)} + \ldots + f_2(t)\ddot{x} + f_1(t)\dot{x}$
(E) $+ f_0(t)x = p(t)$

linear
ODE
inhomogeneous
time-dep. coeffs.
n'th order

Theorem: local existence & global uniqueness
i) For specified IC $x(t_0), \dot{x}(t_0), \ldots x^{(n-1)}(t_0)$
(E) has a solution for t suff. close to t_0
ii) $x_1(t)$ & $x_2(t)$ two solutions
(same IC)
Then $x_1(t) = x_2(t)$ $\forall t$

Note: i) later we see construction
ii) can guess solutions

Ex $\dot{x} = ax$ $x(t) = Ae^{\alpha t}$
$\alpha = a$
$A = x(0)$

Note: - 1-parameter family of solutions
- parameter determined by IC

Ex $\ddot{x} = ax$ $\quad x(t) = Ae^{\alpha t}$

$a > 0 \quad \alpha = \pm\sqrt{a}$

$x(t) = Ae^{\sqrt{a}t} + Be^{-\sqrt{a}t}$

Note: - 2 parameter family of solutions
- parameters determined by IC $x(0), \dot{x}(0)$

Ex. $\ddot{x} = -ax$ $\quad x(t) = Ae^{\alpha t}$

$a > 0 \quad \alpha^2 = -a \quad w = \sqrt{a} \quad \alpha = \pm iw$

$x(t) = Ae^{iwt} + Be^{-iwt}$

$= C\cos wt + D\sin wt$

$= E\cos(wt + \phi_1)$

$= F\sin(wt + \phi_2)$

Ex: $\ddot{x} + 2\gamma\dot{x} + ax = 0$ $\quad x(t) = Ae^{\alpha t}$ $\quad a > 0$, $\gamma > 0$

$\alpha^2 + 2\gamma\alpha + a = 0 \quad \alpha = -\gamma \pm \sqrt{\gamma^2 - a} = \begin{cases} \alpha_1 \\ \alpha_2 \end{cases}$

$x(t) = Ae^{\alpha_1 t} + Be^{\alpha_2 t}$

$= e^{-\gamma t}\left(Ae^{t\sqrt{\gamma^2 - a}} + Be^{-t\sqrt{\gamma^2 - a}}\right)$

special case: $\gamma^2 \ll a$

$x(t) \cong e^{-\gamma t}\left(Ae^{iwt} + Be^{-iwt}\right)$

General Case: $\frac{d^n x}{dt^n} + c_{n-1}\frac{d^{n-1}x}{dt^{n-1}} + \dots + c_1\frac{dx}{dt} + c_0 x = 0$

$x(t) = Ae^{\alpha t}$

$$\alpha^n + c_{n-1}\alpha^{n-1} + \dots + c_1\alpha + c_0 = 0$$

n solutions $\alpha_1, \alpha_2 \dots \alpha_n$

$$x(t) = A_1 e^{\alpha_1 t} + \dots + A_n e^{\alpha_n t}$$

Note: constant coefficients → easy
non-constant coefficients → hard

Ex: $$\ddot{x} + \frac{2p}{t}\dot{x} + \frac{q}{t^2}x = 0$$

$$x(t) = At^{\alpha}$$

$$\alpha(\alpha-1) + 2p\alpha + q = 0$$

$$\alpha^2 + 2\left(p-\tfrac{1}{2}\right)\alpha + q = 0 \qquad \alpha = \begin{Bmatrix}\alpha_1 \\ \alpha_2\end{Bmatrix} = -\left(p-\tfrac{1}{2}\right) \pm \sqrt{\left(p-\tfrac{1}{2}\right)^2 - q}$$

$$x(t) = A_1 t^{\alpha_1} + A_2 t^{\alpha_2}$$

e.g. $q = 0 \Rightarrow \begin{cases}\alpha_1 = 0 & \text{constant mode} \\ \alpha_2 = -(2p-1) & \text{decaying mode } \left(p > \tfrac{1}{2}\right)\end{cases}$

Superposition Principle for linear DE

$$\underbrace{f_n(t)x^{(n)} + f_{n-1}(t)x^{(n-1)} + \dots + f_2(t)\ddot{x} + f_1(t)\dot{x} + f_0(t)x}_{\mathcal{O}(x;t)} = 0$$

$$\mathcal{O}(x;t) = 0$$

$\mathcal{O}$ linear differential operator

$$\mathcal{O}(A_1x_1 + A_2x_2) = A_1\mathcal{O}(x_1) + A_2\mathcal{O}(x_2)$$

Thm: Given linear diff. op. $\mathcal{O}$
If x_1 & x_2 are solutions of $\mathcal{O}(x) = 0$
then $A_1x_1 + A_2x_2$ is also a solution

Q: does this violate uniqueness theorem?

A: no, different IC

<u>inhomogeneous linear DE</u>

(I) $O(x;t) = g(t)$

Thm. General solution of inhomog. eq. (I) is $x_G(t)$
particular solution of inhomog. eq. plus $x_p(t)$
general solution of homog. eq. $x_h(t)$
$O(x,t) = 0$

$$x_G(t) = x_p(t) + x_h(t)$$

Note: O 2nd order → 2d solution space
O nth order → nd " "

Cor. $O(x;t) = c_1 g_1(t) + c_2 g_2(t)$

$$x(t) = c_1 x_1(t) + c_2 x_2(t) + x_h(t)$$

↗

particular solution of $O(x,t) = g_1(t)$
" " $O(x,t) = g_2(t)$

4.2 Simple Harmonic Motion

System: spring of mass m

force $F(x) = -kx$

$$m\ddot{x} = -kx$$

$$\ddot{x} = -\omega^2 x \qquad \omega = \sqrt{\frac{k}{m}}$$

$$x(t) = A\cos(\omega t + \varphi) \qquad \boxed{\text{plot}}$$

$$\text{IC} \rightarrow A, \varphi$$

Ex

$$\left.\begin{aligned} x(0) &= a \\ \dot{x}(0) &= 0 \end{aligned}\right\} \rightarrow \left\{\begin{aligned} A &= a \\ \varphi &= 0 \end{aligned}\right.$$

ω angular frequency

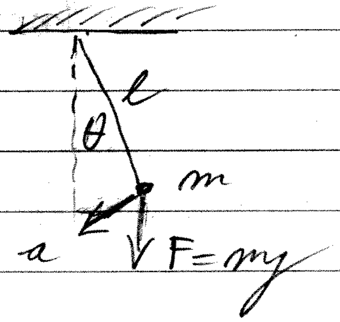
$T = 2\pi/\omega$ period

ν frequency

A amplitude

φ phase

System: Simple pendulum



$$a = l\ddot{\theta}$$

$$F_{\parallel} = -mg\sin\theta$$

$$ml\ddot{\theta} = -mg\sin\theta$$

$$\ddot{\theta} = -\frac{g}{l}\sin\theta \underset{\substack{\uparrow \\ \text{Taylor expansion}}}{\simeq} -\frac{g}{l}\theta = -\omega^2\theta$$

$$\theta(t) = A\cos(\omega t + \varphi)$$

4.3 Damped Harmonic Motion

System: Spring in the presence of frictional force

$F_f = -b\dot{x}$

$b \in \mathbb{R}$

$m\ddot{x} + b\dot{x} + kx = 0$

$\ddot{x} + 2\gamma\dot{x} + \omega^2 x = 0 \quad 2\gamma = b/m \quad \omega^2 = k/m$

$x(t) = e^{-\gamma t}(Ae^{\Omega t} + Be^{-\Omega t}) \quad \Omega = \sqrt{\gamma^2 - \omega^2}$

Case 1 underdamped $\gamma^2 < \omega^2$

$\tilde{\omega} = \sqrt{\omega^2 - \gamma^2}$

$x(t) = e^{-\gamma t}(Ae^{i\tilde{\omega}t} + Be^{-i\tilde{\omega}t})$

$= e^{-\gamma t} C \cos(\tilde{\omega}t + \varphi)$

friction → i) exponential decay of amplitude

ii) decrease in frequency

✓

$\boxed{\text{plot}}$

Case 2 overdamped $\gamma^2 > \omega^2 \Rightarrow \Omega > 0$

$x(t) = Ae^{-(\gamma-\Omega)t} + Be^{-(\gamma+\Omega)t}$

$\rightarrow Ae^{-(\gamma-\Omega)t} \quad t \rightarrow \infty$

$\boxed{\text{plot}}$

Case 3 critical damping $\gamma^2 = \omega^2 \rightarrow \Omega = 0$

→ our guess $x(t) = Ae^{+\alpha t}$ yields only 1 solution

→ 1 solution missing

guess: $x(t) = t e^{-\gamma t}$

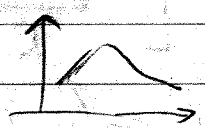
$$\dot{x} = e^{-\gamma t} - \gamma t e^{-\gamma t}$$

$$\ddot{x} = -2\gamma e^{-\gamma t} + \gamma^2 t e^{-\gamma t}$$

$$\ddot{x} + 2\gamma \dot{x} + \omega^2 x = \gamma^2 t e^{-\gamma t} - 2\gamma^2 t e^{-\gamma t} + \omega^2 t e^{-\gamma t} = 0$$

$$x(t) = (A + Bt)e^{-\gamma t}$$

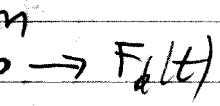
$\boxed{\text{plot}}$



Note: critical damping: fastest decay for fixed ω

4.4 Driven Harmonic Motion

System: $\boxed{\text{sketch}}$



$$\ddot{x} + 2\gamma \dot{x} + \omega^2 x = F\cos(\omega_d t) = \frac{F}{2}\left(e^{i\omega_d t} + e^{-i\omega_d t}\right)$$

Superposition principle →

$$x_G(t) = x_p(t) + x_h(t)$$

unique ↗ $\quad$ ↑ 2 parameter family

↑

guess

↓

$$x_I(t) = A e^{i\omega_d t}$$

$$(-\omega_d^2 + 2i\gamma\omega_d + \omega^2)\, A = \frac{F}{2}$$

$$A = \frac{F}{2}\,\frac{1}{\omega^2-\omega_d^2+2i\gamma\omega_d}$$

$$x_p(t) = \frac{F/2}{\omega^2-\omega_d^2+2i\gamma\omega_d}\, e^{i\omega_d t} + \frac{F/2}{\omega^2-\omega_d^2-2i\gamma\omega_d}\, e^{-i\omega_d t}$$

$$= \frac{F}{(\omega^2-\omega_d^2)^2+4\gamma^2\omega_d^2}\Big[(\omega^2-\omega_d^2)\cos(\omega_d t) + 2\gamma\omega_d \sin(\omega_d t)\Big]$$

$$R \equiv \sqrt{(\omega^2-\omega_d^2)^2+4\gamma^2\omega_d^2}$$

$$\underline{x_p(t)} = \frac{F}{R}\Big[\frac{\omega^2-\omega_d^2}{R}\cos(\omega_d t) + \frac{2\gamma\omega_d}{R}\sin(\omega_d t)\Big]$$

$$= \underline{\frac{F}{R}\cos(\omega_d t - \varphi)}$$

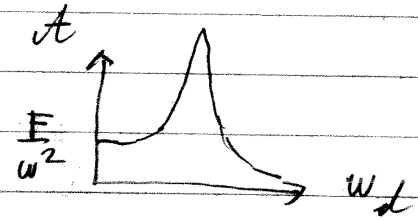
$$\cos\varphi = \frac{\omega^2-\omega_d^2}{R} \qquad \sin\varphi = \frac{2\gamma\omega_d}{R}$$

$$x_f(t) = x_p(t) + x_h(t) \simeq x_p(t) \quad \text{late times}$$

↑ exponential decay

Observations:

(1) Resonance $\quad A = \frac{F}{R}$



$$\gamma \searrow \;\to\; A_{max} \nearrow$$

$$\omega_d \nearrow \;\to\; A_{max} \downarrow$$

$$\tan\varphi = \frac{2\gamma\omega_d}{\omega^2-\omega_d^2}$$

12) Phase lag

a) $\omega_d \ll \omega$ $\tan\varphi = 0$ $\varphi = 0$ in phase

b) $\omega_d = \omega$ $\tan\varphi = \infty$ $\varphi = \frac{\pi}{2}$

particle lags driving force by 1/4 cycle

c) $\omega_d > \omega$ $\tan\varphi < 0 \Rightarrow \varphi > \frac{\pi}{2}$

it is even harder for particle to keep up

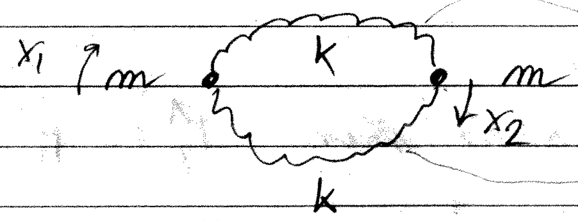
d) $\omega_d \gg \omega$, $\gamma \ll \omega_d$ $\tan\varphi = 0$ from neg. side

$\Rightarrow \varphi = \pi$

exactly out of phase

4.5 Coupled oscillators

System: connected springs   constrained to move on hoop



$F_1 = k(x_2 - x_1)$
$\quad - k(x_1 - x_2)$

$$m\ddot{x}_1 + 2k(x_1 - x_2) = 0$$
$$m\ddot{x}_2 + 2k(x_2 - x_1) = 0$$

2nd order system of 2 coupled homogeneous ODE

Note: exact solution exists
normal modes
↳ linear algebra

Ex: $2\ddot{x} + w^2(5x-3y) = 0$

$2\ddot{y} + w^2(5y-3x) = 0$

$$\left.\begin{aligned} 2(\ddot{x}+\ddot{y}) + 2w^2(x+y) &= 0 \\ 2(\ddot{x}-\ddot{y}) + 8w^2(x-y) &= 0 \end{aligned}\right\} \rightarrow \text{decoupled}$$

$$\left.\begin{matrix} x+y \\ x-y \end{matrix}\right\} \text{normal modes} \qquad \left.\begin{matrix} w \\ 2w \end{matrix}\right\} \text{normal mode frequencies}$$

Systematics — linear algebra

eigenvectors

eigenvalues

Ansatz: $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} A \\ B \end{pmatrix} e^{i\alpha t}$

$$-2A\alpha^2 + 5Aw^2 - 3Bw^2 = 0$$

$$-2B\alpha^2 + 5Bw^2 - 3Aw^2 = 0$$

$$(M) \quad \underbrace{\begin{pmatrix} -2\alpha^2+5w^2 & -3w^2 \\ -3w^2 & -2\alpha^2+5w^2 \end{pmatrix}}_{M} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

if M^{-1} exists then $\begin{pmatrix} A \\ B \end{pmatrix} = M^{-1}\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

Condition for solution: M^{-1} does not exist

$$\Updownarrow$$

$$\det M = 0$$

$$\det M = (-2\alpha^2+5w^2)(-2\alpha^2+5w^2) - 9w^4$$

$$= 4\alpha^4 - 20\alpha^2 w^2 + 16w^4 \overset{!}{=} 0$$

$$\alpha^4 - 5\alpha^2 w^2 + 4w^4 = (\alpha^2 - w^2)(\alpha^2 - 4w^2)$$

solutions: $d_1 = \pm w$ $\quad d_2 = \pm 2w$

normal mode frequencies

insert back into (M)

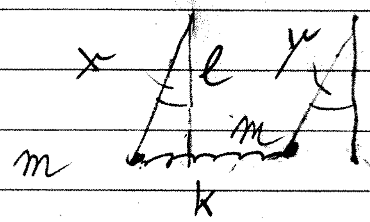
d_1: $\quad 3w^2A - 3w^2B = 0 \qquad B = A$

d_2: $\quad -3w^2A - 3w^2B = 0 \qquad B = -A$

normal modes

$$x(t) = B_1\begin{pmatrix}1\\1\end{pmatrix}\cos(wt+\varphi_1) + B_2\begin{pmatrix}1\\-1\end{pmatrix}\cos(2wt+\varphi_2)$$

Ex Double pendulum



$ml\ddot{x} = -mgx - k(x-y) \qquad \ddot{x} = -w^2x - \frac{k}{ml}(x-y)$

$ml\ddot{y} = -mgy - k(y-x) \qquad \ddot{y} = -w^2y - \frac{k}{ml}(y-x)$

1st normal mode $\quad x = y$

" " freq. $\quad w_1 = w$

2nd normal mode $\quad \ddot{x} - \ddot{y} = -\left(w^2 + \frac{2k}{ml}\right)(x-y)$

$y = -x$

$w_2 = \sqrt{w^2 + \frac{2k}{ml}}$

4.6 Time-Dependent Systems

System $\quad m\ddot{q} = +F(q,\dot{q},t) \qquad p = m\dot{q}$

$$\dot{q} = \frac{1}{m} p$$

$$\dot{p} = +F(q,p,t) = +F_q(q,t) + F_p(p,t)$$

$$= -\alpha(t)q - \beta(t)p$$

$$\begin{pmatrix} \dot{q} \\ \dot{p} \end{pmatrix} = A(t) \begin{pmatrix} q \\ p \end{pmatrix}$$

for system of n coupled DoF $\quad \begin{pmatrix} q_1 \\ \vdots \\ p_n \end{pmatrix} \equiv x$

(H) $\quad \dot{\underline{x}} = A(t)\underline{x} \quad$ homogeneous

(I) $\quad \dot{\underline{x}} = A(t)\underline{x} + B(t) \quad$ inhomogeneous

Theorem: For homogeneous linear system the solution of (H) with initial condition $\underline{x}(s) = \underline{y}$ is

$$\underline{x}(t) = P(t,s)\underline{y}$$

$$P(t,s) = \exp\left(\int_s^t dr\, A(r)\right) \quad \text{propagator}$$

Pf $\quad \dfrac{dx}{dt} \underset{\substack{\uparrow \\ \text{chain rule}}}{=} A(t)\,P(t,s)\underline{y} = A(t)\underline{x}$

Theorem: For an inhomog. linear system solution of (I) with initial conditions $\underline{x}(s) = \underline{y}$ is

$$\underline{x}(t) = P(t,s)y + \int_s^t dr\, P(t,r)B(r)$$

Ex Time-independent system

$$\dot{q} = p$$
$$p = -w^2 q$$

$$\int_0^t A(r)\,dr = \begin{pmatrix} 0 & 1 \\ -w^2 & 0 \end{pmatrix} t$$

$$P(t,0) = \exp\left[\begin{pmatrix} 0 & 1 \\ -w^2 & 0 \end{pmatrix} t \right]$$
$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & t \\ -w^2 t & 0 \end{pmatrix} - \frac{w^2 t^2}{2!} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$
$$- \frac{w^3 t^3}{3!} \begin{pmatrix} 0 & t \\ -w^2 t & 0 \end{pmatrix} + \cdots$$
$$= \begin{pmatrix} \cos(wt) & \frac{1}{w}\sin(wt) \\ -w\sin(wt) & \cos(wt) \end{pmatrix}$$

$$\Downarrow$$

$$\begin{pmatrix} q(t) \\ p(t) \end{pmatrix} = \begin{pmatrix} \cos(wt) & \frac{1}{w}\sin(wt) \\ -w\sin(wt) & \cos(wt) \end{pmatrix} \begin{pmatrix} q(0) \\ p(0) \end{pmatrix}$$
$$= \begin{pmatrix} \cos(wt)q(0) + \frac{1}{w}\sin(wt)p(0) \\ -w\sin(wt)q(0) + \cos(wt)p(0) \end{pmatrix}$$

Ex Time independent system with driving

$$\dot{q} = p$$
$$\dot{p} = -w^2 q + F\cos(w_d t), \quad A = \begin{pmatrix} 0 & 1 \\ -w^2 & 0 \end{pmatrix}$$
$$B = \begin{pmatrix} 0 \\ F\cos(w_d t) \end{pmatrix}$$

$$\underline{x}(t) = \underbrace{P(t,0)\underline{y}}_{\underline{x}_h(t)} + \underbrace{\int_0^t d\tau\, P(t,\tau)\, B(\tau)}_{\underline{x}_p(t)}$$

$$P(t,\tau) = \begin{pmatrix} \cos(w(t-\tau)) & \frac{1}{w}\sin(w(t-\tau)) \\ -w\sin(w(t-\tau)) & \cos(w(t-\tau)) \end{pmatrix}$$

$$\underline{x}_p(t) = \begin{pmatrix} F\int_0^t d\tau\, \frac{1}{w}\sin(w(t-\tau))\cos(w_d\tau) \\ F\int_0^t d\tau\, \cos(w(t-\tau))\cos(w_d\tau) \end{pmatrix}$$

Note: 1) solution obtained without guessing
2) works for any $B(t)$
(guessing does not!)

4.7 Born approximation

$$m\ddot{q} = -\frac{\partial V}{\partial q}$$

$$V(q) = \frac{1}{2}kq^2 + \frac{1}{4}\lambda q^4 \qquad \lambda \ll 1$$

small nonlinearity

$$m\ddot{q} = -kq - \lambda q^3$$

Idea: Perturbation theory in λ
expand in λ

step 1 $\quad m\ddot{q}_0 = -kq_0 \qquad O(\lambda^0)$

find $q_0(t)$

step 2 $\quad q = q_0 + q_1$

$$m\ddot{q}_1 = -kq_1 - \lambda q_0^3 \quad O(\lambda)$$

find $q_1(t)$

step 3 $q = q_0 + q_1 + q_2$

$$m\ddot{q}_2 = -kq_2 - 3\lambda q_0^2 q_1 \qquad O(\lambda^2)$$

find $q_2(t)$

etc.

finding $q_1(t)$

$$\begin{pmatrix} q_1 \\ p_1 \end{pmatrix} = A \begin{pmatrix} q_1 \\ p_1 \end{pmatrix} + \underbrace{\lambda \begin{pmatrix} 0 \\ q_0^3 \end{pmatrix}}_{B}$$

Note: explicit solution exists