PHYS 514
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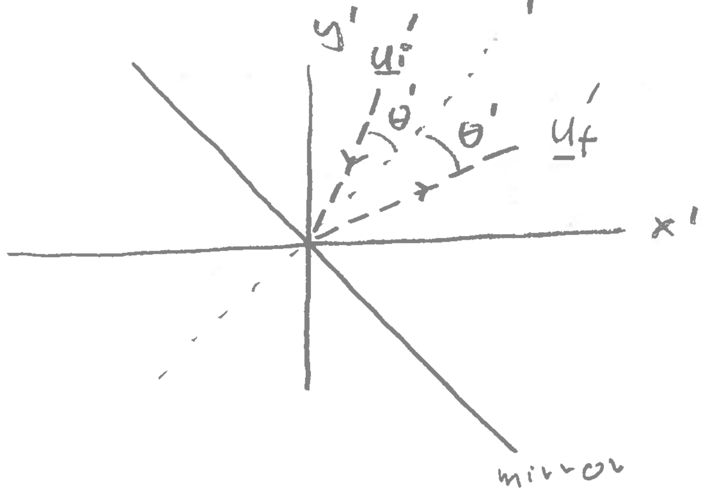
Solutions to Assignment 1

Q1 In S' the setup is



The initial velocity is $\underline{u}_i'$ and the final one is $\underline{u}_f'$. Clearly

$$\tan\left(\frac{\pi}{4} - \theta'\right) = \frac{u'_{f,y}}{u'_{f,x}} \quad \text{and} \quad \tan\left(\frac{\pi}{4} + \theta'\right) = \frac{u'_{i,y}}{u'_{i,x}}$$

where $\frac{\pi}{4}$ is the angle between the normal to the mirror and the x'-axis.

It's easy to see that $u'_{i,y} = -u'_{f,x}$ and

$$u'_{i,x} = -u'_{f,y}$$

Let's now go to the system S which moves with velocity $-v$ in the y'-direction as seen from S'.

The usual velocity addition formulas give that

$$u_{i,y} = \frac{u'_{i,y} + v}{1 + v\, u'_{i,y}}, \qquad u_{i,x} = \frac{u'_{i,x}}{\gamma(1 + v\, u'_{i,y})}$$

$$u_{f,y} = \frac{u'_{f,y} + v}{1 + v\, u'_{f,y}} = \frac{-u'_{i,x} + v}{1 - v\, u'_{i,x}}$$

and

$$u_{f,x} = \frac{u'_{f,x}}{\gamma(1 + v\, u'_{f,y})} = \frac{-u'_{i,y}}{\gamma(1 - v\, u'_{i,x})}$$

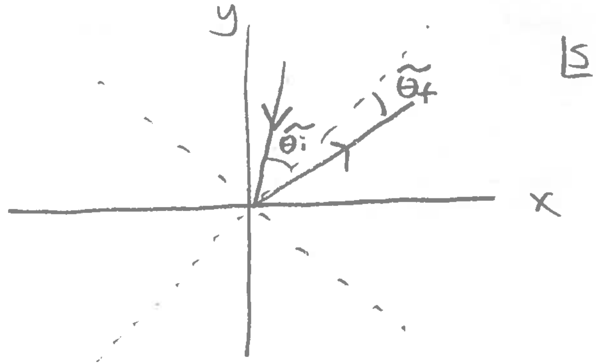
where $\gamma = \frac{1}{\sqrt{1-v^2}}$ and $c = 1$.

Then it's easy to calculate the angles between the velocity vectors and the $x=y$ line (the normal to the $x=-y$ plane). They are

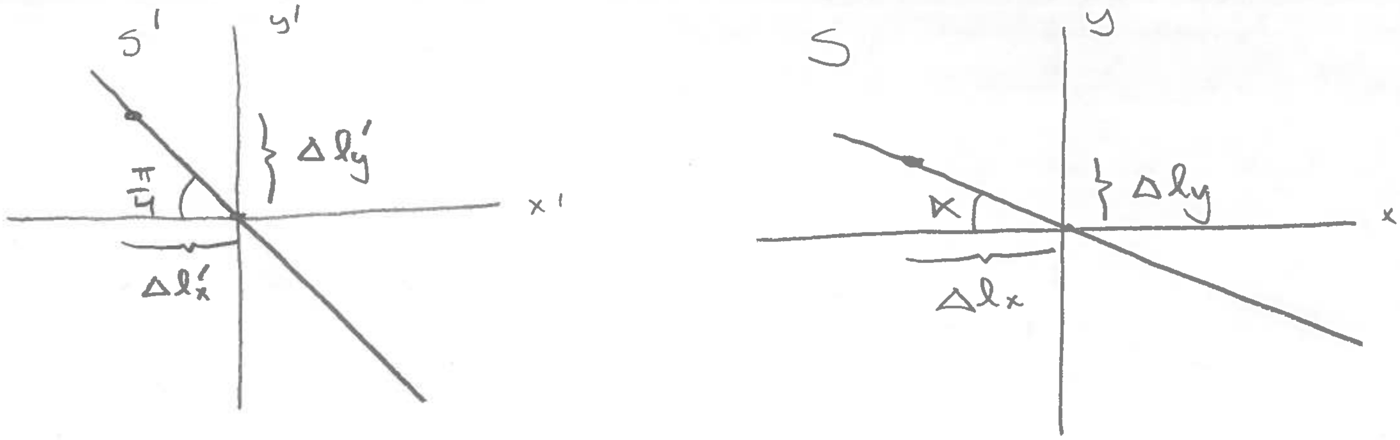
$$\tan\left(\frac{\pi}{4} + \tilde{\theta}_i\right) = \frac{u_{i,y}}{u_{i,x}} = \frac{u'_{i,y} + v}{\sqrt{1-v^2}\, u'_{i,x}}$$

and

$$\tan\left(\frac{\pi}{4} - \tilde{\theta}_f\right) = \frac{u_{f,y}}{u_{f,x}} = \frac{u'_{i,x} - v}{\sqrt{1-v^2}\, u'_{i,y}}.$$



We must also take into account length contraction of the mirror. ③



The usual formulas give that

$$\Delta l_x = \Delta l_x' \quad \text{and} \quad \Delta l_y = \sqrt{1-v^2}\,\Delta l_y'$$

where $\Delta l_x' = \Delta l_y'$. Thus

$$\alpha = \arctan \frac{\Delta l_y}{\Delta l_x} = \arctan \sqrt{1-v^2}.$$

It's then easy to see that the incoming and outgoing angles relative to the normal of the mirror in S are

$$\theta_i = \tilde{\theta}_i - \left(\frac{\pi}{4} - \alpha\right) = \arctan\left(\frac{u_{i,y}' + v}{\sqrt{1-v^2}\,u_{i,x}'}\right) - \frac{\pi}{4} - \frac{\pi}{4} + \alpha$$

$$= \underline{\underline{\arctan\left(\frac{u_{i,y}' + v}{\sqrt{1-v^2}\,u_{i,x}'}\right) + \arctan\sqrt{1-v^2} - \frac{\pi}{2}}}$$

and

$$\theta_f = \tilde{\theta}_f + \left(\frac{\pi}{4} - \alpha\right)$$

$$= \underline{\underline{-\arctan\left(\frac{u_{i,x}' - v}{\sqrt{1-v^2}\,u_{i,y}'}\right) + \frac{\pi}{2} - \arctan\sqrt{1-v^2}}}$$

This can be rewritten using expressions like

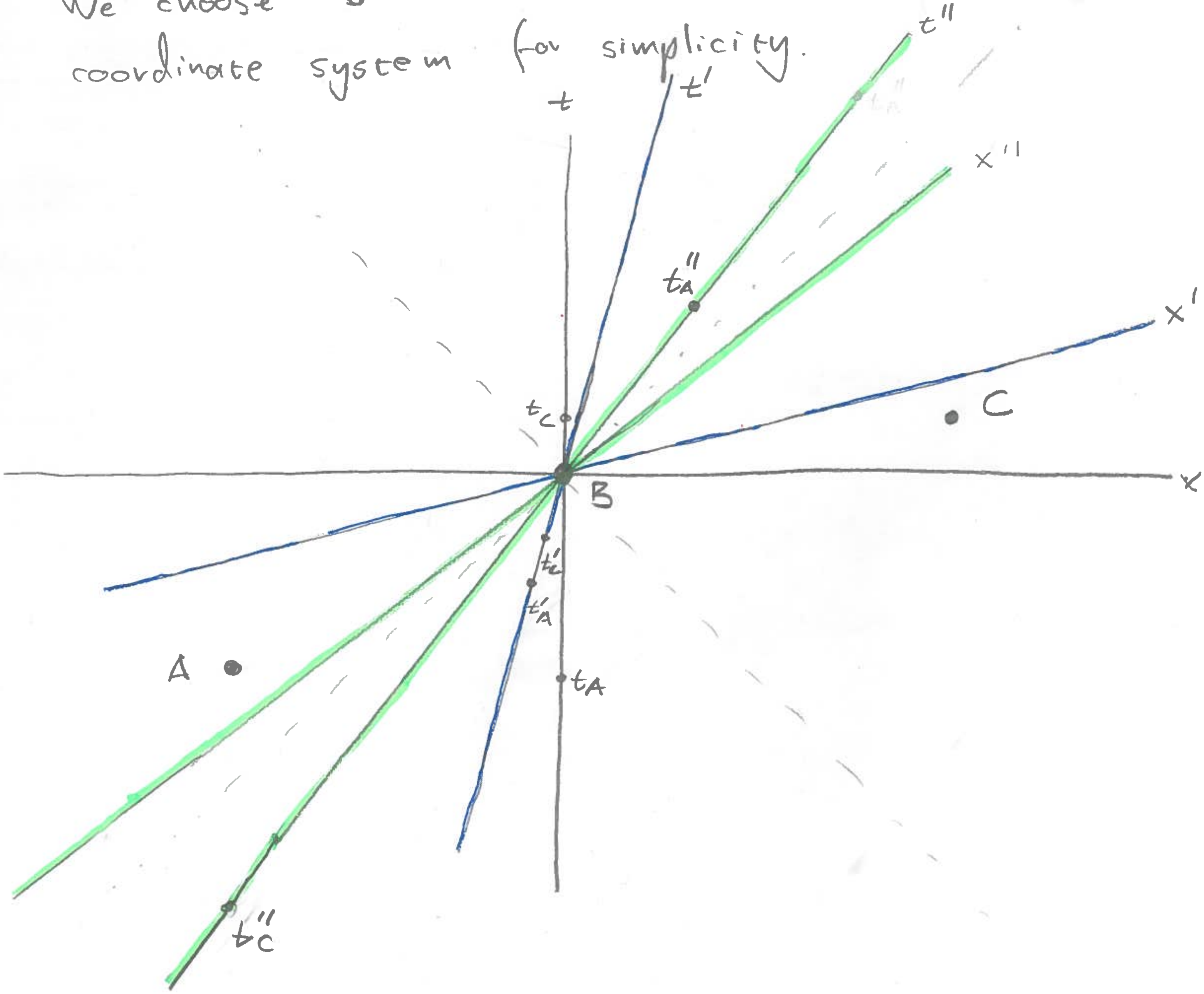
$$\arctan \frac{1}{x} = \frac{\pi}{2} - \arctan x,$$

$$\arctan x + \arctan y = \arctan\left(\frac{x+y}{1-xy}\right)$$

but that doesn't offer much simplification.

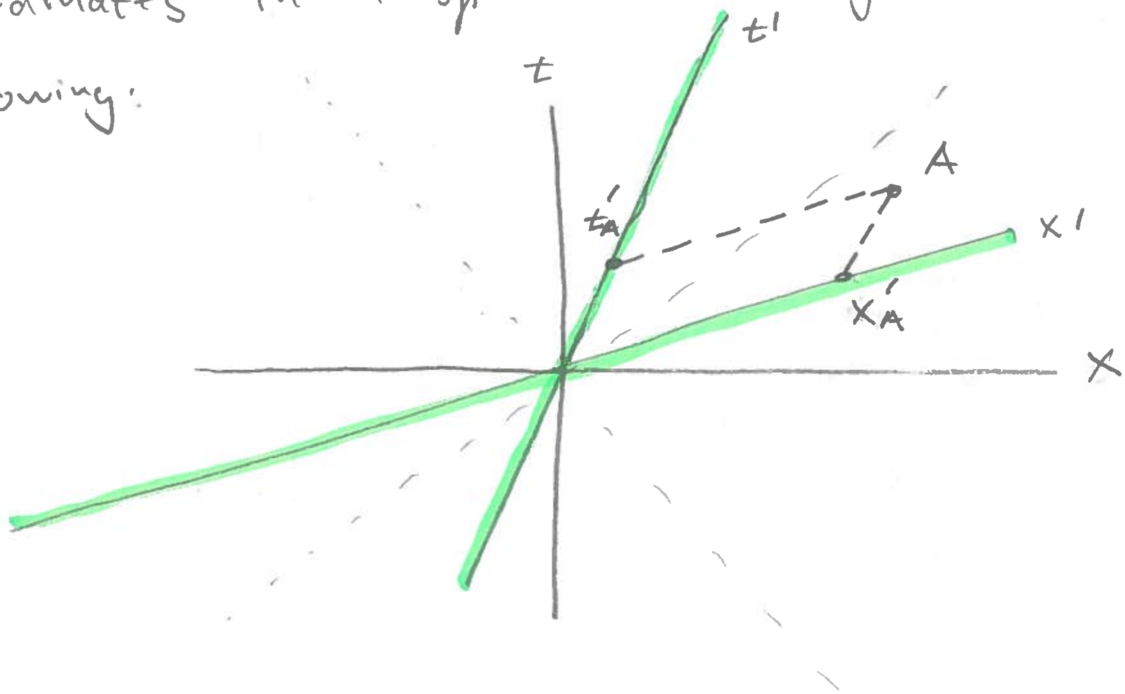
Q2: Yes, this is possible for some combination of three events.

We choose B to be at the center of the coordinate system for simplicity.



We see that in S the order is ABC, in S' it is ACB and in S'' it is CBA.

For clarification: The correct way to see coordinates in a space-time diagram is the following:



Q3: The muon mass is $m = 0.106$ GeV, its energy is $E = 1000$ GeV and the rest-frame lifetime is $\tau = 2.19 \times 10^{-6}$ s. Since $E = \gamma m$ the time dilation formula tells us that the lifetime in the lab frame is

$$t = \gamma \tau = \frac{E\tau}{m} = \underline{\underline{0.021\,s}}$$

It travels distance ct in a storage ring of circumference πd where d is the diameter. Thus it travels

$$\frac{ct}{\pi d} \cdot 2\pi = \underline{\underline{1.3 \times 10^4 \text{ rad}}}$$ before decaying.

Q4: We can analyze this problem directly in the frame of reference S where $E \gg m \gg \nu$ by using momentum conservation.

We can also take a shortcut by doing the analysis in the rest frame of the massive particle, S', and then boosting.

In S'

Before　　　　After

The usual formula for Compton scattering says that

$$\nu'_{after} = \frac{\nu'_{before}}{1 + \frac{\nu'_{before}}{m}(1-\cos\theta)}$$

where θ is the angle between the initial and final direction of motion of the photon. We will clearly maximize the energy transfer if

$\theta = \pi$ so $$\nu'_{after} = \frac{\nu'_{before}}{1 + 2\frac{\nu'_{before}}{m}}.$$

(If $\theta = 0$ there is no scattering which is clearly not the case we're interested in.)

Boosting with velocity v we get that

$$\nu'_{after} = \gamma\, \nu_{after}\, (1-v)$$

and

$$\nu'_{before} = \gamma \nu\, (1+v).$$

So that

$$\gamma\, \nu_{after}\, (1-v) = \frac{\gamma \nu (1+v)}{1 + 2\frac{\gamma}{m} \nu (1+v)}.$$

We have that the boost needed corresponds to

$\gamma = \frac{E}{m}$ and since $E \gg m$ we also have

$v \simeq 1$.

Thus

$$\nu_{after} = \nu \frac{\frac{\gamma(1+v)^2}{1-v^2}}{1 + 2\frac{\gamma}{m}(1+v)\nu}$$

$$\simeq \nu \frac{4\, E^2/m^2}{1 + 4\frac{E}{m^2}\nu} = \nu \frac{4E^2}{m^2 + 4E\nu}.$$

Thus the energy transfer is

$$\nu_{after} - \nu \simeq \nu_{after} \simeq \underline{\underline{\frac{4E^2\nu}{m^2 + 4E\nu}}}$$

(b) Here $E = 10^{20}\,eV$ and

$m = 9.4 \times 10^{8}\,eV.$

There are many ways to understand what is meant by a 3K photon. Let's just use the estimate

$$h\nu = k_B T$$

so that

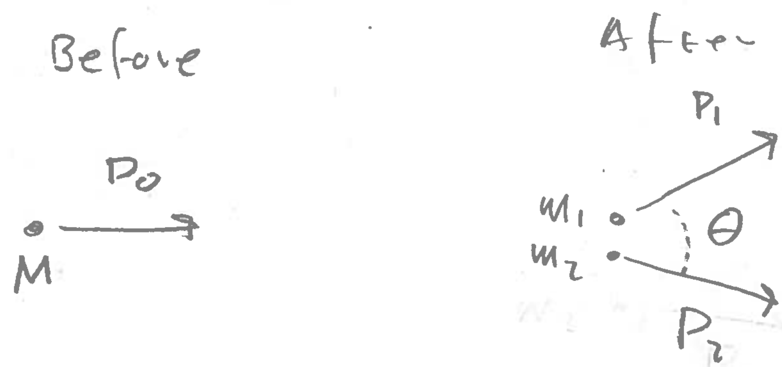
$$\nu = \frac{k_B T}{2\pi} = \frac{8.6\times 10^{-5}\,\frac{eV}{K}\cdot 3K}{2\pi} = 0.41\times 10^{-4}\,eV$$

where we use that $\hbar = \frac{h}{2\pi} = 1$.

Then the energy transfer is

$$\frac{4E^2\nu}{m^2+4E\nu} \simeq \underline{\underline{1.8\times 10^{18}\,eV}}$$

<u>Q5:</u>



Here P_0, P_1, P_2 are four-vectors and

$P_1 = (E_1, \vec{p}_1)$, $P_2 = (E_2, \vec{p}_2)$. We have that

$P_0^2 = -M^2$, $P_1^2 = -m_1^2$, $P_2^2 = -m_2^2$.

Energy-momentum conservation gives $P_0 = P_1 + P_2$

so $P_0^2 = (P_1+P_2)^2 = P_1^2 + 2P_1\cdot P_2 + P_2^2$,

i.e.
$$-M^2 = -m_1^2 - m_2^2 - 2E_1E_2 + 2\mathbf{p}_1\cdot\mathbf{p}_2,$$

i.e.
$$\underline{\underline{M^2 = m_1^2 + m_2^2 + 2E_1E_2 - 2p_1p_2\cos\theta}}$$

<u>Q6:</u> (a) The non-relativistic action is

$$S_{nr} = \int \tfrac{1}{2}mv^2\,dt.$$

Thus $0 = \delta S_{nr} = \int mv^i\delta v^i dt = \int mv^i \frac{d\delta x^i}{dt}dt$

$$= -\int m\frac{dv^i}{dt}\delta x^i dt$$

integration by parts

Clearly the equation of motion is $\dot{v}^i = \frac{dv^i}{dt} = 0$.

(b) The relativistic action is $S = m\int d\tau = \int m\frac{dt}{\gamma}$

where τ is the eigentime and t is the lab time.

Thus $S = \int m\sqrt{1-v^2}\,dt$. In the non-relativistic limit $\sqrt{1-v^2} \simeq 1 - \frac{1}{2}v^2$ so

$S = \int m dt - S_{nr}$, i.e. we get the non-relativist action up to a constant (and an irrelevant minus sign).

In general

$$0 = \delta S = \int m \frac{-v^i}{(1-v^2)^{1/2}} \delta v^i dt$$

$$= -\int m \gamma v^i \frac{d\delta x^i}{dt} dt = \int m \frac{d}{dt}(\gamma v^i) \, dt \, \delta x^i$$

Thus the equation of motion is simply

$\frac{d}{dt}(p) = 0$ where $p = \gamma m \underline{v}$

is the momentum.

Since $\gamma \to 1$ in the non-relativistic limit we get the result of (a).