Homework 5, GR

Problem 1

(a) The Lagrangian density is

$$\mathcal{L} = \sqrt{-g}\left(-\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + A_\mu J^\mu\right)$$

and we're told to ignore the $A_\mu J^\mu$ part.

We have that the energy-momentum tensor obeys

$$T_{\mu\nu} = -2\frac{1}{\sqrt{-g}}\frac{\delta S_M}{\delta g^{\mu\nu}}$$

with $S_M = \int d^4x\sqrt{-g}\left(-\frac{1}{4}g^{\mu\omega}g^{\nu\varkappa}F_{\omega\varkappa}F_{\mu\nu}\right)$.

Looking at perturbations with respect to the metric we get that

$$\delta S_M = \int d^4x\left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}\right)\delta\sqrt{-g}$$

$$+\int d^4x\sqrt{-g}\left(-\frac{1}{4}F_{\omega\varkappa}F_{\mu\nu}\right)\left(\delta g^{\mu\omega}g^{\nu\varkappa} + g^{\mu\omega}\delta g^{\nu\varkappa}\right)$$

$$= \int d^4x \sqrt{-g} \Big(-\frac{1}{4}\left(-\frac{1}{2}\right) F_{\mu\nu} F^{\mu\nu} g_{\alpha\beta}\, \delta g^{\alpha\beta}$$

$$+ \delta g^{\alpha\beta}\, g^{\nu x} \left(-\frac{1}{4} F_{\beta x} F_{\alpha\nu}\right)$$

$$+ \delta g^{\alpha\beta}\, g^{\mu\omega} \left(-\frac{1}{4} F_{\omega\beta} F_{\mu\alpha}\right) \Big)$$

where we used Eq. 4.69 in the textbook to rewrite the first line.

Thus

$$\delta S_M = \int d^4x \sqrt{-g}\, \delta g^{\alpha\beta}$$

$$\times \left(g_{\alpha\beta} \frac{1}{8} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} F_{\alpha\nu} F_\beta{}^\nu \right).$$

Thus $$T_{\alpha\beta} = -2 \frac{1}{\sqrt{-g}} \frac{\delta S_M}{\delta g^{\alpha\beta}} = \underline{\underline{F_{\alpha\nu} F_\beta{}^\nu - \frac{1}{4} g_{\alpha\beta} F_{\mu\nu} F^{\mu\nu}}}.$$

(b) Now the Lagrangian becomes

$$\mathcal{L}_M = \sqrt{-g} \left(-\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + A_\mu J^\mu + \beta R^{\mu\nu} g^{\rho\sigma} F_{\mu\rho} F_{\nu\sigma} \right).$$

(Note that the new term also has a factor $\sqrt{-g}$; otherwise $S_M = \int d^4x\, \mathcal{L}$ wouldn't be a scalar).

Let's first derive Maxwell's equations.

We write $\mathcal{L}_M = \sqrt{-g}\,\hat{\mathcal{L}}_M$. Then

the Euler-Lagrange equation is

$$\frac{\partial \hat{\mathcal{L}}_M}{\partial A_\nu} - \nabla_\mu \left(\frac{\partial \hat{\mathcal{L}}_M}{\partial (\nabla_\mu A_\nu)} \right) = 0$$

(see Eq. 4.49 in the textbook).

It's easy to see that

$$\frac{\partial \hat{\mathcal{L}}_M}{\partial A_\nu} = J^\nu.$$

Furthermore, since

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = \nabla_\mu A_\nu - \nabla_\nu A_\mu$$

we get that

$$\frac{\partial F^{\mu\nu} F_{\mu\nu}}{\partial (\nabla_\alpha A_\beta)} = 2 F^{\mu\nu} \frac{\partial F_{\mu\nu}}{\partial (\nabla_\alpha A_\beta)}$$

$$= 2 F^{\mu\nu} \left(\delta^\mu_\alpha \delta^\nu_\beta - \delta^\mu_\beta \delta^\nu_\alpha \right)$$

$$= 2F^{\alpha\beta} - 2F^{\beta\alpha} = 4 F^{\alpha\beta}$$

and

$$\frac{\partial\, F_{\mu\rho} F_{\nu\sigma}}{\partial(\nabla_\alpha A_\beta)} = (\delta^\alpha_\mu \delta^\beta_\rho - \delta^\alpha_\rho \delta^\beta_\mu) F_{\nu\sigma} + F_{\mu\rho} (\delta^\alpha_\nu \delta^\beta_\sigma - \delta^\beta_\nu \delta^\alpha_\sigma)$$

so that

$$\frac{\partial(\beta R^{\mu\nu} g^{\rho\sigma} F_{\mu\rho} F_{\nu\sigma})}{\partial(\nabla_\alpha A_\beta)}$$

$$= \beta (R^{\alpha\nu} g^{\beta\sigma} - R^{\beta\nu} g^{\alpha\sigma}) F_{\nu\sigma}$$

$$+ \beta (R^{\mu\alpha} g^{\rho\beta} - R^{\mu\beta} g^{\rho\alpha}) F_{\mu\rho}$$

$$= 2\beta (R^{\alpha\nu} g^{\beta\sigma} - R^{\beta\nu} g^{\alpha\sigma}) F_{\nu\sigma}$$

$$= 2\beta (R^{\alpha\nu} F_\nu{}^\beta - R^{\beta\nu} F_\nu{}^\alpha)$$

where we've used that $R^{\omega x} = R^{x\omega}$.

Bringing all the pieces together gives

$$\nabla_\alpha\left(-F^{\alpha\beta} + 2\beta (R^{\alpha\nu} F_\nu{}^\beta - R^{\beta\nu} F_\nu{}^\alpha)\right) - J^\beta = 0.$$

This can be written as

$$\nabla_\alpha F^{\beta\alpha} + 2\beta \nabla_\alpha \left(R^{\alpha\nu} F_\nu{}^\beta - R^{\beta\nu} F_\nu{}^\alpha \right)$$
$$= J^\beta.$$

When $\beta \to 0$ this reduces to Maxwell's equations.

Let's now check that the current is conserved, i.e. $\nabla_\beta J^\beta = 0$.

This is equivalent to showing that

$$\nabla_\beta \nabla_\alpha F^{\beta\alpha} + 2\beta \nabla_\beta \nabla_\alpha \left(R^{\alpha\nu} F_\nu{}^\beta - R^{\beta\nu} F_\nu{}^\alpha \right) = 0.$$

This is easy to see from the fact that both $F^{\beta\alpha}$ and

$$R^{\alpha\nu} F_\nu{}^\beta - R^{\beta\nu} F_\nu{}^\alpha$$

are antisymmetric under interchange of α, β.

For any tensor $B_{\alpha\beta}$ so that $B_{\alpha\beta} = - B_{\beta\alpha}$ we have that

$$\nabla_\alpha \nabla_\beta B^{\alpha\beta}$$

$$= \frac{1}{2} \nabla_\alpha \nabla_\beta \left(B^{\alpha\beta} - B^{\beta\alpha} \right)$$

$$= \frac{1}{2} \left(\nabla_\alpha \nabla_\beta - \nabla_\beta \nabla_\alpha \right) B^{\alpha\beta}$$

$\alpha \leftrightarrow \beta$ in second term

$$= \frac{1}{2} R^\alpha{}_{\lambda\alpha\beta} B^{\lambda\beta} + \frac{1}{2} \underbrace{R^\beta{}_{\lambda\alpha\beta}}_{= -R^\beta{}_{\lambda\beta\alpha}} B^{\alpha\lambda}$$

see eq. 3.114 in textbook

$$= \frac{1}{2} \left(R_{\lambda\beta} B^{\lambda\beta} - R_{\lambda\alpha} B^{\alpha\lambda} \right)$$

$$= R_{\lambda\alpha} B^{\lambda\alpha} = 0$$

because $R_{\lambda\alpha}$ is symmetric but

$B_{\lambda\alpha}$ is antisymmetric.

The only thing left is to show how Einstein's equation changes.

The total Lagrangian is

$$\mathcal{L}_{tot} = \sqrt{-g}\,\left(R + \beta R^{\mu\nu} g^{\rho\sigma} F_{\mu\rho} F_{\nu\sigma}\right) + \mathcal{L}$$

where $\mathcal{L}$ will only contribute to the stress energy-tensor.

As usually $\int d^4x \sqrt{-g}\, R$ gives

$$\frac{1}{\sqrt{-g}} \frac{\delta \int d^4x \sqrt{-g}\, R}{\delta g^{\mu\nu}} = R_{\mu\nu} - \frac{1}{2} R\, g_{\mu\nu}.$$

We can thus focus on

$$S' = \beta \int d^4x \sqrt{-g}\, g^{\rho\sigma} R^{\mu\nu} F_{\mu\rho} F_{\nu\sigma}.$$

Varying with respect to the metric gives

$$\delta S' = (\delta S')_1 + (\delta S')_2 + (\delta S')_3$$

where

$$(\delta S')_1 = \beta \int d^4x \sqrt{-g}\, g_{\rho\sigma} F^{\mu\rho} F^{\nu\sigma}\, \delta R_{\mu\nu},$$

$$(\delta S')_2 = \beta \int d^4x \sqrt{-g}\, R^{\mu\nu} F_{\mu\rho} F_{\nu\sigma}\, \delta g^{\rho\sigma}$$

and $$(\delta S')_3 = \beta \int d^4x\, g^{\rho\sigma} R^{\mu\nu} F_{\mu\rho} F_{\nu\sigma}\, \delta\sqrt{-g}\,.$$

Let's do the easy terms first.

We can leave $(\delta S')_2$ as it is. Using Eq. 4.69 in the textbook we get that

$$(\delta S')_3 = \beta \int d^4x\, g^{\rho\sigma} R^{\mu\nu} F_{\mu\rho} F_{\nu\sigma} \left(-\tfrac{1}{2}\right) \sqrt{-g}\, g_{\alpha\beta}\, \delta g^{\alpha\beta}.$$

In the textbook it is shown (Eq. 4.62) that

$$\delta R^{\rho}{}_{\mu\lambda\nu} = \nabla_\lambda (\delta \Gamma^{\rho}_{\nu\mu}) - \nabla_\nu (\delta\Gamma^{\rho}_{\lambda\mu})$$

so $$\delta R_{\mu\nu} = \nabla_\lambda (\delta \Gamma^{\lambda}_{\nu\mu}) - \nabla_\nu (\delta\Gamma^{\lambda}_{\lambda\mu}).$$

This gives us that

$$(\delta S')_1 = \beta \int d^4x \sqrt{-g}\, g_{\rho\sigma} F^{\mu\rho} F^{\nu\sigma} \times \left[\nabla_\lambda \delta\Gamma^\lambda_{\nu\mu} - \nabla_\nu \delta\Gamma^\lambda_{\lambda\mu} \right]$$

$$= \beta \int d^4x \sqrt{-g}\, g_{\rho\sigma} \left[\nabla_\nu (F^{\mu\rho} F^{\nu\sigma})\, \delta\Gamma^\lambda_{\lambda\mu} - \nabla_\lambda (F^{\mu\rho} F^{\nu\sigma})\, \delta\Gamma^\lambda_{\nu\mu} \right].$$

$$= \beta \int d^4x \sqrt{-g} \left[\nabla_\nu (F^{\mu\rho} F^\nu{}_\rho)\, \delta\Gamma^\lambda_{\lambda\mu} - \nabla_\lambda (F^{\mu\rho} F^\nu{}_\rho)\, \delta\Gamma^\lambda_{\nu\mu} \right]$$

where we used Stoke's theorem to get rid of boundary terms.

Furthemore Eq. 4.64 in the textbook tells us that

$$\delta\Gamma^\lambda_{\nu\mu} = -\tfrac{1}{2} \left[g_{\omega\nu} \nabla_\mu \delta g^{\omega\lambda} + g_{\omega\mu} \nabla_\nu \delta g^{\omega\lambda} - g_{\nu\alpha} g_{\mu\beta} \nabla^\lambda (\delta g^{\alpha\beta}) \right]$$

which means that

$$\delta\Gamma^\lambda_{\lambda\mu} = -\tfrac{1}{2} \left[g_{\omega\lambda} \nabla_\mu \delta g^{\omega\lambda} + g_{\omega\mu} \nabla_\lambda \delta g^{\omega\lambda} - g_{\lambda\alpha} g_{\mu\beta} \nabla^\lambda \delta g^{\alpha\beta} \right].$$

6d

Since $g_{\omega\lambda} \nabla_\mu \delta g^{\omega\lambda}$

$$= \nabla_\mu \left(g_{\omega\lambda} \delta g^{\omega\lambda} \right) = \frac{1}{2} \nabla_\mu \delta \left(g_{\omega\lambda} g^{\omega\lambda} \right)$$

$$= \frac{1}{2} \nabla_\mu \delta(4) = 0$$

we can write

$$\delta \Gamma^\lambda_{\lambda\mu} = -\frac{1}{2} \left[g_{\omega\mu} \nabla_\lambda \delta g^{\omega\lambda} - g_{\lambda\alpha} g_{\mu\beta} \nabla^\lambda \delta g^{\alpha\beta} \right].$$

Therefore

$$(\delta S')_1 = -\frac{\beta}{2} \int d^4x \sqrt{-g}$$

$$\times \Big[\nabla_\nu \left(F^{\mu\rho} F^\nu{}_\rho \right) \left(g_{\omega\mu} \nabla_\lambda \delta g^{\omega\lambda} - g_{\lambda\alpha} g_{\mu\beta} \nabla^\lambda \delta g^{\alpha\beta} \right)$$

$$- \nabla_\lambda \left(F^{\mu\rho} F^\nu{}_\rho \right) \Big(g_{\omega\nu} \nabla_\mu \delta g^{\omega\lambda} + g_{\omega\mu} \nabla_\nu \delta g^{\omega\lambda}$$

$$- g_{\nu\alpha} g_{\mu\beta} \nabla^\lambda \left(\delta g^{\alpha\beta} \right) \Big) \Big]$$

$$= \frac{\beta}{2} \int d^4x \sqrt{-g}$$

$$\Big[\nabla_\lambda \nabla_\nu A^{\mu\nu} \, g_{\omega\mu} \, \delta g^{\omega\lambda} - \nabla_\alpha \nabla_\nu A^{\mu\nu} \, g_{\mu\beta} \, \delta g^{\alpha\beta}$$

$$- \nabla_\mu \nabla_\lambda A^{\mu\nu} \, g_{\omega\nu} \, \delta g^{\omega\lambda} - \nabla_\nu \nabla_\lambda A^{\mu\nu} \, g_{\omega\mu} \, \delta g^{\omega\lambda}$$

$$+ \nabla^2 A^{\mu\nu} \, g_{\nu\alpha} \, g_{\mu\beta} \, \delta g^{\alpha\beta} \Big]$$

where we used Stokes' theorem and have written $A^{\mu\nu} = F^{\mu\rho} F^{\nu}{}_{\rho}$

It's easy to see that

$$\nabla_\lambda \nabla_\nu A^{\mu\nu} g_{\omega\mu} \delta g^{\omega\lambda} = \nabla_\lambda \nabla_\nu A_\omega{}^\nu \delta g^{\omega\lambda}$$

$$= \nabla_\lambda \nabla_\mu A_\omega{}^\mu \delta g^{\omega\lambda} = \nabla_\mu \nabla_\lambda A^{\nu\mu} g_{\omega\nu} \delta g^{\omega\lambda}$$

$$= \nabla_\mu \nabla_\lambda A^{\mu\nu} g_{\omega\nu} \delta g^{\omega\lambda}$$

because $A^{\mu\nu}$ is symmetric. Thus the first and third term cancel.

Similarly the second and fourth term are the same.

Thus

$$(\delta S')_1 = \frac{\beta}{2}\int d^4x \sqrt{-g}\,\Big[-2\nabla_\alpha \nabla_\nu A^{\mu\nu} g_{\mu\beta}\,\delta g^{\alpha\beta}$$

$$+ \nabla^2 A^{\mu\nu} g_{\nu\alpha} g_{\mu\beta}\,\delta g^{\alpha\beta}\Big].$$

Summing everything up we get

$$\delta S' = \frac{\beta}{2}\int d^4x \sqrt{-g}\;\delta g^{\alpha\beta}$$

$$\Big[-2\nabla_\alpha \nabla_\nu (F_{\beta\rho} F^{\nu\rho}) + \nabla^2 (F_{\beta\rho} F_\alpha{}^\rho)$$

$$+ 2R^{\mu\nu} F_{\mu\alpha} F_{\nu\beta} - g_{\alpha\beta} R^{\mu\nu} F_{\mu\rho} F_\nu{}^\rho\Big]$$

Therefore, Einstein's equation becomes

$$R_{\alpha\beta} - \frac{1}{2} R g_{\alpha\beta}$$

$$+\beta\Big(-\nabla_\alpha \nabla_\nu (F_{\beta\rho} F^{\nu\rho}) + \frac{1}{2}\nabla^2 (F_{\beta\rho} F_\alpha{}^\rho)$$

$$+ R^{\mu\nu} F_{\mu\alpha} F_{\nu\beta} - \frac{1}{2} g_{\alpha\beta} R^{\mu\nu} F_{\mu\rho} F_\nu{}^\rho\Big)$$

$$= 8\pi G T_{\alpha\beta}.$$

Problem 2

We write

$$S = \int d^4x \sqrt{-g}\, g^{\mu\nu} R_{\mu\nu}(\Gamma)$$

where the Riemann tensor depends on a connection Γ that we vary seperately from g.

Let's first derive the equation of motion for the connection.

$$\delta S = \int d^4x \sqrt{-g}\, g^{\mu\nu}\, \delta R_{\mu\nu}(\Gamma)$$

We can show exactly like in the textbook (see Eq. 4.62) that

$$\delta R^{\rho}{}_{\mu\lambda\nu} = \nabla_\lambda (\delta \Gamma^{\rho}_{\nu\mu}) - \nabla_\nu (\delta \Gamma^{\rho}_{\lambda\mu})$$

so

$$\delta R_{\mu\nu} = \nabla_\rho\, \delta \Gamma^{\rho}_{\nu\mu} - \nabla_\nu\, \delta \Gamma^{\rho}_{\rho\mu}$$

(Remember: The covariant derivative involves the connection Γ; we have a covariant derivative because $\delta\Gamma$ is the difference of two connections and thus a tensor, see the argument leading up to 3.20 in the text book.)

Thus we get that

$$\delta S = \int d^4x \sqrt{-g}\, g^{\mu\nu} \left(\nabla_\rho \delta\Gamma^\rho_{\nu\mu} - \nabla_\nu \delta\Gamma^\rho_{\rho\mu} \right)$$

$$= \int d^4x \sqrt{-g} \left[\nabla_\rho \left(g^{\mu\nu} \delta\Gamma^\rho_{\nu\mu} \right) - \nabla_\nu \left(g^{\mu\nu} \delta\Gamma^\rho_{\rho\mu} \right) \right]$$

$$- \int d^4x \sqrt{-g} \left(\delta\Gamma^\rho_{\nu\mu} \nabla_\rho g^{\mu\nu} - \delta\Gamma^\rho_{\rho\mu} \nabla_\nu g^{\mu\nu} \right)$$

The former line vanishes by Stokes theorem and we're left with

$$\delta S = - \int d^4x \sqrt{-g}\, \delta\Gamma^\rho_{\nu\mu} \left(\nabla_\rho g^{\mu\nu} - \delta^\nu_\rho \nabla_\alpha g^{\mu\alpha} \right)$$

Thus the equation of motion is

$$\nabla_\rho g^{\mu\nu} - \delta^\nu_\rho \nabla_\alpha g^{\mu\alpha} = 0$$

Acting on this equation with δ^ρ_ν we get

$$0 = \nabla_\rho g^{\mu\rho} - 4 \nabla_\alpha g^{\mu\alpha} = -3 \nabla_\alpha g^{\mu\alpha}.$$

Thus it's easy to see that

$$\nabla_\rho g^{\mu\nu} = 0 .$$

This means that our connection is metric compatible. We furthermore assumed that the connection is torsion-free. But a torsion-free, metric compatible connection can only be the Levi-Civita connection (see p. 99 in the textbook).

Let's now turn to variations in g.

We have that

$$\delta S = \int d^4x \; R_{\mu\nu}(\Gamma)\, g^{\mu\nu}\, \delta\sqrt{-g}$$

$$+ \int d^4x \; R_{\mu\nu}(\Gamma)\, \sqrt{-g}\; \delta g^{\mu\nu}$$

$$= \int d^4x \; R_{\mu\nu}\, g^{\mu\nu} \delta\sqrt{-g}$$

$$+ \int d^4x \; \sqrt{-g} \;\; R_{\mu\nu}\, \delta g^{\mu\nu} .$$

where R is just the usual Riemann tensor (because our connection is the Levi-Civita one).

This is the same as the variation of

$$S = \int d^4x \; R_{\mu\nu} \, g^{\mu\nu} \sqrt{-g}$$

with respect to g except that we've missing the term

$$\int d^4x \sqrt{-g} \; g^{\mu\nu} \, \delta R_{\mu\nu}$$

but that can be shown to vanish (see Eq. 4.65 in the textbook and the subsequent discussion).

Thus exactly the same derivation as for the usual Hilbert action

$$S_H = \int \sqrt{-g} \, R \, d^4x$$

will lead to Einstein's equation

Problem 3

There are multiple ways to solve this problem.

Let's use that in normal coordinates the geodesics are straight lines

$$x^\mu(\lambda) = \lambda a^\mu$$

for some fixed vectors a^μ.

The geodesic equation

$$\frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu_{\omega\chi}\frac{dx^\omega}{d\lambda}\frac{dx^\chi}{d\lambda} = 0$$

then gives that

$$\Gamma^\mu_{\omega\chi}(x)\, a^\omega a^\chi = 0$$

Let's Taylor expand the equation

$$g_{\alpha\mu}\Gamma^\mu_{\omega\chi}(x)\, a^\omega a^\chi = 0$$

in x. Since Γ vanishes at the origin we have to go to first order

We have that

$$g_{\alpha\mu}\Gamma^{\mu}_{\omega\chi} = \frac{1}{2}\left(\partial_\omega g_{\chi\alpha} + \partial_\chi g_{\omega\alpha} - \partial_\alpha g_{\chi\omega}\right)$$

so

$$g_{\alpha\mu}\Gamma^{\mu}_{\omega\chi}$$

$$= \frac{1}{2}\left(g_{\chi\alpha,\omega\nu} + g_{\omega\alpha,\chi\nu} - g_{\chi\omega,\alpha\nu}\right)x^\nu$$

$$+ O(x^2).$$

This is valid along the geodesic $x^\nu = \lambda a^\nu$

so $\quad g_{\alpha\mu}\Gamma^{\mu}_{\omega\chi}(x)\,a^\omega a^\chi = 0$

gives

$$\left(g_{\chi\alpha,\omega\nu} + g_{\omega\alpha,\chi\nu} - g_{\chi\omega,\alpha\nu}\right)a^\nu a^\omega a^\chi = 0.$$

This tells us that

$$\sum_{(\nu,\omega,\chi)}\left(g_{\chi\alpha,\omega\nu} + g_{\omega\alpha,\chi\nu} - g_{\chi\omega,\alpha\nu}\right) = 0$$

all symmetric combinations

or more explicitely

$$2\left(g_{\alpha\kappa,\omega\nu} + g_{\alpha\omega,\nu\kappa} + g_{\alpha\nu,\kappa\omega}\right) - \left(g_{\kappa\omega,\nu\alpha} + g_{\omega\nu,\kappa\alpha} + g_{\nu\kappa,\omega\alpha}\right) = 0. \qquad \text{(I)}$$

We can rewrite this equation by doing

$\alpha \to \kappa \to \omega \to \alpha$ which gives

$$2\left(g_{\kappa\omega,\alpha\nu} + g_{\kappa\alpha,\nu\omega} + g_{\kappa\nu,\omega\alpha}\right) - \left(g_{\omega\alpha,\nu\kappa} + g_{\alpha\nu,\omega\kappa} + g_{\nu\omega,\alpha\kappa}\right) = 0. \qquad \text{(II)}$$

Taking this equation and doing

$\alpha \to \kappa \to \omega \to \alpha$ then gives

$$2\left(g_{\omega\alpha,\kappa\nu} + g_{\omega\kappa,\nu\alpha} + g_{\omega\nu,\alpha\kappa}\right) - \left(g_{\alpha\kappa,\nu\omega} + g_{\kappa\nu,\alpha\omega} + g_{\nu\alpha,\kappa\omega}\right) = 0. \qquad \text{(III)}$$

Summing these three equations (I, II and III)

then gives

$$\boxed{g_{\alpha\kappa,\nu\omega} + g_{\alpha\omega,\nu\kappa} + g_{\kappa\omega,\nu\alpha} = 0} \qquad \text{(A)}$$

Interchanging α and γ in (I)

gives

$$2(g_{\gamma\chi,\omega\alpha} + g_{\gamma\omega,\alpha\chi} + g_{\alpha\gamma,\chi\omega})$$

$$-(g_{\chi\omega,\gamma\alpha} + g_{\omega\alpha,\chi\gamma} + g_{\alpha\chi,\omega\gamma}) = 0.$$

Eq. (A) shows us that the negative terms vanish leading to

$$\boxed{g_{\alpha\chi,\omega\gamma} + g_{\alpha\omega,\gamma\chi} + g_{\alpha\gamma,\chi\omega} = 0} \quad \text{(B)}$$

Comparing (A) and (B) we see that

$$\boxed{g_{\chi\omega,\alpha\gamma} = g_{\alpha\gamma,\chi\omega}} \quad \text{(C)}$$

We can now finally show that

$$R_{\alpha\beta\gamma\delta} + R_{\gamma\beta\alpha\delta} = -3\, g_{\alpha\gamma,\beta\delta}.$$

In our normal coordinates

$$R_{\alpha\beta\gamma\delta} = g_{\alpha\lambda}\Gamma^{\lambda}_{\delta\beta,\gamma} - g_{\alpha\lambda}\Gamma^{\lambda}_{\gamma\beta,\delta}$$

$$= \frac{1}{2}\left[g_{\alpha\beta,\delta\gamma} + g_{\alpha\delta,\beta\gamma} - g_{\delta\beta,\alpha\gamma}\right]$$

$$-\frac{1}{2}\left[g_{\alpha\beta,\delta\gamma} + g_{\alpha\gamma,\beta\delta} - g_{\gamma\beta,\alpha\delta}\right]$$

$$= \frac{1}{2}\left(g_{\alpha\delta,\beta\gamma} - g_{\delta\beta,\alpha\gamma} - g_{\alpha\gamma,\beta\delta} + g_{\gamma\beta,\alpha\delta}\right)$$

$$= g_{\alpha\delta,\beta\gamma} - g_{\delta\beta,\alpha\gamma}$$

where we used A.

Thus

$$R_{\alpha\beta\gamma\delta} + R_{\gamma\beta\alpha\delta}$$

$$= g_{\alpha\delta,\beta\gamma} - g_{\delta\beta,\alpha\gamma} + g_{\gamma\delta,\beta\alpha} - g_{\delta\beta,\alpha\gamma}$$

$$= g_{\delta\alpha,\beta\gamma} + g_{\delta\gamma,\alpha\beta} - 2\,g_{\delta\beta,\gamma\alpha}$$

$$= -3\,g_{\delta\beta,\gamma\alpha}$$

where we used B. Finally, using C

we get

$$R_{\alpha\beta\gamma\delta} + R_{\gamma\beta\alpha\delta} = -3\,g_{\beta\delta,\alpha\gamma}.$$

Problem 4

We have that

$$R^{\alpha}{}_{\beta\gamma\delta} = \langle dx^{\alpha}, R(\partial_\gamma, \partial_\delta)\partial_\beta \rangle$$

Since the covariant derivative is metric compatible we get that

$$\nabla_\varepsilon R^{\alpha}{}_{\beta\gamma\delta} = \langle \nabla_\varepsilon dx^{\alpha}, R(\partial_\gamma, \partial_\delta)\partial_\beta \rangle$$

$$+ \langle dx^{\alpha}, (\nabla_\varepsilon R)(\partial_\gamma, \partial_\delta)\partial_\beta \rangle$$

$$+ \langle dx^{\alpha}, R(\nabla_\alpha \partial_\gamma, \partial_\delta)\partial_\beta \rangle$$

$$+ \langle dx^{\alpha}, R(\partial_\gamma, \nabla_\alpha \partial_\delta)\partial_\beta \rangle$$

$$+ \langle dx^{\alpha}, R(\partial_\gamma, \partial_\delta)\nabla_\alpha \partial_\beta \rangle.$$

Working in a normal coordinate system $\nabla_\varepsilon dx^{\alpha} = 0$ and $\nabla_\alpha \partial_\beta = 0$ (because the Christoffel symbols vanish).

Thus

$$\nabla_\varepsilon R^{\alpha}{}_{\beta\gamma\delta} = \langle dx^{\alpha}, \nabla_\varepsilon R(\partial_\gamma, \partial_\delta)\partial_\beta \rangle.$$

Problem 5

I will take as given that

$$R(X,Y) = -R(Y,X) \qquad \text{I}$$

$$\langle R(X,Y)Z, U\rangle = -\langle R(X,Y)U, Z\rangle \qquad \text{II}$$

$$R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0. \qquad \text{III}$$

We then get that

$$\langle R(X,Y)Z,U\rangle \overset{\text{III}}{=} -\langle R(Y,Z)X,U\rangle - \langle R(Z,X)Y,U\rangle$$

$$\overset{\text{II}}{=} \langle R(Y,Z)U,X\rangle + \langle R(Z,X)U,Y\rangle$$

$$\overset{\text{III}}{=} -\langle R(Z,U)Y,X\rangle - \langle R(U,Y)Z,X\rangle$$

$$-\langle R(X,U)Z,Y\rangle - \langle R(U,Z)X,Y\rangle$$

$$\overset{\text{I and II}}{=} 2\langle R(Z,U)X,Y\rangle$$

$$+\langle R(U,Y)X,Z\rangle + \langle R(X,U)Y,Z\rangle$$

$$\overset{\text{III}}{=} \quad 2\langle R(Z,U)X,Y\rangle$$
$$- \langle R(Y,X)U,Z\rangle$$

$$\overset{\text{I and II}}{=} \quad 2\langle R(Z,U)X,Y\rangle - \langle R(X,Y)Z,U\rangle.$$

Comparing the first and last expression we see that

$$\langle R(X,Y)Z,U\rangle = \langle R(Z,U)X,Y\rangle.$$

Problem 6

We're supposed to show that $\Phi = -\frac{1}{2} h_{00}$ is equivalent to $\nabla^2 \Phi = R_{00}$, i.e. that

$$R_{00} = -\tfrac{1}{2} \nabla^2 h_{00},$$

in the weak field and static limit.

[I guess the conventions in the problem text didn't match up; it's supposed to be $\Phi = -\frac{1}{2} h_{00}$

We have that

$$R_{00} = R^{\lambda}{}_{0\lambda 0} = \partial_\lambda \Gamma^{\lambda}_{00} - \cancel{\partial_0 \Gamma^{\lambda}_{\lambda 0}}$$
$$+ \Gamma^{\lambda}_{\lambda\omega} \Gamma^{\omega}_{00} - \Gamma^{\lambda}_{0\omega} \Gamma^{\omega}_{\lambda 0}.$$

The term $\partial_0 \Gamma^{\lambda}_{\lambda 0} = 0$ because we're working in the static limit.

Furthermore it's easy to see that if

$$g^{\mu\nu} = \underbrace{\eta^{\mu\nu}}_{\text{flat metric}} + h^{\mu\nu}$$

Then $\Gamma = O(h)$ so terms with

$\Gamma^2 = O(h^2)$

and can be dropped in the weak field limit. We've then left with

$$R_{00} = \partial_\lambda \Gamma^{\lambda}_{00} = \partial_i \Gamma^{i}_{00}.$$

Since

$$\Gamma^i_{00} = \frac{1}{2}\eta^{ik}(\partial_0 h_{k0} + \partial_0 h_{k0} - \partial_k h_{00})$$

$$= -\frac{1}{2}\partial^i h_{00}$$

we get that

$$R_{00} = -\frac{1}{2}\partial_i\partial^i h_{00} = -\frac{1}{2}\nabla^2 h_{00}.$$