Phys 514 - Assignment 8
Solutions

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1. Consider two inspiraling black holes with mass $10M_0$, where M_0 is the mass of the sun. Assume the system is located at a distance from us which is equal to our distance from the centre of our galaxy. Assume that the initial seperation is $100r_s$, where r_s is the Schwarzschild radius. In the weak field approximation, compute the gravitational wave amplitude $h(t)$ at the LIGO site as a function of time, making use of the quadrupole radiation formula. Then, using the formula for the radiated power derived in class, compute the gradual decay of the orbital radius $r(t)$ (using Newtonian physics to relate the energy density radiated to the change in the orbital radius). The approximations cease to be valid once $r(t)$ approaches r_s, so stop the calculation before that point.
Out of 20

Solution

The weak field limit is $g_{\mu\nu} \approx \eta_{\mu\nu} + h_{\mu\nu}$, where $h_{\mu\nu}$ corresponds to a small perturbation about Minkowski space (such as those sourced by distant binary black holes. The derivation of the quadrupole moment is presented in section 7.5 of Carroll, and so won't be rederived here. The final statement is that the trace reversed perturbation is

$$\bar{h}_{ij}(t, \mathbf{x}) = \frac{2G}{r}\frac{d^2 I_{ij}}{dt^2}(t_r)$$

Where $t_r = t - |\mathbf{x} - \mathbf{y}|$ is the retarded time, r is the distance from the source to the observer, $\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}h\eta_{\mu\nu}$, and $I_{ij}(t)$ is the quadrupole moment tensor

$$I_{ij}(t) = \int y^i y^j T^{00}(t, \mathbf{y}) d^3y$$

This is very similar to the example of the binary star system starting on page 305, so we follow closely. The velocity of each black hole is given by equating the centripedal force to the gravitational one

$$\frac{100GM_0^2}{(100r_s)^2} = \frac{10M_0 v^2}{50r_s} \qquad\qquad v = \sqrt{\frac{GM_0}{20r_s}}$$

A single orbit takes $T = 2\pi(50r_s)/v$ and has an angular frequency

$$\Omega = \frac{2\pi}{T} = \frac{1}{100}\sqrt{\frac{GM_0}{5r_s^3}}$$

The paths of the black holes (labelled by a and b) then follow

$$x_a^1 = 50r_s \cos \Omega t \qquad x_a^2 = 50r_s \sin \Omega t$$
$$x_b^1 = -50r_s \cos \Omega t \qquad x_b^2 = -50r_s \sin \Omega t$$

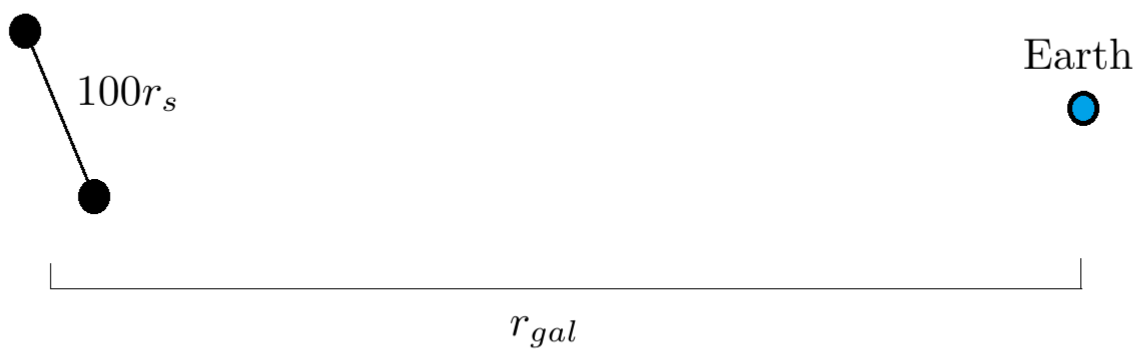

Figure 1: An over-simplified Microsoft paint illustration of the problem

The energy density is localized in space to each of the black holes, and so we get some convenient delta functions

$$T^{00}(t, \vec{x}) = 10M_0 \delta(x^3)[\delta(x^1 - 50r_s \cos \Omega t)\delta(x^2 - 50r_s \sin \Omega t) + \delta(x^1 + 50r_s \cos \Omega t)\delta(x^2 + 50r_s \sin \Omega t)]$$

The quadrupole moment is then easy to find

$$\begin{aligned}
I_{11} &= 50000M_0 r_s^2 \cos^2 \Omega t = 25000M_0 r_s^2(1 + \cos 2\Omega t) \\
I_{22} &= 50000M_0 r_s^2 \cos^2 \Omega t = 25000M_0 r_s^2(1 - \cos 2\Omega t) \\
I_{12} = I_{21} &= 50000M_0 r_s^2 \sin \Omega t \cdot \cos \Omega t = 25000M_0 r_s^2 \sin 2\Omega t \\
I_{i3} = I_{3i} &= 0
\end{aligned}$$

From this, we see that the trace-reversed perturbations are given by

$$\bar{h}_{ij}(t, \vec{x}) = \frac{2G}{r} \cdot 100000M_0 r_s^2 \Omega^2 \begin{bmatrix} -\cos 2\Omega t_r & -\sin 2\Omega t_r & 0 \\ -\sin 2\Omega t_r & \cos 2\Omega t_r & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Which of course is just the same as equation 7.149 in the text

$$\bar{h}_{ij}(t, \vec{x}) = \frac{8G}{r} \cdot MR^2 \Omega^2 \begin{bmatrix} -\cos 2\Omega t_r & -\sin 2\Omega t_r & 0 \\ -\sin 2\Omega t_r & \cos 2\Omega t_r & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

with $M \to 10M_0$ and $R \to 50r_s$.

Since we have separated the constant and oscillatory parts of $\bar{h}_{ij}$, the amplitude will just be the prefactor.

$$h(t) = \frac{8GMR(t)^2\Omega^2}{r} = \frac{2G^2M^2}{rR(t)}$$

Or, for our specific case at the beginning of the inspiral

$$h = \frac{4G^2M_0^2}{r \cdot r_s}$$

This amplitude grows with time, as the radius of orbit of the black holes ($R(t)$) shrinks.

The power radiated for a binary orbit is given by equation 7.193 in the textbook

$$P = -\frac{2}{5}\frac{G^4M^5}{R^5(t)}$$

The total energy for an object in a binary orbit is

$$E = \frac{1}{2}Mv^2 - \frac{GM^2}{2R}$$

In the Newtonian limit, the velocity is given by $v^2 = GM/2R$ so our expression becomes

$$E = -\frac{GM^2}{4R} \qquad\qquad P = \frac{dE}{dt} = \frac{GM^2}{4R^2}\frac{dR}{dt}$$

Equating the two power equations yields

$$\begin{aligned} \frac{GM^2}{4R^2}\frac{dR}{dt} &= -\frac{2}{5}\frac{G^4M^5}{R^5} \\ \frac{dR}{dt} &= -\frac{8}{5}\frac{G^3M^3}{R^3} \end{aligned}$$

Integration yields

$$R^4 = -\frac{32}{5}G^3M^3t + C$$

Our initial condition is that $R(t=0) = 50r_s$ so $C = 6.25 \cdot 10^6 r_s^4$
So our orbit decays as

$$R(t) = \left(6.25 \cdot 10^6 r_s^4 - \frac{32}{5}G^3M^3t\right)^{1/4}$$

Where $M = 10M_0$.

2. In class I justified the ansatz for a cosmological metric of the form

$$ds^2 = dt^2 - a(t)^2(d\psi^2 + f_k^2(\psi)[d\theta^2 + \sin^2\theta d\phi^2])$$

and sketched the derivation of the Einstein tensor in the tetrad basis. Perform the explicit calculation.

Solution

We will write the line element in a slightly more compact form

$$ds^2 = dt^2 - a^2(d\psi^2 + f^2[d\theta^2 \sin^2\theta d\phi^2])$$

As a matter of notation, overdots represent time derivatives, and primes represent derivatives with respect to ψ. Now, to start, our tetrad basis and exterior derivatives are

$$\begin{aligned}
e^t &= dt & \mathrm{d}e^t &= 0 \\
e^\psi &= ad\psi & \mathrm{d}e^\psi &= \dot{a}dt \wedge d\psi \\
e^\theta &= afd\theta & \mathrm{d}e^\theta &= \dot{a}fdt \wedge d\theta + af'd\psi \wedge d\theta \\
e^\phi &= af\sin\theta d\phi & \mathrm{d}e^\phi &= \dot{a}f\sin\theta dt \wedge d\phi + af'\sin\theta d\psi \wedge d\phi + af\cos\theta d\theta \wedge d\phi
\end{aligned}$$

Now we use $\mathrm{d}e^1 = e^b \wedge \omega_b^a$ to deduce the spin connection

$$\begin{aligned}
\mathrm{d}e^t = 0 &= ad\psi \wedge \omega_\psi^t + afd\theta \wedge \omega_\theta^t + af\sin\theta d\phi \wedge \omega_\phi^t \\
\mathrm{d}e^\psi = \dot{a}dt \wedge d\psi &= dt \wedge \omega_t^\psi + afd\theta \wedge \omega_\theta^\psi + af\sin\theta d\phi \wedge \omega_\phi^\psi \\
\mathrm{d}e^\theta = \dot{a}fdt \wedge d\theta + af'd\psi \wedge d\theta &= dt \wedge \omega_t^\theta + ad\psi \wedge \omega_\psi^\theta + af\sin\theta d\phi \wedge \omega_\phi^\theta \\
\mathrm{d}e^\phi = \dot{a}f\sin\theta dt \wedge d\phi + af'\sin\theta d\psi \wedge d\phi + af\cos\theta d\theta \wedge d\phi &= dt \wedge \omega_t^\phi + ad\psi \wedge \omega_\psi^\phi + afd\theta \wedge \omega_\theta^\phi
\end{aligned}$$

After some investigation, the six independent spin tensor components (and their exterior derivatives) are

$$\begin{aligned}
\omega_t^\psi &= \dot{a}d\psi & \mathrm{d}\omega_t^\psi &= \ddot{a}dt \wedge d\psi \\
\omega_t^\theta &= \dot{a}fd\theta & \mathrm{d}\omega_t^\theta &= \ddot{a}fdt \wedge d\theta + \dot{a}f'd\psi \wedge d\theta \\
\omega_t^\phi &= \dot{a}f\sin\theta d\phi & \mathrm{d}\omega_t^\phi &= \ddot{a}f\sin\theta dt \wedge d\phi + \dot{a}f'\sin\theta d\psi \wedge d\phi + \dot{a}f\cos\theta d\theta \wedge d\phi \\
\omega_\psi^\theta &= f'd\theta & \mathrm{d}\omega_\psi^\theta &= f''d\psi \wedge d\theta \\
\omega_\psi^\phi &= f'\sin\theta d\phi & \mathrm{d}\omega_\psi^\phi &= f''\sin\theta d\psi \wedge d\phi + f'\cos\theta d\theta \wedge d\phi \\
\omega_\theta^\phi &= \cos\theta d\phi & \mathrm{d}\omega_\theta^\phi &= -\sin\theta d\theta \wedge d\phi
\end{aligned}$$

Now we can determine the tetrad version of the Riemann tensor, $R_b^a = \mathrm{d}\omega_b^a + \omega_c^a \wedge \omega_b^c$. After a bit of algebra (and some glorious cancellations) we find

$$
\begin{aligned}
R^{\psi'}_{t'} &= \ddot{a} dt \wedge d\psi \\
R^{\theta'}_{t'} &= \ddot{a} f dt \wedge d\theta \\
R^{\phi'}_{t'} &= \ddot{a} f \sin\theta dt \wedge d\phi \\
R^{\theta'}_{\psi'} &= (f'' + \dot{a}^2 f) d\psi \wedge d\theta \\
R^{\phi'}_{\psi'} &= \sin\theta (f'' + \dot{a}^2 f) d\psi \wedge d\phi \\
R^{\phi'}_{\theta'} &= \sin\theta (f'^2 + \dot{a}^2 f^2 - 1) d\theta \wedge d\phi
\end{aligned}
$$

Next up is to compute the normal Riemann tensor $R^{\rho}_{\sigma\mu\nu} = e^{\rho}_{a'} e^{b'}_{\sigma} R^{a'}_{b'\mu\nu}$ where we have

$$
e^{\rho}_{a'} = \mathrm{Diag}(1, a^{-1}, (af)^{-1}, (af\sin\theta)^{-1}) \qquad\qquad e^{b'}_{\sigma} = \mathrm{Diag}(1, a, af, af\sin\theta)
$$

Our tensor is thus

$$
\begin{aligned}
R^{\psi}_{t\mu\nu} &= e^{\psi}_{\psi'} e^{t'}_{t} R^{\psi'}_{t'\mu\nu} \\
&= (a^{-1})(1)(\ddot{a} dt \wedge d\psi) \\
R^{\psi}_{t\psi t} &= -\frac{\ddot{a}}{a}
\end{aligned}
$$

$$
\begin{aligned}
R^{\theta}_{t\mu\nu} &= e^{\theta}_{\theta'} e^{t'}_{t} R^{\theta'}_{t'\mu\nu} \\
&= ((af)^{-1})(1)(\ddot{a} f dt \wedge d\theta) \\
R^{\theta}_{t\theta t} &= -\frac{\ddot{a}}{a}
\end{aligned}
$$

$$
\begin{aligned}
R^{\phi}_{t\mu\nu} &= e^{\phi}_{\phi'} e^{t'}_{t} R^{\phi'}_{t'\mu\nu} \\
&= ((af\sin\theta)^{-1})(1)(\ddot{a} f \sin\theta dt \wedge d\phi) \\
R^{\phi}_{t\phi t} &= -\frac{\ddot{a}}{a}
\end{aligned}
$$

$$
\begin{aligned}
R^{\theta}_{\psi\mu\nu} &= e^{\theta}_{\theta'} e^{\psi'}_{\psi} R^{\theta'}_{\psi'\mu\nu} \\
&= ((af)^{-1})(a)((f'' + \dot{a}^2 f) d\psi \wedge d\theta) \\
R^{\theta}_{\psi\theta\psi} &= -\left(\frac{f''}{f} + \dot{a}^2\right)
\end{aligned}
$$

$$
\begin{aligned}
R^{\phi}_{\psi\mu\nu} &= e^{\phi}_{\phi'} e^{\psi'}_{\psi} R^{\phi'}_{\psi'\mu\nu} \\
&= ((af\sin\theta)^{-1})(a)(\sin\theta (f'' + \dot{a}^2 f) d\psi \wedge d\phi) \\
R^{\phi}_{\psi\phi\psi} &= -\left(\frac{f''}{f} + \dot{a}^2\right)
\end{aligned}
$$

$$
\begin{aligned}
R^{\phi}_{\theta\mu\nu} &= e^{\phi}_{\phi'} e^{\theta'}_{\theta} R^{\phi'}_{\theta'\mu\nu} \\
&= ((af\sin\theta)^{-1})(af)(\sin\theta (f'^2 + \dot{a}^2 f^2 - 1) d\theta \wedge d\phi) \\
R^{\phi}_{\theta\phi\theta} &= -(f'^2 + \dot{a}^2 f^2 - 1)
\end{aligned}
$$

In summary, we have

$$
\begin{aligned}
R^{\psi}_{t\psi t} &= -\frac{\ddot{a}}{a} \\
R^{\theta}_{t\theta t} &= -\frac{\ddot{a}}{a} \\
R^{\phi}_{t\phi t} &= -\frac{\ddot{a}}{a} \\
R^{\theta}_{\psi\theta\psi} &= -\left(\frac{f''}{f} + \dot{a}^2\right) \\
R^{\phi}_{\psi\phi\psi} &= -\left(\frac{f''}{f} + \dot{a}^2\right) \\
R^{\phi}_{\theta\phi\theta} &= -(f'^2 + \dot{a}^2 f^2 - 1)
\end{aligned}
$$

Now recall the metric and inverse metric

$$
\begin{aligned}
g_{\mu\nu} &= \text{Diag}(1, -a^2, -a^2 f^2, -a^2 f^2 \sin^2\theta) \\
g^{\mu\nu} &= \text{Diag}(1, -a^{-2}, -a^{-2} f^{-2}, -a^{-2} f^{-2} \sin^{-2}\theta)
\end{aligned}
$$

We can now compute the Ricci tensor, $R_{\mu\nu} = R^{\sigma}_{\mu\sigma\nu}$

$$
\begin{aligned}
R_{tt} &= R^{t}_{ttt} + R^{\psi}_{t\psi t} + R^{\theta}_{t\theta t} + R^{\phi}_{t\phi t} \\
&= -\frac{\ddot{a}}{a} - \frac{\ddot{a}}{a} - \frac{\ddot{a}}{a} \\
&= -3\frac{\ddot{a}}{a}
\end{aligned}
$$

$$
\begin{aligned}
R_{\psi\psi} &= R^{t}_{\psi t\psi} + R^{\psi}_{\psi\psi\psi} + R^{\theta}_{\psi\theta\psi} + R^{\phi}_{\psi\phi\psi} \\
&= g^{tt} g_{\psi\psi} R^{\psi}_{t\psi t} + R^{\theta}_{\psi\theta\psi} + R^{\phi}_{\psi\phi\psi} \\
&= (1)(-a^2)\left(-\frac{\ddot{a}}{a}\right) - \left(\frac{f''}{f} + \dot{a}^2\right) - \left(\frac{f''}{f} + \dot{a}^2\right) \\
&= a\ddot{a} - 2\dot{a}^2 - 2\frac{f''}{f}
\end{aligned}
$$

$$
\begin{aligned}
R_{\theta\theta} &= R^{t}_{\theta t\theta} + R^{\psi}_{\theta\psi\theta} + R^{\theta}_{\theta\theta\theta} + R^{\phi}_{\theta\phi\theta} \\
&= g^{tt} g_{\theta\theta} R^{\theta}_{t\theta t} + g^{\psi\psi} g_{\theta\theta} R^{\theta}_{\psi\theta\psi} + R^{\phi}_{\theta\phi\theta} \\
&= (1)(-a^2 f^2)(-\left(\frac{\ddot{a}}{a}\right)) + (-a^{-2})(-a^2 f^2)(-\left(\frac{f''}{f} + \dot{a}^2\right)) - (f'^2 + \dot{a}^2 f^2 - 1) \\
&= a\ddot{a} f^2 - f f'' - f^2 \dot{a}^2 - f'^2 - \dot{a}^2 f^2 + 1 \\
&= 1 + f^2(a\ddot{a} - 2\dot{a}^2) - f f'' - f'^2
\end{aligned}
$$

$$
\begin{aligned}
R_{\phi\phi} &= R^t_{\phi t\phi} + R^\psi_{\phi\psi\phi} + R^\theta_{\phi\theta\phi} + R^\phi_{\phi\phi\phi} \\
&= g_{\phi\phi} g^{tt} R^\phi_{t\phi t} + g_{\phi\phi} g^{\psi\psi} R^\phi_{\psi\phi\psi} + g_{\phi\phi} g^{\theta\theta} R^\phi_{\theta\phi\theta} \\
&= (-a^2 f^2 \sin^2\theta)(1)(-\left(\frac{\ddot{a}}{a}\right)) + (-a^2 f^2 \sin^2\theta)(-a^{-2})(-\left(\frac{f''}{f} + \dot{a}^2\right)) + (-a^2 f^2 \sin^2\theta)(-a^{-2} f^{-2})(-(f'^2 + \dot{a}^2 f^2 - 1) \\
&= f^2 \sin^2\theta a\ddot{a} - \sin^2\theta f f'' - \dot{a}^2 f^2 - \sin^2\theta (f'^2 + \dot{a}^2 f^2 - 1) \\
&= \sin^2\theta R_{\theta\theta}
\end{aligned}
$$

The Ricci scalar is given by $R = g^{\mu\nu} R_{\mu\nu}$

$$
\begin{aligned}
R &= g^{tt} R_{tt} + g^{\psi\psi} R_{\psi\psi} + g^{\theta\theta} R_{\theta\theta} + g^{\phi\phi} R_{\phi\phi} \\
&= (1)\left(-3\frac{\ddot{a}}{a}\right) - (a^{-2})(a\ddot{a} - 2\dot{a}^2 - 2\frac{f''}{f}) - 2(a^{-2} f^{-2}) R_{\theta\theta} \\
&= -3\frac{\ddot{a}}{a} - \frac{\ddot{a}}{a} + 2\frac{\dot{a}^2}{a^2} + 2\frac{f''}{f a^2} - \frac{2}{a^2 f^2} - 2\frac{\ddot{a}}{a} + 4\frac{\dot{a}^2}{a^2} + 2\frac{f''}{a^2 f} + 2\frac{f'^2}{a^2 f^2} \\
&= -2\left(3\frac{\ddot{a}}{a} - 3\frac{\dot{a}^2}{a^2} - 2\frac{f''}{a^2 f} - \frac{f'^2}{a^2 f^2} + \frac{1}{a^2 f^2}\right)
\end{aligned}
$$

Computing Einstein tensor, $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$ in Mathematica (I got tired, sorry!) yields

$$
\begin{aligned}
G_{tt} &= -3\frac{\dot{a}^2}{a^2} - \frac{f'^2}{a^2 f^2} + \frac{1}{a^2 f^2} - \frac{2f''}{a^2 f} \\
G_{\psi\psi} &= -2a\ddot{a} + \dot{a}^2 + \frac{f'^2}{f^2} - \frac{1}{f^2} \\
G_{\theta\theta} &= -2a\ddot{a} f^2 + \dot{a}^2 f^2 + f f'' \\
G_{\phi\phi} &= \sin^2\theta(-2a\ddot{a} f^2 + \dot{a}^2 f^2 + f f'')
\end{aligned}
$$

3. Consider the thermal equilibrium distribution for Bose and Fermi particles. For bosons this is the black body spectrum.
a) Show that if a particle species decouples at a given time t_1 with a black body distribution, it will maintain the black body distribution with a temperature which is simply redshifted by the expansion of the Universe.
b) Is the spectreal form preserved for any initial distribution?
c) Is the black body spectrum the unique spectrum for which its form is preserved under expansion?

Solution

The distribution functions obeyed by Fermions and Bosons are

$$f(E) = \frac{1}{e^{E/T} \pm 1}$$

Where $+$ corresponds to Fermions and $-$ to Bosons. We have neglected chemical potentials here, as they are usually small (~ 0 in the case of CMB photons). Our question is, given an initial energy E_i and temperature T_i, does the form of $f(E)$ change with the expansion of the universe? To do this, we need to see how particle energy changes with expansion, and so let us look at the evolution of U^0, the time component of the four velocity. We need a metric, and so we will take the usual FRW (in the absence of curvature)

$$ds^2 = -dt^2 + a(t)^2 \left[dx^2 + dy^2 + dz^2\right]$$

The geodesic equation is as usual

$$\frac{dU^\mu}{d\lambda} + \Gamma^\mu_{\rho\sigma} U^\rho U^\sigma = 0$$

Since we are looking for the evolution of the 0 component this reduces to

$$\frac{dU^0}{d\lambda} + \Gamma^0_{\rho\sigma} U^\rho U^\sigma = 0$$

The connection coefficients can be found by

$$\Gamma^0_{\mu\nu} = -\frac{1}{2}(\partial_\mu g_{\nu 0} + \partial_\nu g_{0\mu} - \partial_0 g_{\mu\nu})$$

The only nonzero symbol has $\mu = \nu = i$, so we have

$$\Gamma^0_{ii} = \dot{a}a = \frac{\dot{a}}{a} g_{ii}$$

The geodesic equation is thus

$$\frac{dU^0}{d\lambda} + \frac{\dot{a}}{a} g_{ii} U^i U^i = 0$$
$$\frac{dU^0}{d\lambda} + \frac{\dot{a}}{a} |\mathbf{U}|^2 = 0$$

Now recall that

$$(U^0)^2 - |\mathbf{U}|^2 = 0 \qquad \text{Massless}$$
$$(U^0)^2 - |\mathbf{U}|^2 = 1 \qquad \text{Massive}$$

So we can clearly see that $U^0 dU^0 = |\mathbf{U}|d|\mathbf{U}|$ so our geodesic equation becomes

$$\frac{1}{U^0}\frac{d|\mathbf{U}|}{d\lambda} + \frac{\dot{a}}{a}|\mathbf{U}| = 0$$

Noting that $U^0 = dt/d\lambda$ yields

$$|\dot{\mathbf{U}}| + \frac{\dot{a}}{a}|\mathbf{U}| = 0$$

Solving this yields $|\mathbf{U}| \sim a^{-1}$, which, in the case of a massless particle, implies that $U^0 \sim a^{-1}$ as well. Since this is related to the particles energy, we can assert that (for massless particles) $E \sim a^{-1}$. This means

$$\frac{E_1}{E_2} = \frac{a_2}{a_1}$$

Lets compare the distributions of particles at T_1, E_1 and T_2, E_2

$$\begin{aligned} f(E_1, T_1) &= \frac{1}{e^{E_1/T_1} \pm 1} \\ &= \frac{1}{e^{\frac{a_2 E_2}{a_1 T_1}} \pm 1} \end{aligned}$$

Now since we also know that $T \sim a^{-1}$, we have

$$\frac{T_1}{T_2} = \frac{a_2}{a_1}$$

So we have

$$\begin{aligned} f(E_1, T_1) &= \frac{1}{e^{\frac{a_2 E_2}{a_1 T_1}} \pm 1} \\ &= \frac{1}{e^{E_2/T_2} \pm 1} = f(E_2, T_2) \end{aligned}$$

Therefore, the spectrum retains its shape with the expansion of spacetime.

b) No, the reason why the spectrum was preserved in part (a) was due to the fact that the spectrum was fully characterized by the ratio E/T. Any spectrum characterized only by this ratio (or any power of this ratio) will be preserved under expansion, but not otherwise. For example, a spectrum of Bremsstrahlung produced photons follows an E^{-1} distribution, and will be distorted by expansion.

c) No, any spectrum with only an E/T dependence will be preserved, since temperature and energy redshift in the same way. Maxwell Boltzmann, Fermi Dirac, and Bose Einstein spectrums will all be preserved (for massless particles, and no chemical potential).

4. In class I mentioned that it is not possible in Standard Big Bang cosmology to explain the observed spatial flatness of the universe today. To understand this problem, compute the temperature evolution of $\Omega - 1$ where $\Omega = \rho/\rho_c$, and ρ_c is the energy density of a spatially flat universe. Show that this quantity decreases rapidly as the temperature increases. What does this mean for the initial conditions in the very early universe if these are able to reproduce what we see today?

Solution

Recall that the density parameter is

$$\Omega = \frac{8\pi G}{3H^2}\rho = \frac{\rho}{\rho_c}$$

So

$$\rho_c = \frac{3H^2}{8\pi G}$$

If we recall the Friedmann equation, we have

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G\rho}{3} - \frac{\kappa}{a^2}$$

Since $H = \dot{a}/a$, we can easily rewrite this as

$$\Omega - 1 = \frac{\kappa}{H^2 a^2} = \frac{\kappa}{\dot{a}^2}$$

Where κ represents the curvature of the universe ($\kappa < 0$: open, $\kappa = 0$: flat, $\kappa > 0$: closed).

For simplicity, we will consider the universe at the interface of matter-radiation equality. Call the scale factor at this time $a_{eq} = 1$. In the radiation domination era, $a \sim t^{1/2}$ (so $H \sim t^{-1}$). Since $T \sim a^{-1}$ from a previous problem, we can note that

$$\frac{T_{eq}}{T} = \frac{a}{a_{eq}} = a$$

Where T_{eq} is the temperature of the universe at matter-radiation equality. We also have the ratio

$$\frac{a_{eq}}{a} = \left(\frac{t_{eq}}{t}\right)^{1/2} \qquad \longrightarrow \qquad a = \left(\frac{t}{t_{eq}}\right)^{1/2} \qquad \dot{a} = \frac{1}{2}t_{eq}^{-1/2}t^{-1/2} = \frac{1}{2}t_{eq}^{-1}a^{-1} = \frac{1}{2t_{eq}}\frac{T}{T_{eq}}$$

To see the temperature evolution of the density parameter, lets compute the time derivative

$$\begin{aligned}
\frac{d}{dt}(\Omega - 1) &= \frac{d}{dt}\left(\frac{\kappa}{\dot{a}^2}\right) \\
\dot{\Omega} &= 4\kappa t_{eq}^2 T_{eq}^2 \frac{d}{dt}(T^{-2}) \\
&= -8\kappa t_{eq}^2 T_{eq}^2 \frac{\dot{T}}{T^3}
\end{aligned}$$

Lets compare this with the density parameter itself

$$\Omega = 4t_{eq}^2 \kappa \frac{T_{eq}^2}{T^2} \qquad \dot{\Omega} = -8\kappa t_{eq}^2 \frac{T_{eq}^2}{T^2} \cdot \frac{\dot{T}}{T}$$

If we ignore numerical factors of t_{eq}, T_{eq}, κ, and note that if we start from some early time and evolve forward, $T > 0$, $\dot{T} < 0$ we get the following scaling of the density parameter and its derivative

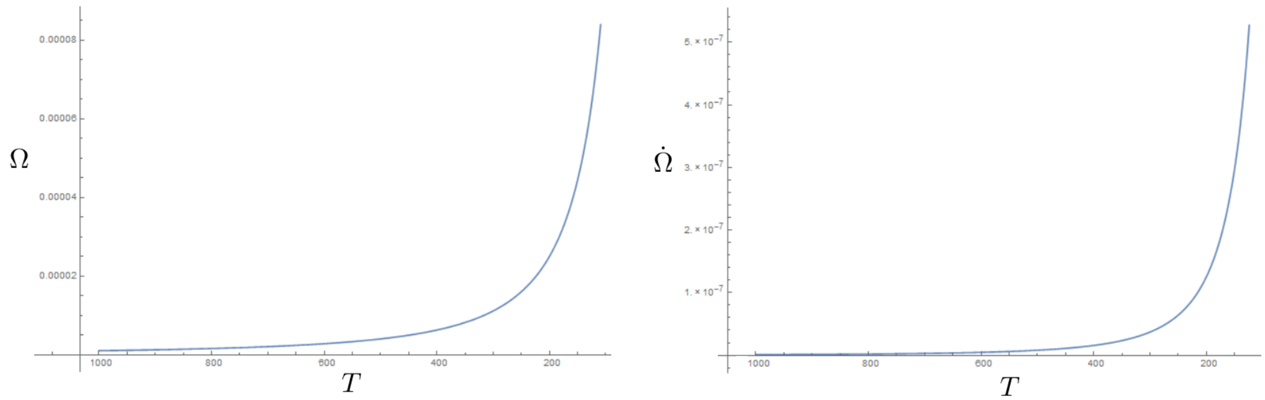

Figure 2: Temperature evolution of the density parameter and its derivative, in scaled units. Plotted with an initial temperature $T = 1000$ and final temperature $T = 10$ (in these scaled units). Note the temperature decreases along the x axis.

From the above figure, it is clear that Ω blows up from its initial value as the temperature decreases. Given that the temperature today is $T \sim 2.7K$, the range of parameters plotted is extremely conservative. Since the density parameter Ω is observed to be so small today (consistent with a flat universe), the universe would have had to be much more flat at very early times. Is there a natural mechanism that can conserve the spatial flatness of the universe? Such a topic is an open problem in cosmology.