**402-0389 Physics of the Very Early Universe 2019**
**Problem Set 4 (for Week 4)**

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**Warm-up exercises**:

*Black-body radiation (Warm-up I)*: Show that the observed momentum of a photon emitted by black-body radiation is proportional to the temperature of the radiation.

*Photon redshifting in an expanding universe (Warm-up II)*: Show that the momentum of a photon $p$ in an expanding FRW universe redshifts as $p \propto a^{-1}$ (this is purely a background calculation, so no perturbations are needed)

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**The metric potential and ISW I**: In class we discussed the ISW effect. Here we want to better understand some of the associated phenomenology. Consider the equations

$$\ddot{\delta} + 2H\dot{\delta} - \tfrac{3}{2}\Omega_m H^2 \delta = 0$$
$$k^2\Phi = 4\pi G a^2 \rho_m \delta$$

Solve the first equation in an expanding FRW universe (computing the density perturbation $\delta(a)$ and $\delta(t)$) for the cases of matter-domination ($\Omega_m \sim 1$) and $\Lambda$-domination ($\Omega_m \ll 1$). When do perturbations grow and how quickly? How does the gravitational potential $\Phi$ evolve in both cases? Does it grow/decay/stay constant with time? What does this mean for the ISW term we computed? Will there be a negative or a positive correlation between observed hot spots in the CMB with observed (foreground) galaxies? (think about CMB photons entering and leaving the gravitational well of a given galaxy on the way to our detectors)

**The metric potential and ISW II (optional)**: What changes, if (during $\Lambda$-domination) the above Poisson equation is modified to

$$k^2\Phi = 4\pi G a^2 \mu(a) \rho_m \delta,$$

where $\mu = 1 + \Omega_\Lambda a^2$? Assume the solution for $\delta$ is as before. Qualitatively, how does $\Phi$ evolve now? (Consider late times $t \to \infty$) How does this affect the ISW term? How about the above CMB-galaxy correlations?

**Perturbative connection coefficients**: For the metric

$$ds^2 = a^2(\tau)\left[-(1+2\Phi)d\tau^2 + (1-2\Psi)\delta_{ij}dx^i dx^j\right]$$

compute the connection coefficients $\Gamma^0_{00}$, $\Gamma^0_{i0}$ and $\Gamma^0_{ij}$ up to linear order in the potentials $\Phi, \Psi$. Use this to verify the following relation (which we used in calculating perturbed photon geodesics in class)

$$\Gamma^0_{\mu\nu}\frac{P^\mu P^\nu}{p^2}(1+2\Phi) = -2\mathcal{H} + \dot{\Psi} - \dot{\Phi} - 2\hat{p}^i\partial_i\Phi,$$

where a dot denotes a derivative wrt. conformal time $\tau$ and $\partial_i \equiv \partial/\partial x^i$.

**Baryon loading**: Here we want to understand the effect of baryons on the CMB spectrum better. We have the following evolution equation for $\Theta$:

$$c_s^2 \frac{\partial}{\partial \tau}(c_s^{-2}\dot{\Theta}) + c_s^2 k^2 \Theta = -\frac{k^2}{3}\Psi$$

and assume $\Phi, \Psi, R$ are all constant (notation as in the class) and $c_s^2 = 1/(3+3R)$. Can you re-phrase this in the form $\ddot{X} + c_s^2 k^2 X = 0$? What is $X$? Write down the equation of motion for the position of a mass $m$ attached to a spring with spring constant $k$ in a constant gravitational field. How does the amplitude and zero point of the oscillations shift, when the mass changes? What does this mean for the effect of baryons on $\Theta$?

**The effect of damping**: In the class we considered an evolution equation for the photon density perturbation in a photon-baryon fluid. The friction term in that equation leads to a damping of oscillations. To better understand how this works, consider the damped harmonic oscillator

$$m\ddot{x} + b\dot{x} + kx = 0.$$

Solve this when $k/m > b^2/(4m^2)$. What is the frequency of oscillations? How does the solution differ from the $b = 0$ case? What else changes when $b \neq 0$, apart from the frequency? What does this imply for the effect of the friction term for the photon density perturbation?